<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>卫星リンクとブリューニアン性の関係</td>
</tr>
<tr>
<td>著者(s)</td>
<td>谷野 隆也</td>
</tr>
<tr>
<td>引用</td>
<td>数理解析研究所講究録 (1985), 542: 27-36</td>
</tr>
<tr>
<td>発行日</td>
<td>1985-01</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/98775">http://hdl.handle.net/2433/98775</a></td>
</tr>
<tr>
<td>型式</td>
<td>部門別論文</td>
</tr>
<tr>
<td>出版者</td>
<td>京都大学出版局</td>
</tr>
</tbody>
</table>

京都大学学術情報リポジトリ
Kyoto University Research Information Repository
Satellite Links with Brunnian Properties

Taizo Kanenobu (金泰泰造)

We shall work throughout in the PL category. An n-link with m components is a locally flat oriented submanifold of the oriented \((n+2)\)-sphere \(S^{n+2}\) homeomorphic to \(m\) disjoint copies of \(S^n\). An n-knot is an n-link with one component. A trivial n-link is one whose components bound disjoint locally flat \((n+1)\)-disks \(B^{n+1}\) in \(S^{n+2}\). An n-link \(L\) is splittable if there exists an \((n+2)\)-disk \(B^{n+2}\) in \(S^{n+2}\) satisfying \(L \cap B^{n+2} \neq \emptyset\), \(L \cap \partial B^{n+2} = \emptyset\), and \(L \cap (S^{n+2} - B^{n+2}) \neq \emptyset\), where \(\partial B^{n+2}\) is the boundary of \(B^{n+2}\).

Let \(\mathcal{A}\) be the family of those subsets \(S\) of \(I = \{1, 2, \ldots, m\}\) for which the sublink \(L_S = \bigcup_{i \in S} L_i\) of an n-link \(L = L_1 \cup L_2 \cup \ldots \cup L_m\) does not split. Then we call \(L\) has the Brunnian property of type \(\mathcal{A}\). For the convenience we assume that \(\emptyset, \{i\} \notin \mathcal{A}\) for all \(i \in I\). In this family of subsets \(\mathcal{A}\), the following condition must be satisfied:

\((*)\) If \(S, T \in \mathcal{A}\) and \(S \cap T \neq \emptyset\), then \(S \cup T \in \mathcal{A}\).

Conversely we prove:
Theorem. Suppose $n \geq 1$ and $m \geq 2$. Let $\mathfrak{A}$ be a family of subsets of $I$ satisfying the condition (*). Then there exists an $n$-link with $m$ components with the Brunnian property of type $\mathfrak{A}$.

This theorem is previously obtained by H. Debrunner [3] for $n \geq 2$, using a ribbon $n$-link. Our example is a satellite link, which is defined in a similar way that a satellite knot is defined in [11, pp.110-113] and [7]. As partial results, the following are known: Let $\mathfrak{B}_k$, $2 \leq k \leq m$, be the family of all the subsets of $I$ consisting of $k$ or more elements. An $n$-link with the Brunnian property of type $\mathfrak{B}_k$ is one such that no sublink with $k$ or more components is splittable but every sublink with less than $k$ components is completely splittable. For $n = 1$ and $k = m$, such links were given by H. Brunn [1], see also [11, pp.67-69]; for $n = 1$ and $k \leq m$, by H. Debrunner [2]. R. H. Fox [3, problem 38] asked whether examples existed for $n = 2$ and $k \geq m$, and T. Yanagawa [13] answered by constructing such examples using ribbon 2-link. For $n \geq 1$ and $k = m$, see also [11, pp.197-199].

A group $G$ is indecomposable (relative to free product) if $G = A \ast B$ implies $A = 1$ or $B = 1$. To prove that a link is unsplittable, we use the following fact, cf. [10, Theorem 27.1]:

Proposition. An $n$-link $L$ is nonsplittable if its group $\pi_1(S^{n+2} - L)$ is indecomposable. If $n = 1$, then the converse is valid.
Hence any 1-knot group is indecomposable. Moreover an n-knot group with a nontrivial center ([5]) is indecomposable ([9, p.195]).

Proof of Theorem. Let $O = O_1 \cup O_2 \cup \ldots \cup O_m$ be a trivial n-link with m components. Let $x_1 \in \pi_1(S^{n+2} - 0)$ be a meridian of $O_i$. Let $S = \{i_1, i_2, \ldots, i_k\}$, $I, 1 \leq i_1 < i_2 < \ldots < i_k \leq m$. We write $F_S$ for the free group with basis $\{x_{i_1}', x_{i_2}', \ldots, x_{i_k}'\}$; thus $\pi_1(S^{n+2} - 0_S) = F_S$. Let $\alpha_S = [x_{i_1}', x_{i_2}', \ldots, x_{i_k}'],$ where $[x_{i_1}', x_{i_2}'] = x_{i_1}^{-1}x_{i_2}^{-1}x_{i_1}x_{i_2}$ and $[x_{i_1}', \ldots, x_{i_{j-1}}]', x_{i_j}'] = [[x_{i_1}', \ldots, x_{i_{j-1}}]', x_{i_j}'].$

Let $\alpha = \{S_1, S_2, \ldots, S_r\}$ and let $\alpha_i = \alpha_{S_i}$. If $n = 1$, then $\alpha_i$ can be represented by mutually disjoint simple closed curves $l_i$ in $S^3 - N(O)$, where $N(O)$ is a tubular neighborhood of 0 in $S^3$, such that the $(#S_i + 1)$-component link $O_{S_i} \cup l_i$ has the Brunnian property of type $b_{#S_i + 1}$ and that the r-component link $l_1 \cup l_2 \cup \ldots \cup l_r$ is trivial. We can always find such $l_i$ as illustrated in the figure, which consists of four circles $O_1 \cup O_2 \cup O_3 \cup O_4$ and three curves $l_1 \cup l_2 \cup l_3$, where $l_1$, $l_2$ and $l_3$ represent $[x_1', x_2', x_3']$, $[x_3', x_4']$ and $[x_2', x_4']$, respectively, see [7, p.67]. If $n \geq 2$, we can also find mutually disjoint simple closed curves $l_i$ in $S^{n+2} - N(O)$; each $\alpha_i$ is represented by a unique isotopy class of $l_i$ ([6, Corollary 8.1.2 and Theorem 10.1]).
Hence in any case, if let $V_i$ be disjoint tubular neighborhoods of the $\ell_i$ in $S^{n+2} - N(0)$, then $S^{n+2} - \text{int } V_i$ is homeomorphic to $S^n \times D^2$, where $\text{int } V_i$ is the interior of $V_i$. Let $K$ be an $n$-knot such that $\pi_1(S^{n+2} - K)$ is not infinite cyclic and indecomposable. Let $h_1 : S^{n+2} - \text{int } V_i \to N(K)$ be a homeomorphism. Then $S^{n+2} - \text{int } h_1(V_i)$ is homeomorphic to $S^n \times D^2$, $2 \leq i \leq r$. In the same way, we inductively define homeomorphisms $h_j : S^{n+2} - \text{int } V_{i-1}^j + N(K)$, $1 \leq j \leq r$, where $V_0^i = V_i$ and $V_i^1 = h(V_i^{j-1})$, $j+1 \leq i \leq r$. Let $\ell_0^i = \ell_i$ and $\ell_i^j = h_j(\ell_i^j)$. Let $L_0^i = O_i$ and $L_i^j = h_j(L_i^{j-1})$, where $L_0^i = O_i$ and $L_i^j = h_j(L_i^{j-1})$. We show that the iterated satellite link $L = L^r_i (L_i^r = L_i^r)$ has the Brunnian property of type $\mathcal{C}$. If $S_i \not\in T \subset I$, then $\ell_i^j$ and $O_T$ split and if $S_i \in S_j = \emptyset$, then $\ell_i^j$ and $O_{S_j}$ split. Thus, if $T \not\in \mathcal{C}$, then $L_T$ is splittable. Moreover, to show the contrary, we have only to prove that $L$ is nonsplittable assuming $I \in \mathcal{C}$.

Let $S_1 = I$ and $m > \#S_2 \geq \ldots \geq \#S_r$. Applying the van Kampen theorem, we have the diagrams of inclusion homomorphisms:

\[
\begin{array}{ccc}
\pi_1(S^{n+2} - \text{int } N(K)) & \xrightarrow{\eta} & \pi_1(\emptyset N(K)) \\
\pi_1(S^{n+2} - N(K)) & \xrightarrow{\xi^j} & \pi_1(S^{n+2} - L_j) \\
\pi_1(N(K) - L_j) & \xrightarrow{\theta^j} & \pi_1(N(K) - L_j)
\end{array}
\]
for $1 \leq j \leq r$. Note that $\pi_1(S^{n+2} - \text{int } N(K)) \cong \pi_1(S^{n+2} - K)$.

Since $O_{S_j}$ and $\ell_i$ split for $1 \leq i \leq j$, $L_{S_{j+1}}^j = O_{S_{j+1}}$, and so $\pi_1(S^{n+2} - L_{S_{j+1}}^j) = F_{S_{j+1}}^j$. By deleting the components which are not contained in $S_{j+1}$, we have an epimorphism $\psi^j : \pi_1(S^{n+2} - L^j) \to F_{S_{j+1}}^j$.

If both $\eta$ and $\theta^j$ are injective, then both $\xi^j$ and $\omega^j$ are also injective ([9, Sec. 4.2]), that is $\pi_1(S^{n+2} - L_j)$ is the free product of $\pi_1(S^{n+2} - K)$ and $\pi_1(N(K) - L_j)$ with an amalgamated subgroup $\pi_1(\partial N(K))$ [9, p.207]. Further suppose that both $\pi_1(S^{n+2} - K)$ and $\pi_1(N(K) - L_j)$ are indecomposable, then $\pi_1(S^{n+2} - L_j)$ is indecomposable [9, p.246].

**Case 1.** $n = 1$. Let $\pi_1(\partial N(K)) = \langle u, \lambda \mid [u, \lambda] = 1 \rangle$, where $u$ is a meridian and $\lambda$ is a longitude. Since $K$ is knotted, $\eta$ is injective ([11, Theorem 4B2]). Let $f^j : \pi_1(N(K) - L_j) \to \pi_1(S^3 - L_j^{-1})$ be an isomorphism and $\zeta^{-1} : \pi_1(S^3 - L_j^{-1}) \to \pi_1(S^3 - L_j^{-1})$ be an inclusion homomorphism. Then $\psi^{-1}f\zeta^{-1}f\psi^j(u) = a_j$, which has infinite order in $F_{S_j}$ ([9, Sec. 1.4]). Furthermore $f\psi^j(\lambda)$ is a meridian of $L_j^{j+1}$, and so $\theta^j$ is injective. Thus $\xi^j$ and $\omega^j$ are injective.

Since any proper sublink of $O \cup \ell_1$ is trivial and $\ell_1$ represents a nontrivial element $a_1$ in $F_I$, $O \cup \ell_1$ is nonsplittable, and so $\pi_1(N(K) - L_j) \cong \pi_1(S^3 - O \cup \ell_1)$ is indecomposable. In the same way, $L_{S_j}^{j-1} \cup \ell_{S_j}^{j-1} = O_{S_j} \cup \ell_j$ is
nonsplittable. Suppose that $L^{j-1}$ is nonsplittable. Then $L^{j-1} \cup L^{j-1}_j = (L^{j-1}_S \cup L^{j-1}_j) \cup L^{j-1}_j$ is also nonsplittable, and so $\pi_1(N(K) - L^j) \cong \pi_1(S^3 - L^{j-1} \cup L^{j-1}_j)$ is indecomposable. Hence by induction on $j$, $\pi_1(S^3 - L)$ is indecomposable.

**Case 2.** $n \geq 2$. Let $\pi_1(\partial N(K)) = \langle u \rangle$. Then $\eta(u)$ is a meridian of $N(K)$, and so $\eta$ is injective. Since the inclusion homomorphism $\pi_1((S^{n+2} - \text{int} V_j^{j-1}) - L^j) \rightarrow \pi_1(S^{n+2} - L^j)$ is isomorphic, we have an isomorphism $g^j : \pi_1(N(K) - L^j) \rightarrow \pi_1(S^{n+2} - L^j)$ and $\psi^j g^j \theta^j(\mu) = \alpha_{j+1}$, which has infinite order in $F_{S^1_j}$, and so $\theta^j$ is injective. Thus $\xi^j$ and $\omega^j$ are injective.

If $\pi_1(N(K) - L^1) \cong \pi_1(S^{n+2} - L^1)$ is indecomposable, then by induction on $j$, $\pi_1(S^{n+2} - L)$ is indecomposable. Hence the proof is reduced to the lemma below.

**Sublemma.** Let $H_m = \langle x_1, x_2, \ldots, x_m \mid [x_1, x_2, \ldots, x_m] = 1 \rangle$. If $m \geq 2$, then $H_m$ is indecomposable.

**Proof.** We prove by induction on $m$. $H_2$ is free abelian of rank 2, and is indecomposable. Assume that $H_{m-1}$ is indecomposable. Let $H_m = A \ast B$. Since $\beta = [x_1, x_2, \ldots, x_{m-1}]$ and $x_m$ commute, either both $\beta$ and $x_m$ are in a conjugate of $A$ or $B$, or $\beta$ and $x_m$ are both powers of the same element [9, Corollary 4.1.6]. Considering the exponent sums on generators, the latter case cannot occur. Thus by an inner
automorphism of $H_m$, we may suppose that $\beta \in A$ and $x_m \in A$. Let $N$ be the normal subgroup generated by $\beta$ and $x_m$ in $A$. Then we have $H_{m-1} \cong A/N \ast B$, cf. [8, Problem 4.1.5]. By inductive hypothesis, we obtain $A/N = 1$ or $B = 1$. If $A/N = 1$, then $H_m \cong A \ast H_{m-1}$, and so the rank (i.e., minimum number of generators) of $A$ is one [8, p.192], a contradiction. This completes the proof.

**Lemma.** If $n \geq 2$, then $G = \pi_1(S^{n+2} - L^1)$ is indecomposable.

**Proof.** Let $G = C \ast D$. Then by the condition, $\pi_1(S^{n+2} - K)$ is contained in a conjugate of $C$ or $D$ [8, p.245]. We may suppose that $\pi_1(S^{n+2} - K)$ is contained in $C$. Then the HNN extension of $G$ with an associated subgroup $\langle u \mid \rangle \cong Z$ [8, p.179]

$$G^* = \langle G, x_{m+1} \mid x_{m+1}^{-1}ux_{m+1} = u \rangle$$

is a nontrivial free product $C^* \ast D$, where $C^*$ is an HNN extension of $C$

$$C^* = \langle C, x_{m+1} \mid x_{m+1}^{-1}ux_{m+1} = u \rangle.$$

On the other hand, since $u = [x_1, x_2, \ldots, x_m]$, $G^*$ is the free product of $H_{m+1}$ and $\pi_1(S^{n+2} - K)$ with an amalgamated subgroup $\langle u \mid \rangle$. Now both $H_{m+1}$ and $\pi_1(S^{n+2} - K)$ are indecomposable, so is $G^*$, and this contradiction completes the proof.
Remark 1) A satellite n-link built from the trivial link 0 and the simple closed curve \( \ell \) representing \( \bigcap_{S \in \mathcal{L}_k} \alpha_S \), where \( \mathcal{L}_k \) is the family of all the subsets of \( I \) consisting of \( k \) elements, has the Brunnian property of type \( \mathcal{B}_k \).

2) If \( n = 1 \), then the Alexander polynomial of our link \( L \) is zero by [12, Theorem 5].

References


Department of Mathematics
Kyushu University 33
Fukuoka, 812
Japan