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CANONICAL DUALITY for u.s.d-SEQUENCES

by

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We treat here the following QUESTION:

If a sequence of elements **a** in a commutative ring A forms an unconditioned strong d-sequence (abbreviated to u.s.d-seq.) on an A-module E, then does it form a u.s.d-seq. on the canonical module K_E of A, (here we use the term 'canonical module' very roughly.)

This raised naturally from an elementary proof to the following problem given by the auther ($[S_5]$):

If a Buchsbaum local ring A possesses the canonical module K_A, then K_A is also a Buchsbaum module.

Our main theorem is :

THEOREM. Let A be a commutative ring with $1 \neq 0$ and E an A-module. Assume that a sequence $\mathbf{a}=a_1,\ldots,a_s$ of elements in A form a u.s.d-sequence on E. Then for any injective A-module I, the sequence form a u.s.d-sequence on

$$\mathrm{Hom}_A(H^s_{\mathbf{a}}(E),\ I),$$

where $H^s_{\mathbf{a}}(E)$ stands for the limit of the direct system of Koszul (co-)homology modules

$$H^i(a_1^n,\ldots,a_s^n;E)$$

and mappings

$$\phi^{n,n+1}:H^i(\mathbf{a}^n;E) \longrightarrow H^i(\mathbf{a}^{n+1};E),$$

where $\mathbf{a}^m$ denotes the system of elements $a_1^m,\ldots,a_s^m$.

As an easy but useful corollay to our main theorem, we have the following, which treat essentially the (S_2)-fication of a ring that plays an important role in the argument of Sharp's Conjecture by Aoyama and Goto in this same volume.

Corollary. Let A be a complete local ring with the canonical module K_A. Then if a sequence of elements **a** in A forms a u.s.d-sequence on the A, then it also forms a u.s.d-seq. on the A-module

$$\mathrm{Hom}_A(K_A, K_A).$$

We must give here the

DEFINITION. Let A and E be as in the theorem above. A sequence of elements in A is called a d-sequence on E if for each i=1,...,s and for any j with $i \leqq j \leqq s$ the following holds,

$$(a_1,\dots,a_{i-1})E : a_i a_j = (a_1,\dots,a_{i-1})E : a_j$$

A sequence **a** is called a <u>strong</u> d-sequence on E, if for any integers $n_1,\dots,n_s>0$, the sequence

$$a_1^{n_1},\dots,a_s^{n_s}$$

forms a d-sequence on E.

If besides each of the properties is stable under any permutation of the sequence, the term <u>unconditoned</u> is attached.

DEFINITION. Let A be a Noetherian local ring with the maximal ideal $\mathbf{m}$ and the residue field $\mathbf{k}$ and E a finitely generated A-module. A finitely generated A-module K is called the canonical module of E denoted by K_E if the completion of K is isomorphic to

$$\mathrm{Hom}_A(H^s_{\mathbf{m}}(E), I_A(\mathbf{k})),$$

where s=dimA and $I_A(k)$ denotes the injective envelope of $\mathbf{k}$ over A.

Together with the characterization of Buchsbaum modules by the d-sequence property of system of parameters, the Main Theorem leads the canonical duality theory of Buchsbaum modules:

THEOREM, ([S_5]). Let A be a Noetherian local ring with its maximal ideal $\mathbf{m}$ and the residue field $\mathbf{k}$. Assume that a finitely generated A-module E possesses the canonical module K_E.

If E is a Buchsbaum module, then K_E is also a Buchsbaum module.

In this note, to introduce the reader to the theory of u.s.d-sequences, we quote some results stated in [S_6] without proofs with the exception for the Main theorem.

u.s.d-sequences

Throughout this section, let A denote a commutative ring with $1\neq 0$ and E an A-module, unless specified otherwise. For a system of elements $\mathbf{a}=a_1,\ldots,a_s$ of A, let $K_*(\mathbf{a};E)$, $Z_*(\mathbf{a};E)$, $B_*(\mathbf{a};E)$ and $H_*(\mathbf{a};E)$ denote the complex generated by $\mathbf{a}$ over E, the cycle, the boundary and the homology module, respectively.

We begin with :

LEMMA. Let $\mathbf{a}=a_1,\ldots,a_s$ be a d-sequence on E, then:

(1) For any $i=2,\ldots,s$, $a_i,\ldots,a_s$ is a d-sequence on $E/(a_1,\ldots,a_{i-1})E$.

(2) If $i\leqq j$ then we have

$$(0:_E a_i)\subseteqq(0:_E a_j)$$

and

$$(0:_E a_1^n)=(0:_E a_1)$$

for any n>0.

(3)If therefore **a** is a u.d-sequence on E, then

$$(0:_E(\mathbf{a})A)=(0:_E a_i)$$

for any $i(1\leqq i\leqq s)$.

Consequently, if **a** is a u.s.d-sequence on E, then

$$(0:_E(a_1^{n_1},\dots,a_s^{n_s})A)=(0:_E a_i)$$

for any $i(1\leqq i\leqq s)$.

(4)If $\mathbf{a}=a_1,\dots,a_s$ is a u.s.d-sequence on E, then

$$H_q^0(E)\cap qE=0,$$

where $q=(\mathbf{a})A$.

LEMMA,(Goto's lemma [S_5]). Assume that $a_1,\dots,a_s$ is a u.s.d-sequence on E/bE for some b A. Then for any integers $n_1,\dots,n_s>0$ we have,

$$(a_1^{n_1},\dots,a_s^{n_s})E:_E b=\sum_{J\subseteqq 1,\dots,s}\left(\prod_{j\in J}a_j^{n_j-1}\right)\left[\left(\sum_{j\in J}a_jE\right):_E b\right].$$

The next theorem is one of the fundamental facts for the local cohomology with respect to a u.s.d-sequence. Essentially the proof had already been given in [S_3]Prop.4.

Theorem. If $\mathbf{a}=a_1,\ldots,a_s$ is a u.s.d-sequence on E, and $q=(\mathbf{a})A$. Then:

$$qH_p(a_1^n,\ldots,a_s^n;E)=0,$$

for any n and p>0.

Consequently, if p<s, then

$$qH_q^p(E)=0.$$

Proof of the main Theorem .

Let $q=(\mathbf{a})A$ and $L = \mathrm{Hom}_A(H_q^s(E),I)$.

We must show that for each i and j with $1\leqq i\leqq j\leqq s$, the following holds

$$(a_1,\ldots,a_{i-1})L:_L a_i a_j \subseteqq (a_1,\ldots,a_{i-1})L:_L a_j \;.$$

There exists an exact sequence

$$(\#):0 \longrightarrow H_q^{s-1}(E) \longrightarrow H_q^{s-1}(E/a_1E) \longrightarrow$$
$$H_q^s(E) \xrightarrow{\;.a_1\;} H_q^s(E) \longrightarrow 0$$

and the I-dual sequence which is also exact,

$$(\#2):0 \longrightarrow L \xrightarrow{.a_1} L \xrightarrow{T^*} L' \longrightarrow \mathrm{Hom}_A(H^{s-1}_q(E),I) \longrightarrow 0 ,$$

where $L'=\mathrm{Hom}_A(H^{s-1}_q(E/a_1E),I)$.

We at first treat the leading two elements in the sequence. From (#2) it follows that a_1 is regular on L. Also a_2 must be regular on L' by the same reason, because $a_2,\ldots,a_s$ form a u.s.d-sequence on E/a_1E, hence a_2 acts regularly on the submodule L/a_1L of L'.

Note that we have already finished for the case where i=1 and 2, in general.

In order to go further, we prepare the following lemma which is the key:

LEMMA ([S_5]). Let A,E and **a** be as the statement of the theorem (1.1) and $s\geq 3$. Let P denote the A-linear mapping

$$H^{s-1}_q(E) \longrightarrow H^{s-1}_q(E/a_1E)$$

induced from the natural mapping

$$E \longrightarrow E/a_1E$$

and I be <u>any</u> A-mdule.

Suppose that $g_2,\dots,g_s$ are A-linear mappings of $H^{s-1}_q(E/a_1E)$ into I satisfying the following equation:

$$a_2g_2+\dots+a_sg_s=0.$$

Then for each $i=2,\dots,s$ the composition $g_i\circ P=0$.

Let us continue the proof of the Main Theorem. The remaining cases are $s\geqq 3$ and $i\geqq 3$. Let

$$f \in (a_1,\dots,a_{i-1})L\underset{L}{:}a_ia_j.$$

Then we have

$$a_ia_jf \in (a_1,\dots,a_{i-1})L,$$

and by operating T^*

$$a_ia_jT^*(f) \quad (a_1,\dots,a_{i-1})L'=(a_2,\dots,a_{i-1})L'.$$

By the induction hypothesis on the length s of the sequence, we may conclude that

$$(\#3) \qquad a_jT^*(f)=\textstyle\sum_{l=2}^{i-1}a_lg_l$$

for some g_l's L'. By the lemma above, for each $l=2,\dots,i-1$, we have

$$P^*(g_l)=g_l\circ P=0.$$

By the exactness of (#2), for each $l=2,\dots,i-1$, there exists f_l L such that $T^*(f_l)=g_l$. Substituting them to (#3), it follows,

$$T^{*}(a_j f) = \sum_{l=2}^{i-1} a_l T^{*}(f_l)$$

and hence

$$T^{*}(a_j f - \sum_{l=2}^{i-1} a_l f_l) = 0.$$

Again by exactness of (#2), there must exist f_1 L such that

$$a_1 f_1 = a_j f - \sum_{l=2}^{i-1} a_l f_l,$$

namely as required we have

$$a_j f \in (a_1, \ldots, a_{i-1})L.$$

(Q.E.D.)

Let us give you a brief Proof of the Corollary stated in the introduction.

Proof of Corollary. Since we may consider A as a homomorphic image of a complete Grenstein local ring R of the same dimension as A, $K_A = \mathrm{Hom}_R(A,R)$ and it follows by the local duality theorm that

$$\mathrm{Hom}_A(K_A, K_A) = K_{(K_A)}.$$

Consequently the assertion follows directly from the main theorem.

We close this note with an example which sustains the best possibility of our main theorm.

Example. The d-sequence property is not necessarily inherited by the canonical module, even in the case where A has the finite local cohomology.

Indeed let $(A,\mathfrak{m},\mathbf{k})$ be a local ring of dimension $d>2$ and depth $d-1$ such that $H^{d-1}_{\mathfrak{m}}(A)=A/\mathfrak{m}^2$ .Let $\mathbf{a}=a_1,\ldots,a_d$ be a s.o.p of A. If furthermore we choose for some a_i with $i<d$, say $i=1$, a_1 is not contained in $\mathfrak{m}^2$, then, with $A'=A/(a_1,\ldots,a_{d-1})$,

$$H^0_{\mathfrak{m}}(A')=[0:(a_1,\ldots,a_{d-1})]_{A/\mathfrak{m}^2}=[0:a_1]_{A/\mathfrak{m}^2}=\mathfrak{m}/\mathfrak{m}^2.$$

This means that A' is a Buchsbaum ring of dimension 1, hence $\mathbf{a}$ forms a d-sequence on A. On the other hand,

$$H^2_{\mathfrak{m}}(K_A)=\mathrm{Hom}_A(H^{d-1}_{\mathfrak{m}}(A),E_A(\mathbf{k}))=\mathrm{Hom}_A(A/\mathfrak{m}^2,E_A(\mathbf{k}))$$

and

$$H^0_{\mathfrak{m}}(K_A/(a_1,a_2)K_A)=[0:(a_1,a_2)]_{H^2_{\mathfrak{m}}(K_A)}=\mathrm{Hom}_A(A/(a_1,a_2)+\mathfrak{m}^2,E_A(\mathbf{k})).$$

Let us choose a_1,a_2,a_3 so that they form a part of a minimal generating system of $\mathfrak{m}$ from the first. If they were d-sequence on K_A, then we have

$$a_3 H^2_{\mathfrak{m}}(K_A/(a_1,a_2)K_A) = 0,$$

namely, a_3 belongs to the annihilator $(a_1,a_2)+\mathfrak{m}^2$. But it is impossible, because dim A >2.

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