数理解析研究所講究録
543巻 1985年 1-12

Cohomology modules defined by an unconditioned strong d-sequence*)

東京理科大学研究生　山岸規久道 (Kikumichi Yamagishi)

The notion of a d-sequence is first introduced by Huneke [1], who has proved important results on the properties of ideals generated by a d-sequence. We presently realize that d-sequences play roles comparable to regular sequences in the theory of the ordinary Koszul complex, [2][3]. However it is impossible for us to study the behaviours of d-sequences in the cohomology theory because the cohomology functors are just given by the direct limits of the Koszul cohomologies. So we will try to do it introducing new sequences "unconditioned strong d-sequences" instead of d-sequences, see [4]. In this lecture we will discuss several properties of the cohomology modules defined by a such sequence.

§1. Let A be a commutative ring and E an A-module. Let $a_1, a_2, \dots, a_s$ be a sequence of elements in A and denote by q the ideal of A generated by $a_1, a_2, \dots, a_s$. For convenience' sake we will use the following notations:

(i) $q_k = (a_1, \dots, a_k)$ for $1 \leqq k \leqq s$ and $q_0 = (0)$;

(ii) $q_I = (a_i \mid i \in I)$ for $I \subset \{1, .., s\}$ and $q_\emptyset = (0)$.

*) This shall appear in the joint work [5] with Goto.

Recall that $a_1, a_2, \dots, a_s$ is called a d-sequence**) on E if the equality

$$q_{i-1}E : a_i a_j = q_{i-1}E : a_j$$

holds for every $1 \leqq i \leqq j \leqq s$ [1].

Definition (1.1)([4]). We will say that $a_1, a_2, \dots, a_s$ form an unconditioned strong d-sequence (henceforth a USD-sequence) on E if $a_1^{n_1}, a_2^{n_2}, \dots, a_s^{n_s}$ form a d-sequence on E in any order for every $n_1, n_2, \dots, n_s > 0$.

Our first result is stated as follows:

Theorem (1.2). The following are equivalent.

(1) $a_1, a_2, \dots, a_s$ form a USD-sequence on E .

(2) $q\,H_p(a_{i_1}^{n_1}, \dots, a_{i_k}^{n_k}; E) = (0)$ for every $1 \leqq k \leqq s$, $1 \leqq i_1 < \dots < i_k \leqq s$, $n_1, \dots, n_k > 0$ and $p > 0$.

(3) $q\,H_1(a_{i_1}^{n_1}, \dots, a_{i_k}^{n_k}; E) = (0)$ for every $1 \leqq k \leqq s$, $1 \leqq i_1 < \dots < i_k \leqq s$ and $n_1, \dots, n_k > 0$.

Proof. (1) $\Rightarrow$ (2) As it holds that

$$(a_i^{n_i} \mid i \in I)E : a_j^{n_j} \subset (a_i^{n_i} \mid i \in I)E : q$$

for every $I \subsetneqq \{1, \dots, s\}$, $j \notin I$ and $n_i, n_j > 0$ $(i \in I)$, this implication comes at once. (2) $\Rightarrow$ (3) Clear. (3) $\Rightarrow$ (1) It is enough to show that

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**) "d-" means "determinantial" (an informal talk of Huneke at Nihon University).

$(a_1^{n_1}, \ldots, a_{i-1}^{n_{i-1}})E : a_i^{n_i} a_j^{n_j} \subset (a_1^{n_1}, \ldots, a_{i-1}^{n_{i-1}})E : a_j^{n_j}$ for $1 \leqq i \leqq j \leqq s$ and $n_1, \ldots, n_i, n_j > 0$. Let x be an element of E and assume that

$$a_i^{n_i} a_j^{n_j} x = a_1^{n_1} x_1 + \ldots + a_{i-1}^{n_{i-1}} x_{i-1}$$

with $x_k \in E$. Consider the element

$$\alpha = x_1 e_1 + \ldots + x_{i-1} e_{i-1} + (- a_j^{n_j} x) e_i$$

of $K_1(a_1^{n_1}, \ldots, a_i^{n_i}; E)$. Then α is a cycle, whence by (3) we have that $q\alpha \subset B_1$, thus we get

$$a_j(a_j^{n_j} x) \in (a_1^{n_1}, \ldots, a_{i-1}^{n_{i-1}})E$$

as $a_j \in q$. Therefore we can express that

$$a_j^{n_j+1} x = a_1^{n_1} y_1 + \ldots + a_{i-1}^{n_{i-1}} y_{i-1}$$

with $y_k \in E$. Put

$$\beta = y_1 e_1 + \ldots + y_{i-1} e_{i-1} + (- x) e_i$$

in $K_1(a_1^{n_1}, \ldots, a_{i-1}^{n_{i-1}}, a_j^{n_j+1}; E)$. Then as β is a cycle it yields that $q\beta \subset B_1$ by (3) again. This shows $a_j^{n_j} x$ is contained in $(a_1^{n_1}, \ldots, a_{i-1}^{n_{i-1}})E$.

Corollary (1.3). Suppose that A is Noetherian and E is finitely generated over A and that q is contained in the Jacobson radical of A. Then $a_1, a_2, \ldots, a_s$ form a USD-sequence on E if and only if $q H_1(a_1^{n_1}, \ldots, a_s^{n_s}; E) = (0)$ for all $n_1, \ldots, n_s > 0$.

§2. Let $K^{\bullet}(a_1, \ldots, a_s; E)$ (simply say $K^{\bullet}$) be the complex defined by

$$K^p(a_1, \ldots, a_s; E) = K_{s-p}(a_1, \ldots, a_s; E)$$

for each $p \in \mathbb{Z}$. $Z^{\cdot}$, $B^{\cdot}$ and $H^{\cdot}$ are the cycle, the boundary and the homology of $K^{\cdot}$. Notice that there exists an isomorphism of complexes

$$K^{\cdot} = \mathrm{Hom}_A(K.(a_1, \ldots, a_s; A), E) \quad .$$

So it allows us to call $K^{\cdot}$ the Koszul co-complex generated by $a_1, \ldots, a_s$ over E .

We put

$$H^p_{a_1,\ldots,a_s}(.) = \varinjlim_m H^p(a_1{}^m, \ldots, a_s{}^m; .)$$

for all $p \in \mathbb{Z}$, and call it the p-th cohomology functor defined by $a_1, \ldots, a_s$. If there is no confusion, we simply denote it by $H^p_{\underline{a}}(.)$.

To end of this note, let us assume that $a_1, a_2, \ldots, a_s$ form a USD-sequence on E .

Proposition (2.1). $q\,H^p_{\underline{a}}(E) = (0)$ for all $p \neq s$.

Proof. As $q\,H^p(a_1{}^m, \ldots, a_s{}^m; E) = (0)$ for $m > 0$ by (1.2), this comes at once.

Proposition (2.2). Suppose that A is Noetherian and E is finitely generated over A and that $qE \neq E$. Then either (i) or (ii) has happened:

(i) $\mathrm{ht}_E q = s$, where $\mathrm{ht}_E q = \inf_{p \in \mathrm{Supp}_A E\,,\ p \supset q} \dim_{A_p} E_p$;

(ii) qE is contained in a primary component of zero module (0) which belongs a minimal prime ideal of E .

Moreover one also has $\mathrm{ht}_{E/H^0_{\underline{a}}(E)}\, q = s$.

Example (2.3). Let $A = k[a_1, \ldots, a_s, b_1, \ldots, b_t]$ be a polynomial ring over a field k and put $E = A/I$, where $I = (a_1, \ldots, a_s) \cap (b_1, \ldots, b_t)$. Then $a_1, a_2, \ldots, a_s$ form a USD-sequence on E and $H^0_{\underline{a}}(E) = (b_1, \ldots, b_t)E$. Further $a_1, a_2, \ldots, a_s$ form an $\overline{E}$-regular sequence, where $\overline{E} = E/H^0_{\underline{a}}(E)$.

Let $\emptyset^p : H^p(a_1, \ldots, a_s; E) \longrightarrow H^p_{\underline{a}}(E)$ denote the canonical map for each $p \in \mathbb{Z}$. Then

Proposition (2.4). Each $\emptyset^p$ is surjective for all $p \neq s$.

Remark (2.5). Surjectivity of $\emptyset^p$'s does not necessarily imply the USD-sequenceness of a_i's .

We put

$$M(q_I E) = \sum_{i \in I} [q_{I-\{i\}} E : a_i] + q_I E$$

for each $I \subset \{1, .., s\}$. Of course $M(q_\emptyset E) = (0)$.

By our definition of $K^\bullet$, we have the expression

$$K^p = \bigoplus_{\substack{I \subset \{1, .., s\} \\ \# I = s - p}} E \otimes_A Ae_I$$

with a free basis $\{e_I \mid \# I = s - p\}$ for each $0 \leqq p \leqq s$. Then we define the A-submodule $M^p(a_1, \ldots, a_s; E)$ (simply say M^p)

of K^p such that

$$M^p(a_1, \ldots, a_s; E) = \bigoplus_{\substack{I \subset \{1, \ldots, s\} \\ \# I = s - p}} M(q_{\Lambda - I}E) \otimes_A Ae_I$$

for each $0 \leqq p \leqq s$, where $\Lambda = \{1, \ldots, s\}$. $M^s = M(qE)$ clearly.

It is easy to check that $M^p \supset B^p$ for each p . Moreover we have the following

Theorem (2.6). $\operatorname{Ker} \phi^p = Z^p \cap M^p/B^p$ holds for all $0 \leqq p \leqq s$.

Proof. Let α be an element of Z^p so that $\alpha \in M^p$. Consider the following diagram:

$$\begin{array}{ccc} Z^p & \xrightarrow{\eta} & Z^p(a_1^2, \ldots, a_s^2; E) \\ \downarrow & & \downarrow \\ H^p & \longrightarrow & H^p(a_1^2, \ldots, a_s^2; E) \\ & {}_{\phi^p}\searrow & \swarrow \\ & & H^p_{\underline{a}}(E) \end{array}$$

By our definition of M^p , we see that $\eta(\alpha)$ is contained in $(a_1^2, \ldots, a_s^2) K^p(a_1^2, \ldots, a_s^2; E)$, therefore $\eta(\alpha)$ is a boundary. This shows $\operatorname{Ker} \phi^p \supset Z^p \cap M^p/B^p$. The converse is a routine.

Corollary (2.7). One has the following exact sequences:

$$0 \to \frac{Z^p \cap M^p}{B^p} \to H^p \xrightarrow{\phi^p} H^p_{\underline{a}}(E) \to 0$$

for $0 \leqq p < s$;

$$0 \to \frac{M(qE)}{qE} \to \frac{E}{qE} \to H^s_{\underline{a}}(E) .$$

Theorem (2.8). Let $s \geqq 2$ and $0 \leqq p \leqq s-2$. Then the canonical map

$$\sigma^p : H^p_{\underline{a}}(E) \longrightarrow H^p_{a_1,\ldots,a_{s-1}}(E)$$

is an isomorphism.

Proof. Put

$$H^p(n) = H^p(a_1{}^n, \ldots, a_s{}^n; E)$$
$$H'^p(n) = H^p(a_1{}^n, \ldots, a_{s-1}{}^n; E)$$

for each $n > 0$. Let $\{\xi^p_n, \xi^p_{n,m}\}$ and $\{\xi'^p_n, \xi'^p_{n,m}\}$ denote the direct systems such that:

$$\begin{array}{ccc} H^p(n) & \xrightarrow{\xi^p_n} & H^p_{\underline{a}}(E) \\ {\scriptstyle \xi^p_{n,m}}\downarrow & \nearrow_{\xi^p_m} & \\ H^p(m) & & \end{array} \qquad \begin{array}{ccc} H'^p(n) & \xrightarrow{\xi'^p_n} & H^p_{a_1,\ldots,a_{s-1}}(E) \\ {\scriptstyle \xi'^p_{n,m}}\downarrow & \nearrow_{\xi'^p_m} & \\ H'^p(m) & & \end{array}$$

for $n < m$, and let

$$\varepsilon_n : H^p(n) \longrightarrow H'^p(n)$$

be the canonical map for $n > 0$. Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H'^{p-1}(n) & \longrightarrow & H^p(n) & \xrightarrow{\varepsilon_n} & H'^p(n) & \longrightarrow & 0 \\ & & {\scriptstyle a_s{}^{m-n}\xi'^{p-1}_{n,m}}\downarrow & & {\scriptstyle \xi^p_{n,m}}\downarrow & & {\scriptstyle \xi'^p_{n,m}}\downarrow & & \\ 0 & \longrightarrow & H'^{p-1}(m) & \longrightarrow & H^p(m) & \xrightarrow{\varepsilon_m} & H'^p(m) & \longrightarrow & 0 \end{array}$$

for $n < m$. Since the top and bottom rows in this diagram are exact and since the left hand side of vertical maps are zero maps for each $n < m$, we get the next exact sequence of direct limits:

$$0 \longrightarrow H^p_{\underline{a}}(E) \xrightarrow{\sigma^p} H^p_{a_1,\ldots,a_{s-1}}(E) \longrightarrow 0$$

Theorem (2.9). There exists a canonical A-isomorphism

$$\zeta^p : H^p_{\underline{a}}(E) \longrightarrow \frac{q_pE : a_{p+1}}{q_pE}$$

for every $0 \leqq p < s$.

Proof. In case $p = 0$ there is nothing to say, so we may assume that $p > 0$. Because by (2.8) $H^p_{\underline{a}}(E)$ is canonically isomorphic to $H^p_{a_1,\ldots,a_{p+1}}(E)$, we may further assume that $p = s - 1$. From (2.7) we can find an A-isomorphism such that

$$\lambda : H^{s-1}_{\underline{a}}(E) \longrightarrow \frac{Z^{s-1}}{Z^{s-1} \cap M^{s-1}} .$$

Let $\rho : K^{s-1} \longrightarrow E \otimes_A Ae_s = E$ be the projection to the last coordinate of K^{s-1} (recall $K^{s-1} = K_1 = E \otimes_A (Ae_1 \oplus \ldots \oplus Ae_s)$). Then it is easy to check that this map ρ induces an isomorphism

$$\tau : \frac{Z^{s-1}}{Z^{s-1} \cap M^{s-1}} \longrightarrow \frac{q_{s-1}E : a_s}{M(q_{s-1}E)} .$$

We put $\zeta = \tau \circ \lambda$. Then this ζ is a required one.

Corollary (2.10). Let $0 \leqq p < s$. Then $H^p_{\underline{a}}(E) = (0)$ if and only if the condition

$$q_pE : a_{p+1} = \sum_{i=1}^{p} [(a_1, \ldots, \hat{a}_i, \ldots, a_p)E : a_i] + q_pE$$

holds.

Theorem (2.11). If $n \geqq 2$, the exact sequence

$$0 \to H^p_{\underline{a}}(E) \to H^p_{\underline{a}}(E/a_1{}^nE) \to H^{p+1}_{\underline{a}}(E) \to 0$$

splits for every $0 \leqq p \leqq s - 2$. Therefore one also has

$$H^p_{\underline{a}}(E/(a_1^{n_1}, \ldots , a_k^{n_k})E) = \bigoplus_{i=0}^{k} H^{p+i}_{\underline{a}}(E)^{\binom{k}{i}}$$

for $0 \leqq p < s - k$, $1 \leqq k < s$ and $n_1, \ldots , n_k \geqq 2$.

Proof. We have the commutative diagram such that:

$$\begin{array}{ccc} H^p(a_1^m, \ldots, a_s^m; E) & \xrightarrow{\delta_m} & H^p(a_1^m, \ldots, a_s^m; E/a_1^n E) \\ \downarrow & & \downarrow \\ H^p_{\underline{a}}(E) & \longrightarrow & H^p_{\underline{a}}(E/a_1^n E) \end{array}$$

for each $m \geqq n$. As $m \geqq n$, the map δ_m splits, therefore we know that the canonical map $H^p_{\underline{a}}(E) \longrightarrow H^p_{\underline{a}}(E/a_1^n E)$ also splits. The rest of (2.11) comes by induction.

Corollary (2.12). Let $1 \leqq k < s$ and $n_1, \ldots , n_k \geqq 2$. Then $q\, H^p_{\underline{a}}(E/(a_1^{n_1}, \ldots , a_k^{n_k})E) = (0)$ for all $0 \leqq p < s - k$.

Corollary (2.13). Let $1 \leqq k < s$ and $n_{k+1}, \ldots , n_s \geqq 2$. Then

$$H_p(a_1, \ldots, a_k, a_{k+1}^{n_{k+1}}, \ldots, a_s^{n_s}; E) = \left\{ \bigoplus_{\substack{0 \leqq i \leqq t \\ i < p}} H_{p-i}(a_1, \ldots, a_k; E)^{\binom{t}{i}} \right\} \oplus \left\{ \bigoplus_{\substack{0 \leqq i \leqq t \\ i \geqq p}} H^{i-p}_{\underline{a}}(E/q_k E)^{\binom{t}{i}} \right\}$$

holds for all $1 \leqq p \leqq s$, where $t = s - k$.

§3. Denote by $l_A(.)$ the length of an A-module. Recall that we assume that $a_1, a_2, \ldots, a_s$ form a USD-sequence on E .

Proposition (3.1). The following are equivalent.

(1) All $H^p_{\underline{a}}(E)$ $(p \neq s)$ have finite length.

(2) All $H^p(a_1, \ldots, a_s; E)$ $(p \neq s)$ have finite length.

Proof. By (2.4) it suffices to show (1) $\Rightarrow$ (2). We shall prove by induction on k that $H_p(a_1, \ldots, a_k; E)$ has finite length for all $p > 0$, where $1 \leqq k \leqq s$. If $k = 1$, there is nothing to say because $H_1(a_1; E) = 0 : a_1 = H^0_{\underline{a}}(E)$. Let $k \geqq 2$ and assume that our assertion holds for $k - 1$. Put

$$K. = K.(a_1, \ldots, a_k; E) \quad \text{and} \quad K'. = K.(a_1, \ldots, a_{k-1}; E) .$$

Then we have the exact sequences:

(#) $$0 \to H'_p \to H_p \to H'_{p-1} \to 0$$

for $p \geqq 2$ and

(##) $$0 \to H'_1 \to H_1 \to H^0_{\underline{a}}(E/q_{k-1}E) \to 0$$

Since $H^0_{\underline{a}}(E/q_{k-1}E)$ also has finite length from (2.1) our assertion immediately comes by (#) and (##) applying the hypothesis of induction on k .

If all $H^p_{\underline{a}}(E)$ $(p \neq s)$ have finite length, we define

$$I(E) = \sum_{i=0}^{s-1} \binom{s-1}{i} . h^i(E) ,$$

where $h^i(E) = l_A(H^i_{\underline{a}}(E))$, and

$$\chi(a_1, \ldots, a_s; E) = \sum_{p=0}^{s} (-1)^p \, l_A(H_p(a_1, \ldots, a_s; E)) \quad .$$

Theorem (3.2). Suppose that all $H^p_{\underline{a}}(E)$ $(p \neq s)$ have finite length. Then one has

$$\chi(a_1, \ldots, a_s; E) = l_A(E/qE) - I(E) \quad .$$

Proof. Since

$$l_A(H_p(a_1, \ldots, a_s; E)) = \sum_{i=0}^{s-p} \binom{s}{p+i} . h^i(E)$$

holds for all $p > 0$, we have

$$\begin{aligned} \chi(a_1, \ldots, a_s; E)) &= l_A(E/qE) + \sum_{p=1}^{s} (-1)^p \Big(\sum_{i=0}^{s-p} \binom{s}{p+i} . h^i(E) \Big) \\ &= l_A(E/qE) - \sum_{i=0}^{s-1} \Big(\sum_{p=1}^{s-i} (-1)^{p-1} \binom{s}{p+i} \Big) . h^i(E) \\ &= l_A(E/qE) - I(E) \end{aligned}$$

because $\binom{s-1}{i} = \sum_{p=1}^{s-i} (-1)^{p-1} \binom{s}{p+i}$ for $0 \leqq i < s$ (if $s \geqq 2$).

Let $G_q(E)$ denote the associated graded module of E with respect to q and put $h_i = a_i \bmod q^2$ in $[G_q(A)]_1$ for $1 \leqq i \leqq s$.

Theorem (3.3). Suppose that all $H^p_{\underline{a}}(E)$ $(p \neq s)$ have finite length. Then one has

$$\chi(a_1, \ldots, a_s; E) = \sum_{i=1}^{s-1} \binom{s-1}{i-1} . h^i(E) + l_A([H^s_{\underline{h}}(G_q(E))]_{-s})$$

Proof. As $E \supset M(qE) \supset qE$, we get by (3.2) that

$$\chi(a_1, \ldots, a_s; E) = l_A(E/M(qE)) + l_A(M(qE)/qE) - I(E) \quad .$$

Since we know that

$$l_A(M(qE)/qE) = \sum_{i=0}^{s-1} \binom{s}{i} \cdot h^i(E)$$

and

$$[H_{\underline{h}}^s(G_q(E))]_{-s} = E/M(qE) \quad ,$$

our assertion comes by arithmetical computations.

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