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Invariant subspace problemについて

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C. J. Readの結果「non trivialなinvariant subspaceをもたないBanach空間上の有界線型作用素が存在する」について紹介する．これはすでに［1］に発表されているが，同論文の末尾に「ℓ^1空間上にもこのような作用素Tが作くれる」とある．この改良された結果を述べたノートが手に入ったので講演でもこれをもとに紹介を行なった。

以下は，このノートをもとにしたタイプである．（オリジナルのノートは手書のため読み難くかった）なお，正確に欠けていると思われる点は，筆者の理解する範囲で修正し，［Note］と図を補った．また，内容の順序も一部並べ変えた．このため，最終的にReadによって発表されるものは別の形になっていることと思われる．なおこのノートは伊藤隆司先生（武蔵工大）から越先生（北大）を通じて入手した．記してお礼を述べたい．

C. Read's construction of an operator on the Banach space ℓ^1 having no non-trivial invariant subspace.

(Based on a seminor given by Read in Edinburgh on 7/5/84)

Let $\mathcal{P}$ denote the complex vector space of all polynomials in a single variable x with complex coefficients, and let $\mathcal{P}_n$ denote the subspace of polynomials of degree $\leq$ n, so that $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$. We let p_n denotes the natural projection of $\mathcal{P}$ onto $\mathcal{P}_n$, so that $p_n(\Sigma a_k x^k) = \Sigma a_k x^k$. We also define a norm $\| \; \|_*$ on $\mathcal{P}$ by $\| \Sigma a_k x^k \|_* = \Sigma |a_k|$.

Lemma 1. Let n be a positive integer and let ε, δ, $M > 0$. Then, there exists $K = K(\varepsilon,\delta,M,n) > 0$ such that if $0 \leq m \leq n$ and $g \in \mathcal{P}_n$ with $\| g \|_* \leq M$ and $\| p_m g \|_* \geq \delta$ then we can find $q \in \mathcal{P}_n$ with $\| p_n(qg) - x^m \|_* < \varepsilon$ and $\| q \|_* \leq K$.

Proof. If $g = \sum_{j=k}^{n} \alpha_j x^j$ and $\alpha_k \neq 0$, then considering a linear combination of g, $p_n(xg)$, ... , $p_n(x^n g)$ we find a polynomial $q \in \mathcal{P}_n$ with $p_n(qg) = x^m$ for $k \leq m \leq n$. Since the set $\{ g \in \mathcal{P}_n : \| g \|_* \leq M , \| p_m g \|_* \geq \delta \}$ is compact, we need only a finite number of q_1, ... , q_ℓ to have $\| p_n(q_j g) - x^m \|_* < \varepsilon$ for some q_j. So, $K = \max_j \| q_j \|_*$ has the desired propety.

Let $1 < a_1 < b_1 < a_2 < b_2 \ldots$ be a sequence of positive integers which are required to increase rapidly; in the statements of subsequent lemmas the condtion 'provided the sequence a_1, b_1,... increases fast enough' will be understood.

We put $a_0 = b_0 = 1$ and let $v_1 = v_0 = 0$, $v_n = (n-1)(a_n + b_n)$ for $n > 1$.

We define a basis $f_0, f_1, \ldots$ of $\mathcal{P}$ as follows:

$$f_0 = 1;$$

If $n = r = 1$, or $n \geq 2$ and $1 \leq r \leq n-1$, then

(A) for $ra_n \leq k \leq ra_n + v_{n-r-1}$

$$f_k = a_{n-r}(x^k - x^{k-a_n});$$

(B) for $a_1 + b_1 < k < a_2$, $(r-1)a_n + v_{n-r} < k < ra_n$ $(n \neq 2)$

$$f_k = 2^{[(r-\frac{1}{2})a_n - k]/b_{n-1}} \cdot x^k \quad ;$$

(C) for $k = a_1 + b_1$, $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$ $(n \geq 2)$

$$f_k = x^k - b_n x^{k-b_n} \quad ;$$

(D) for $a_1 < k < a_1 + b_1$, $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$ $(n \geq 2)$

$$f_k = 2^{[(r-\frac{1}{2})b_n - k]/na_n} \cdot x^k \quad .$$

For given $n \geq 1$, formulae (A) - (D) define f_k for $v_{n-1} < k < v_n$. Note that the polynomial f_k always has a non-zero x^k term and no terms of higher order; hence $f_0, \ldots, f_n$ span $\mathcal{P}_n$ for each n. (See Fig. 1).

We define a norm $\| \|_X$ on $\mathcal{P}$ by $\| \Sigma \alpha_k f_k \|_X = \Sigma |\alpha_k|$ and let X be the completion of $\mathcal{P}$ with respect to this norm. So,

X is a Banach space isomorphic to ℓ^1. We also define a linear operator $T : \mathcal{P} \to \mathcal{P}$ by $Tx^k = x^{k+1}$, i.e., T is multiplication by x. Note that $p(T)q = pq$ for polynomials p,q.

We note two consequences of the definition of the norm $\| \ \|_X$. If $1 \le r \le n-1$ and $ra_n \le k \le ra_n + v_{n-r-1}$, then from (A) we get

$$x^k - x^{k-ra_n} = a_{n-r}^{-1} f_k + a_{n-r+1}^{-1} f_{k-a_n} + \dots + a_{n-1}^{-1} f_{k-(r-1)a_n}$$

So

$$\| x^k - x^{k-ra_n} \|_X = a_{n-r}^{-1} + \dots + a_{n-1}^{-1} \le 2/a_{n-r} \qquad \dots\dots(1)$$

assuming $\{a_n\}$ increases fast enough. Similarly, from (C) we get, for $r(a_n + b_n) \le k \le (n-1)a_n + rb_n$,

$$\|x^k - b_n^r x^{k-rb_n} \|_X = 1 + b_n + \dots + b_n^{r-1} \le 2b_n^{r-1} \qquad \dots\dots(2)$$

[Note: (1) and (2) show the corresponding finite dimensional subspaces are nearly isometric each other with respect to the coordinate x^k and the norm $\| \ \|_X$. See Fig. 2].

<u>Lemma 2</u>. $\| Tf_k \|_X \le 2$ for all k.

From Lemma 2 it follows that T extends by continuity to a bounded linear operator on X, with $\|T\|_X \le 2$.

For $m > 1$, we define $\sigma_m = \{ k \in N :$ for some $n > m$ we have $(n-m)a_n \le k \le (n-m)a_n + v_{m-1} \}$. [Note: In other words, $\sigma_m = \bigcup_{n>m} \{ k \in (A):$ with $n = n$ and $r = n-m$ in the definition of $f_k \}$].

Lemma 3. Suppose $m > 2$, $k > (m-1)a_m$ and $b_m + a_m \le s \le b_m + (m-1)a_m$. Then,

(a) if $k \notin \sigma_m$, we have $\|T^s f_k\|_x \le 4$.

(b) if $k \in \sigma_m$ then, writting $k = (n-m)a_n + j$, $0 \le j \le \nu_{m-1}$, we have $\| T^s f_k + a_m x^{j+s} \|_x \le 1$.

We now define a linear mapping $Q_m : \mathcal{P} \to \mathcal{P}_{(m-1)a_m}$ for $m > 2$ [see Fig. 3.3 for Q_m with $m = k$] by

$$Q_m(f_k) = \begin{cases} f_k & \text{if} \quad 0 \le k \le (m-1)a_m \\ 0 & \text{if} \quad k > (m-1)a_m,\ k \notin \sigma_m \\ -a_m x^j & \text{if} \quad k \in \sigma_m \ (\text{i.e., } k=(n-m)a_n+j,\ j \le \nu_{m-1}). \end{cases}$$

The conclusion of Lemma 3 can be restated as follows:

$$m > 2,\ b_m + a_m \le s \le b_m + (m-1)a_m,\ g \in \mathcal{P}$$
$$\Rightarrow \quad \| T^s g - T^s Q_m(g) \|_x < 4 \|g\|_x \quad .....(6)$$

In fact, to prove (6) it suffices to prove it for $g = f_k$; this is trivial if $k \le (m-1)a_m$ and follows from Lemma 3 for $k > (m-1)a_m$.

We can find $C_m \ge a_m$, depending only on $a_1, b_1, \dots, a_m$, such that $\| f_j \|_* \le C_m$ and $\| x^j \|_x \le C_m$ for $j = 0, \dots, (m-1)a_m$. Then, by the defintion of Q_m, we have $\| Q_m(f_k) \|_x \le a_m C_m$ for all k, so $\| Q_m g \|_x \le a_m C_m \| g \|_x$ for $g \in \mathcal{P}$, and so Q_m extends continuously to X and (6) holds for $g \in X$.

Note also that $\| Q_m(f_k) \|_* \le \max(a_m, C_m) \le C_m$, so

$$\|Q_m(g)\|_* \le C_m \|g\|_X, \quad g \in X.$$

Lemma 4. Let $g \in X$ with $\| g \|_X = 1$. Suppose that for some $m > 2$ and $1 \le r < m-2$ we have $\| P_{ra_m} Q_m(g) \|_* \ge 1/a_m$. Then, there is a polynomial ϕ with $\| \phi(T)g - 1 \|_X < 3/a_{m-r-1}$.

THEOREM. Let Y be a closed subspace of X with $TY \subseteq Y$. Then, either $Y = \{0\}$ or $Y = X$.

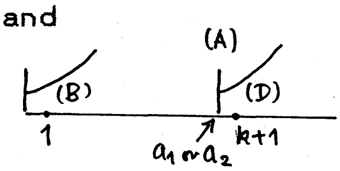
Proof of Lemma 2. We consider k belonging to the four types (A), (B), (C), (D), separately. In the course of the proof we note estimate (3), (4), (5) below for future use.

Case (A): $ra_n \le k \le ra_n + \nu_{n-r-1}$

In this case $Tf_k = a_{n-r}(x^{k+1} - x^{k+1-a_n})$. Now, if $k < ra_n + \nu_{n-r-1}$, we get similarly $Tf_k = f_{k+1}$, so $\| Tf_k \|_X = 1$. This leaves the case $k = ra_n + \nu_{n-r-1}$. If $n = r = 1$ or $n=2$ and $r=1$, then $k+1 = a_n+1 \in (D)$ and $k+1-a_n = 1 \in (B)$, so

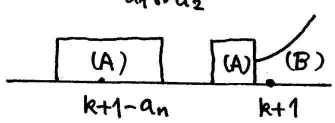
$$\| x^{k+1} \|_X = 2^{(a_n+1-\frac{1}{2} b_n)/a_n} < 1/b_n \quad ; \text{ and}$$

$$\| x^{k+1-a_n} \|_X = 2^{(1-\frac{1}{2} a_1)} < 1/a_1^2 .$$



So, $\| Tf_k \|_X \le 1/a_{n-r}$.

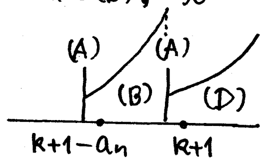
If $n > 2$ and $r < n-1$, then $k+1 \in (B)$, so



$$\| x^{k+1} \|_X = 2^{(\nu_{n-r-1}+1-\frac{1}{2} a_n)/b_{n-1}} < 1/a_n$$

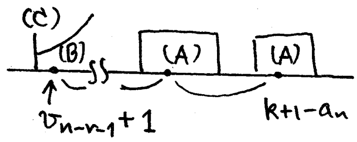
and if $n > 2$ and $r = n-1$, then $k+1 \in (D)$ and $k+1-a_n \in (B)$, so

$$\| x^{k+1} \|_X = 2^{[(n-1)a_n+1-\frac{1}{2} b_n]/na_n} < 1/b_n .$$



$$\| x^{k+1-a_n} \|_x = 2^{(-\frac{1}{2}a_n+1)/b_{n-1}} < 1/a_n$$

Also, if $\begin{cases} r < n-1 \\ n > 2 \end{cases}$, then $v_{n-r-1}+1 \in (B)$, and using (1),



$$\begin{aligned} \| x^{k+1-a_n} \|_x &= \| x^{(r-1)a_n+v_{n-r-1}+1} \|_x \\ &\leq 2/a_{n-r+1} + \| x^{v_{n-r-1}+1} \|_x \\ &= 2/a_{n-r+1} + 2^{(v_{n-r-1}+1-\frac{1}{2}a_{n-r})/b_{n-r-1}} . \end{aligned}$$

[Note: The above estimate is true even if r=1]. So

$$\begin{aligned} &\| Tf_{ra_n+v_{n-r-1}} \|_x \\ &\qquad \leq a_{n-r}\left(4/a_{n-r+1} + 2^{(v_{n-r-1}+1-\frac{1}{2}a_{n-r})/b_{n-r-1}} \right). \\ &\qquad \leq 1/a_{n-r} \qquad \ldots\ldots\ldots(3) \end{aligned}$$

provided $\{ a_n, b_n \}$ increase fast enough. So, in case (A), $\| Tf_k \|_x \leq 1$ always.

<u>Case (B)</u>: $(r-1)a_n + v_{n-r} < k < ra_n$

In this case $Tf_k = 2^{[(r-\frac{1}{2})a_n - k]/b_{n-1}} \cdot x^{k+1}$.

Now if $k < ra_n-1$, we get $Tf_k = 2^{1/b_{n-1}} f_{k+1}$, so

$$\| Tf_k \|_x = 2^{1/b_{n-1}} < 2 .$$

This leaves the case $k = ra_n-1$, then $Tf_k = 2^{(1-\frac{1}{2}a_n)/b_{n-1}} \cdot x^{ra_n}$ and $\| x^{ra_n} \|_x \leq 1 + 2/a_{n-r}$ by (1), so provided a_n is large enough,

$$\| T f_{ra_n-1} \|_x \leq 1/a_n \qquad \ldots\ldots\ldots(4)$$

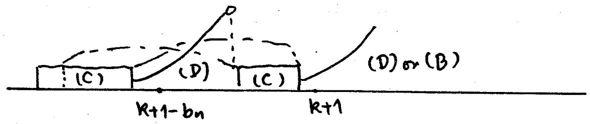
The same proof works for $a_1 + b_1 < k < a_2$ with $n = 2$ and $r=1$.

Case (C) : $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$

In this case $Tf_k = x^{k+1} - b_n x^{k+1-b_n}$.

If $k < (n-1)a_n + rb_n$, then $Tf_k = f_{k+1}$ so $\| Tf_k \|_x = 1$.

This leaves the cases $k = a_1 + b_1$ ($n = r = 1$) and $k = (n-1)a_n + rb_n$. If $r < n-1$, then by (D)



$$Tf_k = 2^{[(n-1)a_n + 1 - \frac{1}{2} b_n]/na_n}(f_{k+1} - b_n f_{k+1-b_n}).$$

If $r = n-1$ or $k = a_1 + b_1$, then $k+1 \in (B)$ and $k+1-b_n \in (D)$, so

$$Tf_k = 2^{k+1-\frac{1}{2} a_{n+1}} \cdot f_{k+1} - 2^{[(n-1)a_n + 1 - \frac{1}{2} b_n]/na_n} \cdot b_n f_{k+1-b_n}$$

or

$$Tf_k = 2^{k+1-\frac{1}{2} a_2} f_{k+1} - 2^{[a_1 + 1 - \frac{1}{2} b_1]/a_1} b_1 f_{k+1-b_1},$$

respectively. So,

$$\| Tf_k \|_x = (1+b_n) \cdot 2^{[(n-1)a_n + 1 - \frac{1}{2} b_n]/na_n} \leq 1$$

if b_n and a_{n+1} large enough.

Case (D) : $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$

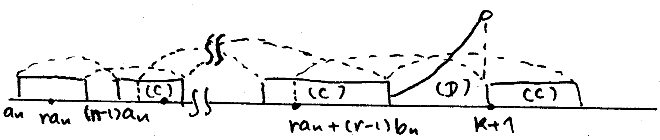
In this case $Tf_k = 2^{[(r-1)b_n - k]/na_n} \cdot x^{k+1}$.

If $k < r(a_n + b_n) - 1$, then $Tf_k = 2^{1/na_n} f_{k+1}$, so $\| Tf_k \|_x < 2$.

This leves the case $k = r(a_n + b_n) - 1$; then,

$$Tf_k = 2^{(1-ra_n - \frac{1}{2} b_n)/na_n} \cdot x^{r(a_n+b_n)}$$

and by (2) and (1) we get



$$\| x^{r(a_n+b_n)} \|_x \leq 2b_n^{r-1} + \| x^{ra_n} \|_x \leq 2b_n^{r-1} + 2/a_{n-r} + 1$$

so if b_n is large enough, we get

$$\| Tf_k \|_x < 1/b_n \quad \text{when} \quad k = r(a_n+b_n)-1 \quad \ldots\ldots(5)$$

The same proof works for $a_1 < k < a_1 + b_1$. This completes the proof.

<u>Proof of Lemma 3</u>. Again we consider separately the cases (A), (B), (C), (D). Case (b) of the statement is covered by case (A)(i) below; the remaining cases cover (a).

<u>Case (A)</u> $ra_n \leq k \leq ra_n + v_{n-r-1}$

In this case $T^s f_k = a_{n-r}(x^{k+s} - x^{k+s-a_n})$ and

$$a_m + b_m + ra_n \leq k+s \leq (m-1)a_m + b_m + ra_n + v_{n-r-1} \quad \ldots(*A)$$

The hypothesis $k > (m-1)a_m$ implies $n > m$. We consider three such cases:

<u>(i) $r = n - m$</u> (<=> $k \in \sigma_m$):

Write $k = ra_n + j$, where $0 \leq j \leq v_{m-1}$; then

$$b_m < j+s < v_{m-1} + (m-1)a_m + b_m < 2b_m ,$$

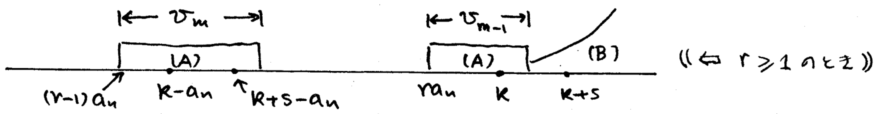
so $ra_n + v_{n-r} < k+s < (r+1)a_n$, i.e., $k+s \in$ (B). We get

$$\| x^{k+s} \|_x = 2^{[\, j + s - \frac{1}{2} a_n \,]/b_{n-1}} \le 2^{[2b_m - \frac{1}{2} a_n]/b_{n-1}} \le 1/a_n$$

if a_n is large enough. Also, since $j + s \le 2b_m < v_m$ (since $m > 2$) [Note: $(r-1)a_n \le s+j+(r-1)a_n \le (r-1)a_n + v_m$, $m = n-(r-1)-1$], we get from (1)

$$\| x^{k+s-a_n} - x^{s+j} \|_x = \| x^{s+j+(r-1)a_n} - x^{s+j} \|_x \le 2/a_{m+1} \,.$$

So,



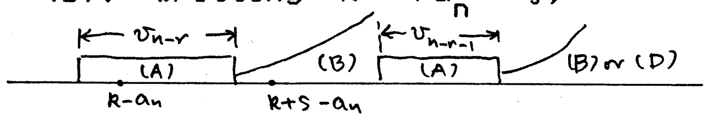
$$\| T^s f_k + a_m x^{s+j} \|_x \le a_m \| x^{k+s} \|_x + a_m \| x^{k+s-a_m} - x^{s+j} \|_x$$
$$\le a_m (1/a_n + 2/a_{m+1}) < 1 \,.$$

<u>(ii) $r > n - m$</u> :

In this case $s > b_m > v_{n-r}$, so

$$ra_n + v_{n-r-1} < k+s < (r+1)a_n \quad\text{and}\quad (r-1)a_n + v_{n-r} < k+s-a_n < ra_n .$$

So, $k+s \in$ (B) or (D) and $k+s-a_n \in$ (B). Writting $k = ra_n + j$, $0 \le j \le v_{n-r-1}$, we get



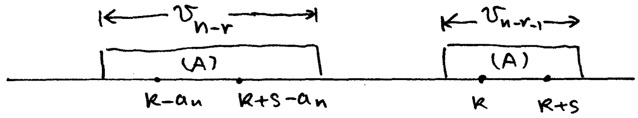
$$\| x^{k+s} \|_x = \| x^{k+s-a_n} \|_x = 2^{[j+s- \frac{1}{2} a_n]/b_{n-1}} \le 2^{(2b_m - \frac{1}{2} a_n)/b_{n-1}} ,$$

or $2^{(k+s-\frac{1}{2}b_n)/na_n} \le 2^{(na_n - \frac{1}{2}b_n)/na_n}$

[Note: $(r+1- \frac{1}{2})a_n - (ra_n+j+s) = (r- \frac{1}{2})a_n - (ra_n+j+s-a_n)$] since $j+s \le v_{n-r-1}+(m-1)a_m+b_m \le v_{m-1}+(m-1)a_m+b_m < 2b_m$. So,

$$\| T^s f_k \|_x \le 2a_{n-r} \cdot 2^{(2b_m - \frac{1}{2} a_n)/b_{n-1}} < 1$$

if a_n is large enough.

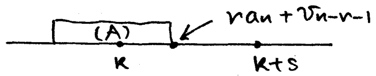


(iii) $r < n - m$:

Write $k = ra_n + j,\ 0 \le j \le v_{n-r-1}$.

(α) $j+s \le v_{n-r-1}$: Then [Note: $k+s \in (A)$] $T^s f_k = f_{k+s}$, so $\| T^s f_k \|_x = 1$.

(β) $j+s > v_{n-r-1}$: Then [Note: $T^{v_{n-r-1}-j} f_k = T^{ra_n+v_{n-r-1}-k} f_k = f_{ra_n+v_{n-r-1}}$ as in (α), since $ra_n+v_{n-r-1} \in (A)$] $T^s f_k = T^{j+s-v_{n-r-1}} Tf_{ra_n+v_{n-r-1}}$. Using (3) and $| T |_x \le 2$, we get



$$\| T^s f_k \|_x \le 2^{j+s-v_{n-r-1}-1} / a_{n-r} < 2^s/a_{n-r} < 2^{2b_m}/a_{n-r} < 1$$

if a_{n-r} is large enough.

Case (B): $(r-1)a_n + v_{n-r} < k < ra_n$

$$a_m + b_m + (r-1)a_n + v_{n-r} < k+s < (m-1)a_m + b_m + ra_n \quad \ldots(*B)$$

The hypothesis $k > (m-1)a_m$ implies $n > m$. [Note: Hence, $n \neq 2$].

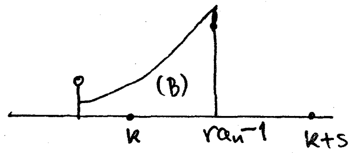
(i) $k+s < ra_n$: Then, $k+s \in (B)$. Hence,

$$T^s f_k = 2^{[(r-\frac{1}{2})a_n - k]/b_{n-1}} \cdot x^{k+s} = 2^{s/b_{n-1}} f_{k+s} .$$

So, $\| T^s f_k \|_x = 2^{s/b_{n-1}} \le 2^{s/b_m} \le 4$, since $s \le 2b_m$.

(ii) $k+s \ge ra_n$: Since $ra_n-1 \in (B)$, as in (i), we have

$$T^s f_k = 2^{(ra_n-1-k)/b_{n-1}} \cdot T^{k+s-ra_n} \cdot Tf_{ra_n-1} .$$



So, using (4) and $\| T \|_x \le 2$ [Note:and $ra_n-k \le s \le 2b_m$], we get

$$\| T^s f_k \|_x \leq 2^{s/b_{n-1}} \cdot 2^{k+s-ra_n\,/a_n}$$

$$\leq 2^{s/b_m} \cdot 2^{s\,/a_n} \leq 4 \cdot 2^{2b_m/a_n} < 1$$

if a_n is large enough, since $n > m$.

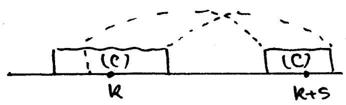
<u>Case (C)</u>: $r(a_n + b_n) \leq k \leq (n-1)a_n + rb_n$

We have $T^s f_k = x^{k+s} - b_n x^{k+s-b_n}$ and

$$b_m + r(a_n + b_n) \leq k+s \leq (m-1)a_m + b_m + (n-1)a_n + rb_n \quad \ldots(*C)$$

The hypothesis implies $n \geq m$. We consider separately the cases $n = m$ and $n > m$.

<u>(i) $n = m$:</u>



From (*C), $k+s \geq (r+1)(a_n + b_n)$.

(α) $k+s \leq (n-1)a_n + (r+1)b_n$ (which implies $r \leq n-2$ and $k+s \in$ (C)): We get $T^s f_k = f_{k+s}$, so $\| T^s f_k \|_x = 1$.

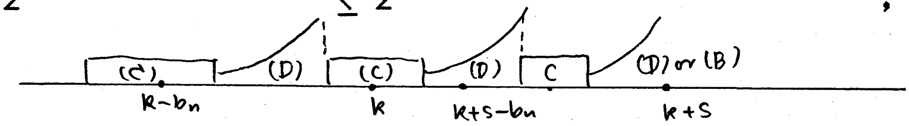
(β) $k+s > (n-1)a_n + (r-1)b_n$: Since $k+s < (r+2)(a_n + b_n)$ from (*C) [Note: if b_n is large enough], we have

if $r < n-2$, then $k+s \in$ (D), so

$$\| x^{k+s} \|_x = 2^{[k+s-(r+\frac{3}{2})b_n]/na_n} \leq 2^{[2(n-1)a_n - \frac{1}{2} b_n]/na_n} ;$$

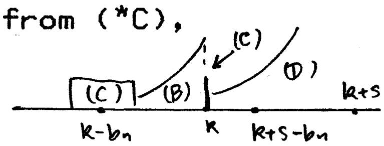
if $r \geq n-2$, then $k+s \in$ (B), so

$$\| x^{k+s} \|_x = 2^{[k+s-\frac{1}{2} a_{n+1}]/b_n} \leq 2^{[nb_n + 2(n-1)a_n - \frac{1}{2} a_{n+1}]/b_n} ;$$



similary, since $k+s \leq 2(n-1)a_n + (r+1)b_n$ from (*C),

if $r < n-1$, then $k+s-b_n \in$ (D), so



$$\| x^{k+s-b_n} \|_x = 2^{[k+s-(r+\frac{3}{2})b_n]/na_n} \leq 2^{[2(n-1)a_n - \frac{1}{2} b_n]/na_n} ;$$

if $r = n-1$, then $k+s-b_n \in$ (B), so

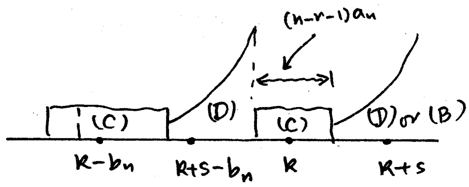
$$\| x^{k+s-b_n} \|_x = 2^{[k+s-b_n - \frac{1}{2} a_{n+1}]/b_n} \leq 2^{[2(n-1)a_n - \frac{1}{2} a_{n+1}]/b_n} .$$

So, if b_n, a_{n+1} are chosen large enough, we get $\| T^s f_k \|_x \leq 1$ in each case.

<u>(ii) n > m</u> :

(α) $k+s \leq (n-1)a_n + rb_n$: Then, $k+s \in$ (C). So, $T^s f_k = f_{k+s}$, and $\| T^s f_k \|_x = 1$.

(β) $k+s > (n-1)a_n + rb_n$: If $r < n-1$, we have $k+s < (r+1)(a_n + b_n)$, so $k+s \in$ (D) and



$$\| x^{k+s} \|_x = 2^{[k+s-(r+\frac{1}{2})b_n]/na_n}$$

$$\leq 2^{[(n-1)a_n + 2b_m - \frac{1}{2} b_n]/na_n} \leq 1/b_n^{\,2} \quad \text{[Note: by (*C)]}$$

and the same estimate holds for $\| x^{k+s-b_n} \|_x$; while if $r = n-1$, then the above estimate holds for $\| x^{k+s-b_n} \|_x$ and, on the other hand, since $k+s \in$ (B),

$$\| x^{k+s} \|_x = 2^{[k+s- \frac{1}{2} a_{n+1}]/b_n}$$

$$\leq 2^{[v_n + 2b_m - \frac{1}{2} a_{n+1}]/b_n} < 1/a_{n+1} \quad \text{[Note: by (*C)].}$$

So, in each case we get $\| T^s f_k \|_x \leq 1$.

<u>Case (D)</u>: $(n-1)a_n + (r-1)b_n < k < r(a_n + b_n)$

We have $T^s f_k = 2^{[(r-\frac{1}{2})b_n - k]/na_n} \cdot x^{k+s}$ and

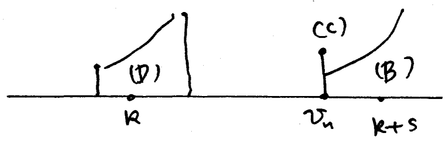
$$a_m + b_m + (n-1)a_n + (r-1)b_n < k+s < (m-1)a_m + b_m + r(a_n + b_n) \quad \ldots(*D)$$

and the hypothesis $k > (m-1)a_m$ again implies $n \geq m$ and we consider separately the cases $n = m$ and $n > m$.

<u>(i) $n = m$</u> :

From (*D) we have $k+s > (n-1)a_n + rb_n$. We subdivide this case into four cases (α)-(δ).

(α) <u>$k+s > (n-1)(a_n + b_n)$</u>:

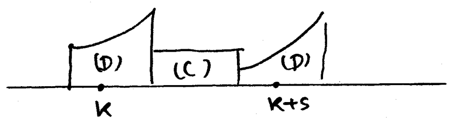


In this case, $k+s \in (B)$ and from (*D)

$$\| x^{k+s} \|_x = 2^{(k+s-\frac{1}{2}a_{n+1})/b_n} \leq 2^{[(n+1)b_n - \frac{1}{2}a_{n+1}]/b_n}.$$

So, $\| T^s f_k \|_x \leq 2^{b_n/a_n + [(n+1)b_n - \frac{1}{2}a_{n+1}]/b_n} \leq 1$ if a_{n+1} is large enough.

In the remaining three cases (β)-(δ), we always assume $(n-1)a_k + rb_n \leq k+s \leq (n-1)(a_n + b_n)$.

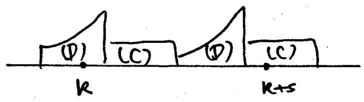


(β) <u>$k+s < (r+1)(a_n + b_n)$, $r < n-1$</u> :

Since $k+s \in (D)$, we have $\| x^{k+s} \|_x = 2^{[k+s-(r+\frac{1}{2})b_n]/na_n}$, so $\| T^s f_k \|_x = 2^{(s-b_n)/na_n} \leq 2$ since $s - b_n \leq (n-1)a_n$.

(γ) <u>$(r+1)(a_n+b_n) \le k+s \le (n-1)a_n+(r+1)b_n$, $r < n-1$</u> :

In this case, $k+s \in (C)$ and, by (2),



$$\| x^{k+s} \|_x \le 2b_n^{\,r} + b_n^{\,r+1} \| x^{k+s-(r+1)b_n} \|_x .$$

Now, $k+s-(r+1)b_n \le (n-1)a_n$, so $\| x^{k+s-(r+1)b_n} \|_x$ is bounded by a function of a_n, so $\| x^{s+k} \|_x \le b_n^{\,r+2}$ if b_n is large enogh. Also, $k \ge (r+1)(a_n+ b_n) - s \ge rb_n - na_n$, so

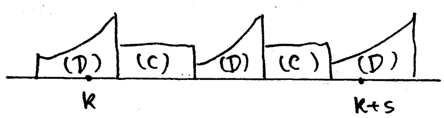
$$\| T^s f_k \|_x \le 2^{[(r-\frac{1}{2})b_n - k]} \cdot b_n^{\,r+2}$$

$$\le 2^{(na_n - \frac{1}{2} b_n)/na_n} \cdot b_n^{\,r+2} \le 1$$

if b_n is large enough.

(δ) <u>$k+s > (n-1)a_n + (r+1)b_n$, $r < n-2$</u> :

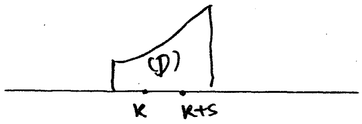
Since $s < 2b_n$, we have $k+s<(r+2)(a_n+ b_n)$, i.e., $k+s \in (D)$. So,



$$\| x^{k+s} \|_x = 2^{[k+s-(r+\frac{1}{2})b_n]/na_n}$$

$$\le 2^{[k+(n-1)a_n-(r+\frac{1}{2})b_n]/na_n} .$$

So, $\| T^s f_k \|_x \le 2^{[(n-1)a_n - b_n]/na_n} \le 1$ if b_n is large enough.

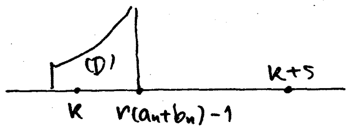


(ii) $n > m$:

(α) $k+s < r(a_n + b_n)$: Then, $s+k \in (D)$, so $x^{k+s} = 2^{[k+s-(r-\frac{1}{2})b_n]/na_n} f_{k+s}$. So, $T^s f_k = 2^{s/na_n} f_{k+s}$ and $\| T^s f_k \|_x = 2^{s/na_n} \leq 2^{2b_m/na_n} \leq 2$ since $n > m$.

(β) $k+s \geq r(a_n + b_n)$: Using $\| T \|_x \leq 2$ and the fact that, by (5), $\| T^s f_{r(a_n+b_n)-1} \|_x \leq 1/b_n$, we get

$$\| T^s f_k \|_x \leq 2^s/b_n \leq 2^{2b_m} / b_n \leq 1$$



[Note: $s+k \leq r(a_n + b_n)-1+s$] if b_n is large enough, since $m < n$. This completes the proof of Lemma 3.

Proof of Lemma 4. Let $h = Q_m(g)$, so $\| h \|_* \leq C_m$. By Lemma 1, we can find $q \in P_{(m-2)a_m}$ with

$$\| P_{(m-2)a_m}(qh) - x^{ra_m} \|_* \leq 1/a_m C_m \text{ ; and}$$

$$\| q \|_* \leq K = K(1/a_m C_m, 1/a_m, C_m, (m-2)a_m).$$

Let $\phi = x^{a_m} q \in P_{(m-1)a_m}$. Then,

$$\| P_{(m-1)a_m}(\phi h) - x^{(r+1)a_m} \|_* \leq 1/a_m C_m ,$$

so

$$\| P_{(m-1)a_m}(\phi h) - x^{(r+1)a_m} \|_x \leq 1/a_m \quad .....(7)$$

Let $\Phi = b_m^{-1} x^{b_m} \phi$. Since $\| x^t - b_m^{-1} x^{t+b_m} \|_x = 1/b_m$ for

$a_m \leq t \leq (m-1)a_m$, we get

$$\| P_{(m-1)a_m+b_m}(\phi h) - P_{(m-1)a_m}(\phi h) \|_X$$
$$= \| b_m^{-1} x^{b_m} P_{(m-1)a_m}(\phi h) - P_{(m-1)a_m}(\phi h) \|_X$$
$$\leq \| \phi h \|_*/b_m = \| qh \|_*/b_m \leq KC_m/b_m \leq 1/a_m \quad .....(8)$$

provided b_m is large enough, as K and C_m depend only on $a_1, b_1, \ldots, a_m$. Now the degree of ϕh does not exceed $2(m-1)a_m + b_m \leq 2(a_m + b_m)$, so (D) [Note: the maximum error is at $k=2(m-1)a_m+b_m$ and we have $\| \phi h \|_* \leq \| \phi \|_\infty \| h \|_* \leq \| \phi \|_* \| h \|_*$] gives

$$\| \phi h - P_{(m-1)a_m+b_m}(\phi h) \|_X \leq 2^{[2(m-1)a_m - \frac{1}{2} b_m]/ma_m} \| \phi h \|_*$$
$$\leq 2^{[2(m-1)a_m - \frac{1}{2} b_m]/ma_m} \cdot KC_m/b_m$$
$$\leq 1/a_m \quad .....(9)$$

again if b_m is large enough. Now apply (6), noting that

$$\phi(x) = \sum_{s=a_m+b_m}^{(m-1)a_m+b_m} \lambda_s x^s$$

where $\sum |\lambda_s| \leq K/b_m$, and that $T^s Q_m(g) = x^s h$ and $\phi(T)Q_m(g) = \phi h$, we get

$$\| \phi(T)g - \phi h \|_X = \| \phi(T)g - \phi(T)Q_m(g) \|_X$$
$$\leq \sum_{s=a_m+b_m}^{(m-1)a_m+b_m} |\lambda_s| \| T^s g - T^s Q_m(g) \|_X$$

$$\leq K\cdot 4\| g \|_X / b_m = 4K/b_m < 1/a_m \quad .....(10)$$

Now from (1), we get $\| x^{(r+1)a_m} - 1 \|_X \leq 2/a_{m-r-1}$ and combining this with (7), (8), (9), (10) gives

$$\| \phi(T)g - 1 \|_X < 2/a_{m-r-1} + 4/a_m < 3/a_{m-r-1}.$$

This proves the lemma.

Proof of THEOREM. Suppose $Y \neq \{0\}$. Let $g \in Y$ with $\| g \|_X = 1$ and any $\varepsilon > 0$ be given. To show that $Y = X$, it suffices to show that we can find a polynomial ϕ with $\| \phi(T)g - 1 \|_X < \varepsilon$. We choose $k > 1$ with $3/a_k < \varepsilon$. In view of Lemma 4, it will be sufficient to find $r \geq 1$ and m with $m-r-1 \geq k$ such that $\| P_{ra_m} Q_m g \|_* \geq 1/a_m$. So, we suppose on the contrary that

$$r \geq 1,\ m-r-1 \geq k \implies \| P_{ra_m} Q_m g \|_* < 1/a_m \quad .....(11)$$

and we shall deduce a contradiction.

For each m we can find $D_m > 0$, depending only on a_1, b_1, ... , b_m , such that $\| x^j \|_X \leq D_m$, $0 \leq j \leq v_m$.

Now we write $g = \sum_{j=0}^{\infty} \alpha_j f_j$, where $\sum |\alpha_j| = \| g \|_X = 1$ and for $n > 1$ we can write

$$\sum_{j=0}^{(n-1)a_n} \alpha_j f_j = \sum_{j=0}^{(n-1)a_n} \beta_{nj} x^j \quad .$$

We have also, for $n > 2$,

$$Q_n\Big(\sum_{j>(n-1)a_n} \alpha_j f_j\Big) = \sum_{j=0}^{\nu_{n-1}} \lambda_{nj} x^j$$

where

$$\lambda_{nj} = -a_n \sum_{m=n+1}^{\infty} \alpha_{j+(m-n)a_m} \quad \text{.......(12)}$$

and so

$$Q_n g = \sum_{j=0}^{(n-1)a_n} \beta_{nj} x^j + \sum_{j=0}^{\nu_{n-1}} \lambda_{nj} x^j \quad \text{......(12)}$$

From (11) and (12) we can deduce

$$r \geq 1,\ m-r-1 \geq k \implies \sum_{j=\nu_{m-1}+1}^{ra_m} |\beta_{mj}| < 1/a_m \quad \text{.....(13)}$$

From the definition of the polynomial f_j we can see that if $ra_m + \nu_{m-r-2} < j \leq ra_m + \nu_{m-r-1}$, then [Note: $j \in (A)$ and no x^j term in f_ℓ for $ra_m + \nu_{m-r-1} < \ell \leq (m-1)a_m$] $\beta_{mj} = a_{m-r}\alpha_j$, hence from (13) with r replaced by $r+1$ we get

$$r \geq 1,\ m-r-2 \geq k \implies \sum_{j=ra_m+\nu_{m-r-2}+1}^{ra_m+\nu_{m-r-1}} |\alpha_j| < 1/a_{m-r}a_m \quad \text{.....(14)}$$

[Note: (14) covers the shaded area in Fig. 3.1] and combining this with (12) we get

$$n \geq k+2 \implies \sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\lambda_{nj}| < |a_n| \sum_{m=n+1}^{\infty} \sum_{j} |\alpha_{j+(m-n)a_m}|$$

$$< \sum_{m=n+1}^{\infty} 1/a_m < 2/a_{n+1} \quad \text{....(15)}$$

From (11) with $r = 1$ we get, if $n \geq k+2$, using (12)

$$\| p_{a_n} Q_n g \|_x < 1/a_n \quad , \text{ so } \quad \sum_{j=0}^{\nu_{n-1}} |\beta_{nj} + \lambda_{nj}| < 1/a_n \quad .....(16)$$

and from (15) and (16) we get

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\beta_{nj}| < 2/a_n \; .$$

Now, if we write

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} \alpha_j f_j = \sum_{j=0}^{\nu_{n-1}} \beta'_{nj} x^j$$

then the correspondence between $(\alpha_{\nu_{n-2}+1}, \ldots , \alpha_{\nu_{n-1}})$ and $(\beta'_{\nu_{n-2}+1}, \ldots , \beta'_{\nu_{n-1}})$ is one-to-one, so there is a constant D'_{n-1}, depending only on $a_1, \ldots , b_{n-1}$, such that

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\alpha_j| \leq D'_{n-1} \sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\beta'_{nj}| \; .$$

Now we observe that if $\nu_{n-1} < j \leq (n-1)a_n$, then f_j has no x^i term for $\nu_{n-2} < i \leq \nu_{n-1}$. Hence, $\beta_j = \beta'_j$ for $\nu_{n-2} < j \leq \nu_{n-1}$, whence

$$\sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\alpha_j| \leq D'_{n-1} \sum_{j=\nu_{n-2}+1}^{\nu_{n-1}} |\beta_{nj}| < 2D'_{n-1}/a_n < 1/\sqrt{a_n}$$

if a_n is large enough, as D'_{n-1} depends only on $a_1, \ldots ,$ b_{n-1} . Putting $m = n-1$, we conclude that

$$m \geq k+1 \quad ==> \quad \sum_{j=\nu_{m-1}+1}^{\nu_m} |\alpha_j| \leq 1/\sqrt{a_{m+1}} \quad \ldots\ldots(17)$$

[Note: (17) covers the shaded area in Fig. 3.2]. Using (17) in (12) we find

$$n \geq k+1 \quad ==> \quad \sum_{j=0}^{\nu_{n-1}} |\lambda_{nj}| \leq a_n \sum_{m=n+1}^{\infty} \sum_{j=0}^{\nu_{n-1}} |\alpha_{j+(m-n)a_m}|$$

$$\leq a_n \sum_{m=n+1}^{\infty} \sum_{j=\nu_{m-1}}^{\nu_m} |\alpha_j|$$

$$\leq a_n \sum_{m=n+1}^{(n-1)a_n} 1/\sqrt{a_{m+1}} < 1/a_{n+1}$$

and combining this with (16) gives

$$\sum_{j=0}^{\nu_{n-1}} |\beta_{nj}| < 2/a_n \quad \text{for} \quad n \geq k+2$$

$$\text{i.e.,} \quad \| p_{\nu_{n-1}} (\sum_{0}^{(n-1)a_n} \alpha_j f_j) \|_* < 2/a_n \quad \ldots\ldots(18)$$

Now from (17) we get

$$\sum_{j=\nu_{n-1}+1}^{(n-1)a_n} |\alpha_j| \leq 1/\sqrt{a_{n+1}} < 1/a_n \ , \ (n \geq k+1)$$

and noting that if $\nu_{n-1} < j \leq (n-1)a_n$ then $\| p_{\nu_{n-1}} f_j \|_* \leq a_{n-1}$ [Note: $p_{\nu_{n-1}} f_j = 0$ ($j \in$ (B), $1 \leq r \leq n-1$), $-a_{n-1}$ ($j \in$ (A), $r = 1$), 0 ($j \in$ (A), $2 \leq r \leq n-1$)], we conclude that

$$\| p_{\nu_{n-1}} (\sum_{j=\nu_{n-1}+1}^{(n-1)a_n} \alpha_j f_j) \|_* \leq \sum_{j=\nu_{n-1}+1}^{(n-1)a_n} |\alpha_j| \| p_{\nu_{n-1}} (f_j) \|_*$$

$$< a_{n-1}/a_n$$

and combining this with (18) gives

$$\| \sum_{j=0}^{\nu_{n-1}} \alpha_j f_j \|_* < (a_{n-1} + 2)/a_n \text{ , if } n \geq k+2$$

whence

$$\sum_{j=0}^{\nu_{n-1}} |\alpha_j| = \| \sum_{j=0}^{\nu_{n-1}} \alpha_j f_j \|_X < D_{n-1}(a_{n-1} + 2)/a_n \text{ ,}$$

which tends to zero as $n \to \infty$ if $\{a_n\}$ increasing fast enough. Since $\sum |\alpha_j| = 1$, this is the desired contradiction.

References

1. C.J. Read, A solusion of the invarant subspace problem on a Banach space, Bull. London Math. Soc. 16 (1984), 337-401.

Case	k
f_0	[0
(B)	(⋮
(A)	[a_1
(B)	(⋮
(C)	[a_1+b_1
(B)	(⋮
(A)	[a_2
(D)	(⋮
(C)	[v_2
(B)	(⋮
(A)	[a_3
(B)	(⋮
(A)	[$2a_3$
(D)	(⋮
(C)	[a_3+b_3 ⋮ $2a_3+b_3$
(D)	(⋮
(C)	[v_3
(B)	(⋮

Case	k
(C)	[v_{n-1}
(B)	(⋮
(A)	[a_n … a_n+v_{n-2}
(B)	(⋮
(A)	[$2a_n$ … $2a_n+v_{n-3}$
	⋮
(A)	[$(n-4)a_n$ … $(n-4)a_n+v_3$
(B)	(⋮
(A)	[$(n-3)a_n$ … $(n-3)a_n+v_2$
(C)	(⋮
(A)	[$(n-2)a_n$
(C)	(⋮
(A)	[$(n-1)a_n$

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Case	k
(D)	(⋮
(C)	[a_n+b_n … $(n-1)a_n+b_n$
(D)	(⋮
(C)	[$2(a_n+b_n)$ … $(n-1)a_n+2b_n$
(D)	(⋮
(C)	[$3(a_n+b_n)$ … $(n-1)a_n+3b_n$
	⋮
(C)	[$(n-3)(a_n+b_n)$ ⋮ $(n-1)a_n+(n-3)b_n$
(D)	(⋮
(C)	[$(n-2)(a_n+b_n)$ … $(n-1)a_n+(n-2)b_n$
(D)	(⋮
(C)	[v_n

Fig. 1

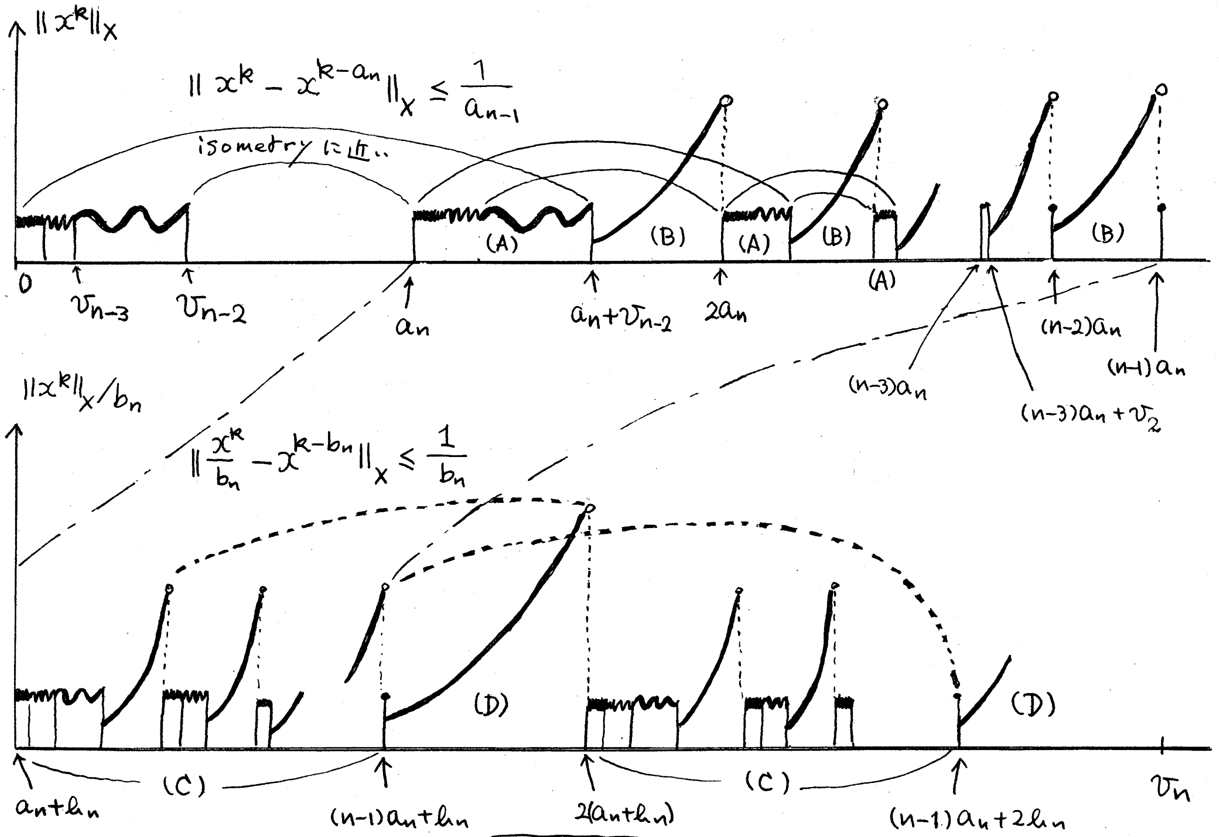

Fig. 2

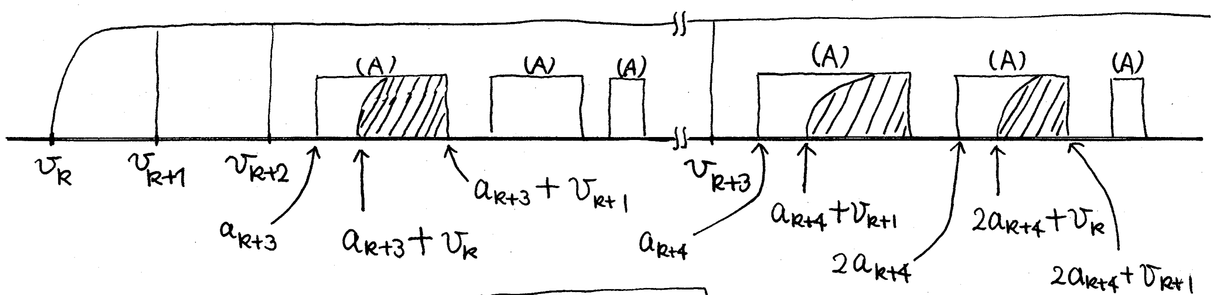

Fig. 3. 1

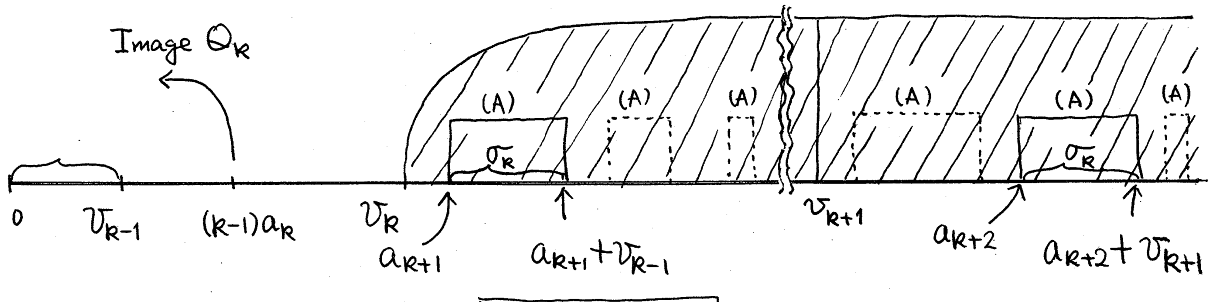

Fig. 3. 2