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On Certain Nonlinear Differential Equations of Second Order in Time

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0. Introduction and Theorem.

Let H be a real Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$, and V a real reflexive Banach space with norm $|\cdot|_V$ such that $V \subset H$, V is dense in H, and the inclusion mapping of V into H is continuous.
The dual space of H is dentifed with itself. The dual space of V is written as V^*. We use the same symbol $(\cdot,\cdot)$ as the inner product of H to denote the pairing between V and V^*.

Let ψ be a proper, convex and lower semicontinuous function from V to $(-\infty,\infty]$ and let $\partial\psi$ be its subdifferential operator defined by

$$\partial\psi x = \{f \in V^*;\ \psi(y) - \psi(x) \geqq (f,\ y-x) \quad \text{for any} \quad y \in V\}.$$

We assume that $\partial\psi$ is a single valued, everywhere defined and bounded operator from V to V^*, and that $\psi(\cdot)$ satisfies the following coersiveness condition

$$(0.0) \qquad \lim_{|u|_V \to \infty} \psi(u)/\ |u|_V = \infty.$$

Let ϕ be a lower semicontinuous proper convex function from H to $(-\infty,\infty]$ and $\partial\phi$ be the subdifferential of ϕ. Then we shall consider the following equation

$$(0.1)\qquad \begin{cases} \dfrac{d^2}{dt^2}u + \partial\psi u + \partial\phi u \ni f(\cdot,u) \\[2ex] u(0) = a, \quad \dfrac{d}{dt}u(0) = b \qquad \text{on} \quad [0,T] \end{cases}$$

where T is any positive number.

In [1] we showed the existence of solutions of (0.1) in the case of $\partial\psi$ = positive self-adjoint operator A. In this paper we purpose to prove the existence of a solution of the problem (0.1).

By $\partial\phi_\lambda$ and ϕ_λ we denote the Yosida approximations of $\partial\phi$ and ϕ respectively (i.e $\partial\phi_\lambda x = \lambda^{-1}(1-J_\lambda)^{-1}x$ and $\phi_\lambda(x) = (2\lambda)^{-1}\,|x-J_\lambda x|_H^{\,2} + \phi(J_\lambda x)$ where $J_\lambda = (1+\lambda\partial\phi)^{-1}$).

Next we shall introduce the assumptions.
Let X_1 and X_2 be real Banach spaces.

Assumption 1. The following inclusion relasions hold:

$$V \subset X_1 \subset H \subset X_2 \quad \text{and} \quad X_1 \subset \{\text{the dual space of } X_2\}$$

where each inclusion mapping is continuous. Moreover X_1 is separable and the inclusion mapping from V to X_1 is compact. H is dense in X_2.

Assumption 2. There exists $z \in V$ such that

$$(\partial\phi_\lambda x,\ x-z) \geqq c_1 |\partial\phi_\lambda x|_{X_2} - c_2$$

for $x \in V$, $|x|_V \leqq R$ and $|\phi(x)| \leqq R$ where c_1 and c_2 are constants depending only on R.

Assumption 3. The continuous function f from $[0,T]\times H$ to H satisfies for any $t \in [0,T]$ and $x,y \in H$

$$|f(t,x)-f(t,y)|_H \leqq h(t)|x-y|_H ,$$

$$|\frac{d}{dt}f(t,x)|_H \leqq h(t)(1 + |x|_H)$$

where h is a function belonging to $L_1(0,T)$.

Assumption 4. The closure of $D(\phi)\bigcap V$ in H is equal to the closure of $D(\phi)$ in H.

Assumption 5. For any sequence of functions $\{u_n\}$ in $W^1_\infty(0,T;H)\bigcap L_\infty(0,T;V)$ such that $\{u_n\}$ is bounded in $L_\infty(0,T;V)$ and converges to some u in the strong topology of $L_2(0,T;H)$ as $n \to \infty$, a subsequence $\{u_{n_j}\}$ can be extracted so that

$$\partial\phi u_{n_j} \to \partial\phi u \quad \text{in the weak star topology of } L_\infty(0,T;V^*).$$

Clearly V and X_1 are dense in H. By assumption H is dense in X_2. We use the same notation $(\cdot,\cdot)$ as the inner product of H to denote the pairing between V, X_1, X_2

and their carresponding duals.

Now we define the solution of (0.1).

Definition. We say that a function $u \in C([0,T];X_1) \bigcap W^1_\infty(0,T;H)$ is a solution of the equation (0.1) when it satisfies the following requirements;

1) For any $t \in [0,T]$ $u(t) \in D(\phi) \bigcap V$.

2) There exist weak right and left derivatives $\frac{d^\pm}{dt}u(t) \in H$ for any $t \in [0,T]$. Moreover for any $t \in [0,T]$

$$\left|\frac{d^\pm}{dt}u(t)\right|_H^2 + 2\psi(u(t)) + 2\phi(u(t))$$

$$\leqq |b|_H^2 + 2\psi(a) + 2\phi(a)$$

$$+ 2\int_0^T (f(s,u(s)), \frac{d}{ds}u(s))ds$$

(with necessary modifications at 0 and T).

3) There exists a linear functional F on $C([0,T];X_1)$ such that

$$F(v-u) \leqq \int_0^T \phi(v(s))ds - \int_0^T \phi(u(s))ds$$

for any $v \in C([0,T];X_1)$ and

$$\int_0^T (\frac{d}{ds}u(s), \frac{d}{ds}v(s))ds + \int_0^T (f(s,u(s)) - \partial\psi u(s), v(s))ds$$

$$+ (b, v(0)) - (\frac{d^-}{dt}u(T), v(T)) = F(v)$$

for any $v \in C([0,T];X_1) \bigcap L_1(0,T;V) \bigcap W^1_\infty(0,T;H)$.

4) The initial conditions are satisfied in the following sense

$$u(0) = a, \quad b - \frac{d}{dt}u(0) \in \partial I_{K_0} a$$

where K_0 is the closure of the domain of ϕ, I_{K_0} is the indicator function of K_0 and ∂I_{K_0} is the subdifferential of I_{K_0}.

We introduce the theorem.

Theorem. Let a and b be given elements satisfying

$$a \in V \bigcap D(\phi), \quad b \in H.$$

Then under the assumptions 1, 2, 3, 4 and 5 we have at least one solution of (0.1).

Now let $\psi(\cdot)$ satisfy the coersiveness condition (0.0). Then if Ψ is the convex function from H to $(-\infty,\infty]$ defined by

$$\Psi(v) = \begin{cases} \psi(v) & \text{if} \quad v \in V \\ \infty & \text{if} \quad v \in H-V = \{g \in H;\ g \notin V\}, \end{cases}$$

Ψ is lower semicontinuous on H by the coersive condition and the subdifferential operator $\partial\Psi$ is defined. Then $\partial\Psi(v) = \partial\psi(v) \in H$ for $v \in V$. For any $\mu > 0$ we denote the Yosida approximation of Ψ and that of $\partial\Psi$ respectively

by

$$\Psi_\mu(x) = (2\mu)^{-1} \, |x - \bar{J}_\mu x|_H^{\,2} + \psi(\bar{J}_\mu x)$$

and

$$\partial\Psi_\mu(x) = \partial\Psi(\bar{J}_\mu x),$$

where $\bar{J}_\mu = (I + \mu\partial\Psi)^{-1}$ and I is the identity operator on H. Then $\partial\Psi_\mu$ is the subdifferential operator of Ψ_μ.

To prove the above theorem we consider the following approximate equations for $\lambda > 0$ and $\mu > 0$

$$(0.2) \qquad \begin{cases} \dfrac{d^2}{dt^2}u_{\lambda,\mu} + \partial\Psi_\mu u_{\lambda,\mu} + \partial\phi_\lambda u_{\lambda,\mu} = f(\cdot, u_{\lambda,\mu}) \\[2ex] u_{\lambda,\mu}(0) = a\,, \qquad \dfrac{d}{dt}u_{\lambda,\mu}(0) = b. \end{cases}$$

If μ tend to zero we shall get solutions of the following equation.

$$(0.3) \qquad \begin{cases} \dfrac{d^2}{dt^2}u_\lambda + \partial\psi u_\lambda + \partial\phi_\lambda u_\lambda = f(\cdot, u_\lambda) \\[2ex] u_\lambda(0) = a\,, \qquad \dfrac{d}{dt}u_\lambda(0) = b. \end{cases}$$

Next we shall investigate the convergence of the solutions of the approximate equations (0.3) and prove the theorem.

1. Convergence of approximate solutions

In this section we study the convergence of the solutions

of (0.2) and (0.3). First we show some properties of the approximate solutions.

Lemma 1. For any $\lambda, \mu > 0$ we have solution of the problem (0.2) such that $u_{\lambda,\mu} \in W_1^2(0,T;H)$.

Proof. Noting that $\partial\Psi_\mu$ and $\partial\phi_\lambda$ hold the Lipschitz continuity and using the assumption 3 we can prove this lemma.

Lemma 2. For the solutions of (0.2) we hold the following equality and inequality

1) $$\left|\frac{d}{dt}u_{\lambda,\mu}(t)\right|_H^2 + 2\Psi_\mu(u_{\lambda,\mu}(t)) + 2\phi_\lambda(u_{\lambda,\mu}(t))$$

$$= |b|_H^2 + 2\Psi_\mu(a) + 2\phi_\lambda(a) + 2\int_0^t \left(f(s,u_{\lambda,\mu}(s)), \frac{d}{ds}u_{\lambda,\mu}(s)\right)ds$$

2) $$\left|\frac{d}{dt}u_{\lambda,\mu}(t)\right|_H^2 + 2\Psi_\mu(u_{\lambda,\mu}(t)) + 2\phi_\lambda(u_{\lambda,\mu}(t))$$

$$\leq C_1(|b|_H^2 + |a|_H^2 + \Psi_\mu(a) + \phi_\lambda(a) + 1).$$

Proof. Taking the inner product of both sides of (0.2) with $\frac{d}{dt}u_\lambda$ and integrating the resultant equality over $[0,t]$, we have 1) of this lemma. Combining 1) of this lemma and the assumption 3 and using Gronwall's lemma we have 2) of the lemma.

2) of Lemma 1 and the assumption 2 imply that $|u_{\lambda,\mu}|_V$ is uniformly bounded. Combining that the inclusion mapping from V to H is compact and the above mention resultant, and using the assumption 5 and 2) of Lemma 1 we now that there

exists a subsequence $\{u_{\lambda,\mu_j}\}$ such that

$$\begin{cases} u_{\lambda,\mu_j} \to u_\lambda & \text{in } C([0,T];H) \text{ (uniformly)} \\ \frac{d}{dt}u_{\lambda,\mu_j} \to \frac{d}{dt}u_\lambda & \text{in weak } L_2(0,T;H) \\ \partial\Psi_{\mu_j}(u_{\lambda,\mu_j}) \to \partial\psi(u_\lambda) & \text{in weak star } L_\infty(0,T;V^*). \end{cases}$$

Then we have a following lemma.

Lemma 3. There exist solutions of (0.3) $u_\lambda \in C([0,T];H)$ such that the following conditions are satisfied:

1) $u_\lambda \in W^1_\infty(0,T;H) \bigcap \text{weak-}C([0,T];V)$.

2) The right derivative $\frac{d^+}{dt}u_\lambda(t)$ and the left derivative $\frac{d^-}{dt}u_\lambda(t)$ exist on $[0,T]$ both in the weak topology of H and in the strong topology of V^*(with necessary modifications at 0 and T).

3) We have

$$2^{-1}\left|\frac{d^\pm}{dt}u_\lambda(t)\right|_H^2 + \Psi(u_\lambda(t)) \leqq 2^{-1}|b|_H^2 + \Psi(a) + \int_0^t \left(\frac{d}{dt}u_\lambda(s),\ f(s,u_\lambda(s))-\partial\phi_\lambda u_\lambda(s)\right)ds$$

for any $t \in [0,T]$(with necessary modifications at 0 and T).

4) For any $v \in W^1_1([0,T];H) \bigcap C([0,T];V)$ we hold

$$0 = \int_0^T \left(\frac{d}{ds}u_\lambda(s),\ \frac{d}{ds}v(s)\right)ds -$$

$$\int_0^T (\partial\Psi u_\lambda(s) - f(s, u_\lambda(s)) + \partial\phi_\lambda u_\lambda(s),\ v(s))ds + (b,\ v(0)) -$$

$$(\frac{d^-}{dt}u_\lambda(T),\ v(T)).$$

5) The initial condition is satisfied :

$$u_\lambda(0) = a \quad \text{and} \quad \frac{d^+}{dt}u_\lambda(0) = b.$$

Next to study the convergence of the solutions of (0.3) we shall use the methoeds and results of [1].

Combining the 3) of Lemma 3 and the method of 2) of lemma 1 and noting that $\frac{d}{dt}\phi_\lambda(u_\lambda(t)) = (\partial\phi_\lambda u_\lambda(t),\ \frac{d}{dt}u_\lambda(t))$ we have the following lemma.

Lemma 4. We hold the following inequality

1) $$|\frac{d}{dt}u_\lambda(t)|_H^2 + 2\Psi(u_\lambda(t)) + 2\phi_\lambda(u_\lambda(t))$$

$$\leqq |b|_H^2 + 2\Psi(a) + 2\phi_\lambda(a) + 2\int_0^t (f(s, u_\lambda(s)),\ \frac{d}{ds}u_\lambda(s))ds$$

2) $$|\frac{d}{dt}u_\lambda(t)|_H^2 + 2\Psi(u_\lambda(t)) + 2\phi_\lambda(u_\lambda(t))$$

$$\leqq C_1(|b|_H + |a|_H + \Psi(a) + \phi_\lambda(a) + 1).$$

Lemma 5. There exists a constant independent of λ such that

$$\int_0^T |\partial\phi_\lambda u_\lambda(s)|_{X_2} ds \leqq \text{Constant}.$$

Proof. In the inequality of assumption 2 we put $x = u_\lambda(t)$. From Lemma 4 the constants c_1 and c_2 are

independent of λ. In 3) of Lemma 3 we replace v by $(u_\lambda - z)$ and use the assumption 2. Combining Lemma 4 and the above mention-resultant we get this lemma.

Lemma 6. We have a continuous function u from $[0,T]$ to H such that a subsequence $\{u_{\lambda_j}\}$ of the sequence $\{u_\lambda\}$ converges uniformly to u in H as $\lambda_j \to 0$.

Proof. See Lemma 4 in [1].

For simplicity we denote this subsequence by $\{u_\lambda\}$.

Lemma 7. The sequence $\{\partial\Psi u_\lambda\}$ converges to $\partial\psi u$ in the weak star topology of $L_\infty(0,T;V^*)$.

Proof. Using the same method as Lemma 3 we can prove the lemma.

3. Proof of Thoerem.

Replacing A by $\partial\psi$, using the same method as them of Lemma 5,6,7,8,9,10 and 11 in [1] and combining Lemma 6 and 7 in this paper we can prove this theorem.

4. Example.

We consider the following equation

$$\left(\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u + |u|^{p+1}u \right)(u - r) = 0,$$

$$(4.1)\quad \begin{cases} \dfrac{\partial^2}{\partial t^2}u - \dfrac{\partial^2}{\partial x^2}u + |u|^{p+1}u \geqq 0 \\ \text{in the sence of distributions in } [0,1]\times[0,T], \\ u(x,t) \geqq r(x), \qquad u(x,0) = u_0(x), \quad \text{for } x \in [0,1], \\ \dfrac{\partial}{\partial t}u(x,t) = u_1(x), \qquad \text{a.e in } [0,1] \\ u(0,t) = u(1,t) = 0 \qquad \text{for } t \in [0,T] \end{cases}$$

where r is continuous given function such that $r(0) < 0$, $r(1) < 0$.

Set $K = \{f \in L_2(0,1);\ f(x) \geqq r(x)\}$. The equation (4.1) is rewritten as the following equation in $L_2(0,1)$

$$(4.2)\quad \begin{cases} \dfrac{d^2}{dt^2}u + \partial\psi u + \partial I_K u \ni 0 \\ u(0) = u_0, \quad \dfrac{d}{dt}u(0) = u_1. \end{cases}$$

where $\psi(u) = \int_0^1 (2^{-1}|\text{grad } u|^2 + (p+1)^{-1}|u|^{p+1})dx$ for any $u \in \overset{\circ}{W}{}_2^1(0,1) = \{ u \in W_2^1(0,1);\ u(0) = u(1) = 0 \}$.

We can apply our main theorem to this equation if only r is a continuous function satisfying $r(0) < 0$, $r(0) < 0$ to deduce the existence of a soluton of (4.2).

Biblograhy

[1] K. Maruo: Existence of Solutions of some nonlinear wave equations, to appear