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Inverse problems for hyperbolic equations; the uniqueness and stability

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§1. Introduction.

There are important and interesting inverse problems in geophisics. Among others, reflection seismology attempts to determine the inside structure of the earth, from the seismographic profiles on the surface.

In the one-dimensional model, this problem is governed by the hyperbolic equation

$$\rho(z)\partial_t^2 y - \partial_z(\mu(z)\partial_z y) = 0 \qquad (z>0,\ t>0), \tag{1.1}$$

with the displacement $y=y(z,t)$. The variables z and t stands for the space and the time, and $\rho=\rho(z)$ and $\mu=\mu(z)$ are the rock density and the elastic coefficient, respectively. The coefficients (ρ,μ) represent the inside structure of the earth, and are assumed to be unknown. We consider the boundary condition

$$-\mu(0)\partial_z y_{|z=0} = g(t) \qquad (t>0) \tag{1.2}$$

with the initial condition

(1.3) $\qquad y_{|t=0} = \partial_t y_{|t=0} = 0 \qquad (z>0),$

where $g=g(t)$ is the excitation communicated to the medium surface, supposed to be known. Our task is to determine the unknown coefficients (ρ,μ) by the inhomogeneous term $g(t)$ as well as by the "seismogram"

(1.4) $\qquad f(t) = y_{|z=0} \qquad (0\leqq t\leqq T),$

which means the recording of the vibratory state of the surface.

—— That is the problem which Bamberger-Chavent-Lailly [1] studied, and we want to start with its review.

§2. Identification and identifiability.

For each (ρ,μ) and g , there exists a unique solution $y=y(z,t)$ of (1.1)-(1.3). We put

(2.1) $\qquad Y(\rho,\mu;t) = y_{|z=0}.$

Bamberger-Chavent-Lailly [1] suppose that the upper and the lower bounds $\rho_\pm$, $\mu_\pm$ of ρ,μ are given:

$$0<\rho_-\leqq\rho(z)\leqq\rho_+<\infty, \qquad 0<\mu_-\leqq\mu(z)\leqq\mu_+<\infty,$$

and specifies a set of admissible parameters (ρ,μ) by Σ:

$$\Sigma = \{(\rho,\mu)\in L^\infty(0,\infty)^2 \mid \rho_-\leqq\rho(z)\leqq\rho_+, \ \ \mu_-\leqq\mu(z)\leqq\mu_+, \text{ a.e. } z\}.$$

Their ultimate purpose is to construct $(\rho^*,\mu^*)\in\Sigma$ such that

(2.2) $\qquad Y(\rho^*,\mu^*;t) = f(t) \qquad (0\leqq t\leqq T),$

for given seismogram $f=f(t)$.

They formulate this problem as the optimization problem

$$(2.3) \qquad \min_{(\rho,\mu)\in\Sigma} J(\rho,\mu),$$

where

$$(2.4) \qquad J(\rho,\mu) = \int_0^T (Y(\rho,\mu;t) - f(t))^2 dt.$$

To show the solvability of (2.3), they introduce a topology τ in Σ, which is strong enough to make J continuous and is simultaneously weak enough to make Σ compact. Therefore, the optimization problem (2.3) has a solution $(\rho^*,\mu^*)\in\Sigma$. If $J(\rho^*,\mu^*)=0$, the equality (2.2) is satisfied.

Ill-posedness, the lack of stability, has been reported both theoretically and numerically. See the references of [1]. Consequently, the topology τ is rather weak compared with the ordinary ones, for example, L^p topology.

This kind of constructive approach is called the "identification". While there is the "idendifiability" problem, stated as follows. Let (ρ_0,μ_0) be the *genuine* coefficients whose seismogram is $f=f(t)$. Then,

(1) (uniqueness) Does $J(\rho^*,\mu^*)=0$ imply $(\rho^*,\mu^*)=(\rho_0,\mu_0)$?

(2) (stability) Is (ρ^*,μ^*) near from (ρ_0,μ_0), if $J(\rho^*,\mu^*)$ is small ?

In this way, identification concerns with the existence of the unknown coefficients (ρ^*,μ^*) such that (2.2), while identifiability examines their uniqueness and stability. We want to study the latter in this article.

The approach [1] gives some kind of stability. In fact, if the uniqueness (1) is satisfied, the stability (2) holds in the τ-topology by the compactness of Σ . However, our approach is quite different. Without assuming that the upper or the lower bounds of (ρ,μ) are given, we want to derive an estimate to show the stability (2). Consequently, we don't use the compactness argument.

§3. Formulation of the problem.

The equation which we study here is slightly different from that of [1]. As is noted in [1], it is impossible to determine both (ρ,μ) from (f,g) . Namely, the uniqueness (1) doesn't hold for any (f,g). By the Liouville transformation (e.g. [3]), we transform the equation

(a) $$\rho(z)\partial_t^2 y - \partial_z(\mu(z)\partial_z y) = 0$$

into

(b) $$\partial_t^2 u + (-\partial_x^2 + p(x))u = 0,$$

and pay attention to only one coefficient p , in stead of two coefficients (ρ,μ). Furthermore, we consider (b) on the compact interval $[0,1]$ for the space variable x :

(3.1) $$\partial_t^2 u + (-\partial_x^2 + p(x))u = 0 \qquad (0<x<1,\ -\infty<t<\infty).$$

In stead of the non-homogeneous boundary condition (1.2) and the homogeneous initial condition (1.3), we consider the homogeneous boundary condition

$$(3.2)\qquad (-\partial_x + h)u_{|x=0} = (\partial_x + H)u_{|x=1} = 0 \qquad (-\infty<t<\infty)$$

and the inhomogeneous boundary condition

$$(3.3)\qquad u_{|t=0} = a_0(x), \qquad \partial_t u_{|t=0} = a_1(x) \qquad (0<x<1),$$

respectively. As before, the coefficient $P=(p,h,H)\varepsilon C^0[0,1]\times R\times R$ is unknown. Furthermore, we suppose that the initial value $a=(a_0,a_1)\varepsilon H^1_x(0,1)\times L^2_x(0,1)$ is also unknown and that instead *two* boundary values

$$(3.4)\qquad u_{|x=0} = f_0(t), \qquad u_{|x=1} = f_1(t) \qquad (-T\leqq t\leqq T)$$

are observed and known for some $T>0$.

In this way, our identifiability problem is formulated as follows: Let $P=(p,h,H)\varepsilon C^0[0,1]\times R\times R$ and $a=(a_0,a_1)\varepsilon H^1_x(0,1)\times L^2_x(0,1)$ be the unknown parameters, and let $u=u(x,t)\varepsilon C^0_t((-\infty,\infty)\to H^1_x(0,1))\cap C^1_t((-\infty,\infty)\to L^2_x(0,1))$ be the solution of (3.1)-(3.3). Define f_0,f_1 as in (3.4). Now, let

$$(3.5)\qquad \partial_t{}^2 v + (-\partial_x{}^2 + q(x))v = 0 \qquad (0<x<1, \quad -\infty<t<\infty)$$

be another equation with

$$(3.6)\qquad (-\partial_x + j)v_{|x=0} = (\partial_x + J)v_{|x=1} = 0 \qquad (-\infty<t<\infty)$$

and

$$(3.7)\qquad v_{|t=0} = b_0(x), \qquad \partial_t v_{|t=0} = b_1(x) \qquad (0<x<1),$$

where $Q=(q,j,J)\varepsilon C^0[0,1]\times R\times R$ and $b=(b_0,b_1)\varepsilon H_x^1(0,1)\times L_x^2(0,1)$. The functions ε_0, ε_1 defined by

$$\varepsilon_0(t) = v|_{x=0} - u|_{x=0}, \qquad \varepsilon_1(t) = v|_{x=1} - u|_{x=1} \qquad (-T\leqq t\leqq T) \tag{3.8}$$

stand for the errors of identification. Then, the problems are;

(P1, uniqueness) Does $\varepsilon_0=\varepsilon_1=0$ $(-T\leqq t\leqq T)$ imply $(Q,b)=(P,a)$?

(P2, stability) Is (Q,b) near from (P,a), if $(\varepsilon_0,\varepsilon_1)$ is small ?

Compared with the original equation (1.1)-(1.3), the equation (3.1)-(3.3) may look simpler. However, our problems contain the essential difficulties of the original ones.

§4. Summary.

Putting $a_0=a_1=0$ in (3.3), we get $u\equiv 0$ for each $P=(p,h,H)$. Hence $\varepsilon_0=\varepsilon_1=0$ if $b_0=b_1=0$, for ***any*** $Q=(q,j,J)$. In other words, the uniqueness (P1) doesn't hold without any assumptions on the unknown equation (3.1)-(3.3).

Notation 1. For $P=(p,h,H)\varepsilon C^0[0,1]\times R\times R$, $A=A_P$ denotes the realization in $L^2(0,1)$ of the differential operator $-\partial_x^{\,2}+p(x)$ with the boundary condition $(-\partial_x+h)|_{x=0}=(\partial_x+H)|_{x=1}=0$. The set $\sigma(A_P)=\{\lambda_n\}_{n=0}^{\infty}$ denotes its eigenvalues: $-\infty<\lambda_0<\lambda_1<\cdots\to\infty$. The L^2-normalized eigenfunction corresponding to λ_n is denoted by ϕ_n.

Notation 2. For $P=(p,h,H)\varepsilon C^0[0,1]\times R\times R$ and $a=(a_0,a_1)\varepsilon H^1(0,1)\times L^2(0,1)$, the equation (3.1) with (3.2)-(3.3) is denoted by $E(P,a)$.

Definition 1. We say $E(P,a)\varepsilon G$ if

$$(a_n^0)^2 + (a_n^1)^2 \neq 0 \quad (n=0,1,2,\cdots), \tag{4.1}$$

where

$$a_n^0 = (a_0,\phi_n), \qquad a_n^1 = (a_1,\phi_n). \tag{4.2}$$

Henceforth, (,) denotes the L^2-inner product.

Theorem 1 (Uniqueness). If $E(P,a)\varepsilon G$, $T\geqq 2$ and $\varepsilon_0(t)=\varepsilon_1(t)=0$ $(-T\leqq t\leqq T)$, then $(Q,b)=(P,a)$ follows.

Remark 1. The condition $E(P,a)\varepsilon G$ is necessary for the uniqueness $(Q,b)=(P,a)$. It is open whether $T\geqq 2$ is also necessary or not. In view of the property of finite propagation of hyperbolic equations, it is obvious that a sufficient large $T>0$ must be taken.

In this way, G is a class of "good" unknown equations, and provides us with good data $f_0=u|_{x=0}$, $f_1=u|_{x=1}$. For $E(P,a)\varepsilon G$ and only for $E(P,a)\varepsilon G$, the uniqueness (P1) holds.

To establish the stability estimate, we furthermore introduce the following

Definition 2. For $\alpha>1/2$, we say $E(P,a)\varepsilon G_\alpha$ if $p\varepsilon C^\alpha[0,1]$ and

$$M_\alpha^{-1}(n^2+1)^{-\alpha} \leqq (n^2+1)(a_n^0)^2 + (a_n^1)^2 \leqq M_\alpha(n^2+1)^{-\alpha} \qquad (n=0,1,2,\cdots) \tag{4.3}$$

hold for some $M_\alpha>0$.

By $a=(a_0,a_1)\varepsilon H^1(0,1)\times L^2(0,1)$, the relation

$$\underbrace{\sum_{n=0}^{\infty}\{(n^2+1)(a_n^0)^2 + (a_n^1)^2\}}_{<\,\infty} \approx ||a_0||^2_{H^1(0,1)} + ||a_1||^2_{L^2(0,1)}$$

holds, and $\alpha>1/2$ must be satisfied. In fact, we have

$$D((A_P + \lambda)^{1/2}) = H^1(0,1)$$

for sufficiently large $\lambda>0$ and also the asymptotic behavior

$$\omega_n \equiv \lambda_n^{1/2} = n\pi + O(1/n) \qquad (n\to\infty). \tag{4.4}$$

See [3].

The condition $E(P,a)\varepsilon G_\alpha$ assures us of some crucial regularity for the data $f_0=u|_{x=0}$, $f_1=u|_{x=1}$. Actually, the theory of non-harmonic Fourier series [5] gives

$$f_0,\ f_1 \ \varepsilon\ H_t^\beta(-T,T) \qquad \text{for} \quad 0\leqq\beta<\alpha+1/2 \tag{4.5}$$

and

$$f_0,\ f_1 \not\varepsilon\ H_t^\delta(-T,T), \tag{4.6.1}$$

for

$$\delta = \begin{cases} \alpha+1/2 & (\text{if } \alpha+1/2 \neq 1,2,\cdots) \\ \alpha+1/2+\varepsilon & (\text{otherwise}) \end{cases} \tag{4.6.2}$$

with $\varepsilon>0$. See Lemma 3 of §6.

We assume that at the stage of identification described in the preceeding section, approximation is done *over* this natural regularity.

Theorem 2 (stability). Suppose $E(P,a)\varepsilon G_\alpha$ $(\alpha>1/2)$ and $T\geqq 2$. Then, for each $\kappa>0$, there exists $C(\kappa)>0$ such that

$$(4.7)\qquad \|q-p\|_{L^2(0,1)} + |j-h| + |J-H| \leqq C(\kappa)\{\|\varepsilon_0\|_{H_t^{\alpha+2}(-T,T)} + \|\varepsilon_1\|_{H_t^{\alpha+2}(-T,T)}\}$$

for $(Q,b)=(q,j,J;b_0,b_1)\varepsilon C^0[0,1]\times R\times R\times H^1(0,1)\times L^2(0,1)$ with

$$(4.8)\qquad \|q\|_{L^2(0,1)} + |j| + |J| \leqq \kappa .$$

It seems to be difficult to realize such an identification as $\|\varepsilon_0\|_{H_t^{\alpha+2}(-T,T)}$, $\|\varepsilon_1\|_{H_t^{\alpha+2}(-T,T)} \to 0$, in spite of $f_0,f_1 \not\in H_t^\delta(-T,T)$ $(\delta>\alpha+1/2)$. In this context, Theorem 2 has no practical meaning. Nevertheless, it does have some sense. Actually, we can show that the norm $\|\varepsilon_0\|_{H_t^{\alpha+2}} + \|\varepsilon_1\|_{H_t^{\alpha+2}}$ is best possible in (4.7), which proves the significant ill-posedness of the problem. Furthermore, we can emphasize the importance of the irregularity of the data f_0, f_1 in the identification. Hyperbolic equations preserve the irregularity of initial values, and the gap of the exponents of the Sobolev spaces between the errors $\varepsilon_0,\varepsilon_1$ in (4.7) and the data $f_0=u|_{x=0}$, $f_1=u|_{x=1}$ in (4.5) is only $3/2+\varepsilon$ $(\varepsilon>0)$. In this connection, we would like to call (4.7) the "semi-wellposedness". —— We cannot expect such an estimate any more for parabolic equations because of their smoothing property.

§5. Deformation formula.

The solution $u=u(x,t)$ which we handle with is rather irregular, and we must be careful in later caluculations. In any case, $u=u(x,t)\in C_t^0((-\infty,\infty)\to H_x^1(0,1))\cap C_t^1((-\infty,\infty)\to L_x^2(0,1))$ is continuous by Sobolev's imbedding.

Let $\Omega\subset(0,1)\times(-\infty,\infty)$ be a domain.

Definition 3. A continuous function $u=u(x,t)$ satisfies

$$(5.1.1)\qquad \partial_t^{\,2}u + (-\partial_x^{\,2} + p(x))u = 0$$

$$(5.1.2)\qquad (-\partial_x + h)u_{|x=0} = 0$$

in the generalized sense in Ω if

$$(5.2)\qquad \int_{-\infty}^{\infty} dt\int_0^1 dx\; u(x,t)\{\phi_{tt}(x,t) - \phi_{xx}(x,t) + p(x)\phi(x,t)\} = \int_{-\infty}^{\infty} dt\; u(0,t)\{\phi_x(0,t) - h\phi(0,t)\}$$

holds for each $\phi=\phi(x,t)\in C_0^2(R^2)$ with $\text{supp}\,\phi\cap[0,1]\times(-\infty,\infty)\subset\overline{\Omega}\cap[0,1)\times(-\infty,\infty)$.

Obviously, the solution $u=u(x,t)$ of $E(P,a)$ satisfies (5.1) in the generalized sense.

For $T>0$, let

$$(5.3)\qquad \Omega_T = \{(x,t)\mid 0<x<\min(1,T),\ -T+x<t<T-x\}.$$

We then have

Proposition 1. For given $f=f(t)\varepsilon C^0[-T,T]$, there exists a unique $u=u(x,t)\varepsilon C^0(\overline{\Omega}_T)$ such that (5.1) in the generalized sense in Ω_T and

(5.4) $$u_{|x=0} = f(t) \qquad (-T\leqq t\leqq T).$$

For the proof, see [4], for instance.

Our basic idea is to combine two solutions $u=u(x,t)$ of $E(P,a)$ and $v=v(x,t)$ of $E(Q,b)$ through the Gel'fand-Levitan kernel $K=K(x,y)$ ([2]). More precisely, let

(5.5) $$D = \{(x,y) \mid 0<y<x<1\}.$$

Lemma 1. For given $p,q\varepsilon C^0[0,1]$ and $h,j\varepsilon R$, there exists a unique $K=K(x,y)=K(x,y;q,j;p,h)\varepsilon C^1(\overline{D})$ such that

(5.6.1) $$K_{xx} - K_{yy} + p(y)K = q(x)K$$

in the distributional sense in D with

(5.6.2) $$K(x,x) = (j-h) + \frac{1}{2}\int_0^x (q(s)-p(s))ds \qquad (0\leqq x\leqq 1)$$

and

(5.6.3) $$K_y(x,0) = hK(x,0) \qquad (0\leqq x\leqq 1).$$

Lemma 2. If $u=u(x,t)\varepsilon C^0([0,1]\times(-\infty,\infty))$ satisfies

(5.7.1) $$\partial_t^{\,2}u + (-\partial_x^{\,2} + p(x))u = 0$$

(5.7.2) $$(-\partial_x + h)u_{|x=0} = 0$$

in the generalized sense in Ω_T, then

(5.8) $$V(x,t) = u(x,t) + \int_0^x K(x,y;q,j;p,h)u(y,t)dy \in C^0([0,1]\times(-\infty,\infty))$$

satisfies

(5.9.1) $$\partial_t^2 V + (-\partial_x^2 + q(x))V = 0$$

(5.9.2) $$(-\partial_x + j)V|_{x=0} = 0$$

in the generalized sense in Ω_T and

(5.10) $$V|_{x=0} = u|_{x=0} \qquad (-\infty<t<\infty).$$

The relation (5.8) is called the deformation formula. The point is that the kernel K is independent of t. For the proof of these lemmas and their background, see [4].

§6. Non-harmonic Fourier series.

Recall that $\sigma(A_P)=\{\lambda_n\}_{n=0}^{\infty}$ denotes the eigenvalues of A_P and

(6.1) $$\omega_n \equiv \lambda_n^{1/2} = n\pi + O(1/n) \qquad (n\to\infty).$$

Since the solution $u=u(x,t)\in C_t^0((-\infty,\infty)\to H_x^1(0,1))\cap C_t^1((-\infty,\infty)\to L_x^2(0,1))$ of $E(P,a)$ is given as

(6.2) $$u(x,t) = \sum_{n=0}^{\infty}\{a_n^0 \cos \omega_n t + a_n^1 \sin \omega_n t / \omega_n\}\phi_n(x),$$

we have

(6.3.1) $$f_0(t)\equiv u|_{x=0} = \sum_{n=0}^{\infty}\{a_n^0\cos \omega_n t + a_n^1\sin \omega_n t/\omega_n\}\phi_n(0)$$

(6.3.2) $$f_1(t)\equiv u|_{x=1} = \sum_{n=0}^{\infty}\{a_n^0\cos \omega_n t + a_n^1\sin \omega_n t/\omega_n\}\phi_n(1),$$

where

(6.4) $\quad a_n^0 = (a_0, \phi_n), \qquad a_n^1 = (a_1, \phi_n).$

Noting the relation ([4])

(6.5) $$0 < \operatorname{Inf}\{|\phi_n(0)|,\ |\phi_n(1)|;\ n=0,1,2,\cdots\} \leqq \operatorname{Sup}\{|\phi_n(0)|,\ |\phi_n(1)|;\ n=0,1,2,\cdots\} < +\infty,$$

we consider the following class of sequences $\hat{a}_0 = \{\hat{a}_n^0\}_{n\geqq 0}$, $\hat{a}_1 = \{\hat{a}_n^1\}_{n\geqq 0}$ for the exponent $\beta \geqq 0$.

Notation 3. We say $\hat{a} = (\hat{a}_0, \hat{a}_1) \varepsilon X_\beta$ if

(6.6) $$||\hat{a}||_{X_\beta}^2 \equiv \sum_{n=0}^{\infty} (n^2+1)^\beta \{(n^2+1)(\hat{a}_n^0)^2 + (\hat{a}_n^1)^2\} < \infty.$$

We assume

(6.7) $\quad p \varepsilon C^\gamma[0,1] \qquad \text{for} \quad \gamma = \max(\beta-2, 0).$

Then,

Proposition 2. For $\hat{a} = (\hat{a}_0, \hat{a}_1) \varepsilon X_\beta$ $(\beta \geqq 0)$ and $T>0$,

(6.8) $$f_N(t) = \sum_{n=0}^{N} \{\hat{a}_n^0 \cos \omega_n t + \hat{a}_n^1 \sin \omega_n t / \omega_n\}$$

converges in $H_t^{\beta+1}(-T,T)$ as $N \to \infty$.

The limit

$$f(t) = \sum_{n=0}^{\infty} \{\hat{a}_n^0 \cos \omega_n t + \hat{a}_n^1 \sin \omega_n t / \omega_n\} \equiv \Phi(\hat{a})$$

is called the "non-harmonic Fourier series". The operator

$$\Phi : X_\beta \to H_t^{\beta+1}(-T,T)$$

is bounded. Set

$$(6.9)\qquad Y_\beta = \Phi(X_\beta).$$

Lemma 3.

(i) If $T<1$, Φ is not injective. The relation

$$(6.10)\qquad Y = H_t^{\beta+1}(-T,T) \quad (0\leqq\beta<1/2)$$
$$\subsetneqq H_t^{\beta+1}(-T,T) \quad (1/2\leqq\beta)$$

holds.

(ii) If $T>1$, we have $Y_\beta \subsetneqq H_t^{\beta+1}(-T,T)$. However, Φ: $X_\beta \to Y_\beta$ is an isomorphism:

$$||f||^2_{H_t^{\beta+1}(-T,T)} \approx \sum_{n=0}^{\infty}(n^2+1)^\beta\{(n^2+1)(a_n^0)^2 + (a_n^1)^2\}.$$

(iii) If $T=1$, Φ: $X_\beta \to Y_\beta$ is an isomorphism. The relation (6.10) holds.

(iv) In any case, the relation

$$(6.11)\qquad Y_\beta \cap H_t^{\alpha+1}(-T,T) = Y_\alpha$$

holds, provided

$$(6.12)\qquad \beta-\tfrac{1}{2} \neq 0,1,2,\cdots, \quad \text{and} \quad 0<\alpha-\beta<\tfrac{1}{2}.$$

For the proof Proposition 2 and Lemma 3, see [5].

§7. Outline of the proof of the uniqueness theorem (Theorem 1).

We suppose $E(P,a)\varepsilon G$, $T\geqq 2$ and

$$(7.1)\qquad \varepsilon_0(t) \equiv v_{|x=0} - u_{|x=0} = 0, \quad \varepsilon_1(t) \equiv v_{|x=1} - u_{|x=1} = 0$$

$$(-T \leq t \leq T),$$

where $u=u(x,t)$ and $v=v(x,t)$ are the solutions of $E(P,a)$ and $E(Q,b)$, respectively. Let $K(x,y)=K(x,y;q,j;p,h)$ be the kernel of Lemma 1 of §5, and put

$$(7.2) \qquad V(x,t) = u(x,t) + \int_0^x K(x,y)u(y,t)dy.$$

By virtue of Lemma 2, $V=V(x,t)\varepsilon C^0([0,1]\times(-\infty,\infty))$ satisfies

$$(7.3.1) \quad \partial_t{}^2 V + (-\partial_x{}^2 + q(x))V = 0$$

$$(7.3.2) \quad (-\partial_x + j)V|_{x=0} = 0$$

in the generalized sense in $\Omega_T=\{(x,t)|0<x<\min(1,T), -T+x<t<T-x\}$ and

$$(7.3.3) \quad V|_{x=0} = u|_{x=0} = v|_{x=0} \qquad (-T \leq t \leq T)$$

by (7.1).

Therefore,

$$(7.4) \qquad V = v \qquad (\text{on } \overline{\Omega}_T)$$

holds by Proposition 1 of §5. In particular, we have

$$(7.5) \qquad V|_{x=1} = v|_{x=1} = u|_{x=1} \qquad (-T_1 \leq t \leq T_1)$$

by (7.1), where

$$(7.6) \qquad T_1 = T - 1.$$

Hence by (7.2),

$$(7.7)\qquad \int_0^1 K(1,y)u(y,t)dy = \sum_{n=0}^{\infty} \{a_n^0 \cos \omega_n t + a_n^1 \sin \omega_n t / \omega_n\} \times \int_0^1 K(1,y)\phi_n(y)dy = 0 \qquad (-T_1 \leqq t \leqq T_1).$$

Since $T_1 = T-1 \geqq 1$, (7.7) yields

$$(7.8)\qquad a_n^0 \int_0^1 K(1,y)\phi_n(y)dy = a_n^1 \int_0^1 K(1,y)\phi_n(y)dy = 0 \quad (n=0,1,2,\cdots)$$

by Lemma 3 of §6, so that

$$(7.9)\qquad \int_0^1 K(1,y)\phi_n(y)dy = 0 \qquad (n=0,1,2,\cdots)$$

from the assumption $E(P,a) \varepsilon G$. Hence

$$(7.10)\qquad K(1,y) = 0 \qquad (0 \leqq y \leqq 1).$$

On the other hand, the relation

$$(7.11)\qquad (K(1,1)-J+H)u(1,t) + \int_0^1 K_x(1,y)u(y,t)dy = 0 \quad (-T_1 \leqq t \leqq T_1)$$

follows from (7.2), (7.4),

$$(7.12)\qquad "(\partial_x + H)u|_{x=1} = 0"$$

and

$$(7.13)\qquad "(\partial_x + J)v|_{x=1} = 0".$$

We should be careful about the regularity of u and v, and (7.12)-(7.13) must be taken in the generalized sense. The precise proof of (7.11) is given in [4].

Now, in the same way, the equality

$$(K(1,1)-J+H)u(1,t) + \int_0^1 K_x(1,y)u(y,t)dy$$

$$= \sum_{n=0}^{\infty} \{a_n^0 \cos \omega_n t \quad + \quad a_n^1 \sin \omega_n t/\omega_n\}\{(K(1,1)-J+H)\phi_n(1) + \int_0^1 K_x(1,y)\phi_n(y)dy\} = 0 \quad (-T_1 \leqq t \leqq T_1)$$

gives

$$(7.14) \qquad (K(1,1)-J+H)\phi_n(1) + \int_0^1 K_x(1,y)\phi_n(y)dy = 0 \quad (n=0,1,2,\cdots).$$

The asymptotic behavior

$$(7.15) \qquad \phi_n(1) = \frac{1}{\sqrt{2}}(-1)^n + O(1/n) \qquad (n\to\infty)$$

is known ([3]). On the other hand, Parseval's relation

$$||K_x(1,\cdot)||^2_{L^2(0,1)} = \sum_{n=0}^{\infty} (\int_0^1 K_x(1,y)\phi_n(y)dy)^2 < \infty$$

gives

$$\lim_{n\to\infty} \int_0^1 K_x(1,y)\phi_n(y)dy = 0.$$

So that (7.14) gives

$$(7.16) \qquad K(1,1) - J + H = 0$$

and

$$\int_0^1 K_x(1,y)\phi_n(y)dy = 0 \quad (n=0,1,2,\cdots).$$

Therefore,

$$(7.17) \qquad K_x(1,y) = 0 \qquad (0 \leqq y \leqq 1)$$

holds.

We now recall that $K=K(x,y)$ satisfies (5.6). It is known that the relations (7.10), (7.17), (5.6.1) and (5.6.3) give $K\equiv 0$

on $\bar{D}$. See [4]. Therefore,

(7.18) $(q,j,J) = (p,h,H)$

follows from (5.6.2) and (7.16). Now, (7.1) and (7.18) yield

(7.19) $\sum_{n=0}^{\infty} \{c_n^0 \cos \omega_n t + c_n^1 \sin \omega_n t / \omega_n\}\phi_n(0) = 0 \quad (-T \leqq t \leqq T)$

for

(7.20) $c_n^0 = (a_0 - b_0, \phi_n), \quad c_n^1 = (a_1 - b_1, \phi_n).$

By (6.5), $T \geqq 2$ and Lemma 3, we get

$$c_n^0 = c_n^1 = 0 \quad (n=0,1,2,\cdots),$$

hence

(7.21) $a_0 = b_0, \quad a_1 = b_1.$

In this way,

(7.22) $(Q.b) = (P,a)$

is obtained.

§8. Outline of the proof of the stability theorem (Theorem 2).

Let $E(P,a) \varepsilon G_\alpha$ $(\alpha > 1/2)$, $T \geqq 2$ and

(8.1) $\varepsilon_0(t) = v|_{x=0} - u|_{x=0}, \quad \varepsilon_1(t) = v|_{x=1} - u|_{x=1} \quad (-T \leqq t \leqq T),$

where $u=u(x,t)$ and $v=v(x,t)$ are the solutions of $E(P,a)$ and $E(Q,b)$, respectively. Since $\varepsilon_0 \neq 0$, it is impossible to combine v with u directly as in (7.2), this time.

Let

$$(8.2.1)\qquad L(x,y) = K(x,y;q,j;p,h)$$

$$(8.2.2)\qquad M(x,y) = K(x,y;p,h;q,j).$$

By Lemma 2, the continuous function

$$(8.3)\qquad \tilde{U}(x,t) = v(x,t) + \int_0^x M(x,y)v(y,t)dy$$

satisfies

$$(8.4.1)\qquad \partial_t^2\tilde{U} + (-\partial_x^2 + p(x))\tilde{U} = 0$$

$$(8.4.2)\qquad (-\partial_x + h)\tilde{U}_{|x=0} = 0$$

in the generalized sense in $(0,1)\times(-\infty,\infty)$. Therefore, again by Lemma 2, the continuous function

$$(8.5)\qquad V(x,t) = \tilde{U}(x,t) + \int_0^x L(x,y)\tilde{U}(y,t)dy$$

satisfies

$$(8.6.1)\qquad \partial_t^2 V + (-\partial_x^2 + q(x))V = 0$$

$$(8.6.2)\qquad (-\partial_x + j)V_{|x=0} = 0$$

in the generalized sense in $(0,1)\times(-\infty,\infty)$. Now, we have

$$(8.7)\qquad V_{|x=0} = U_{|x=0} = v_{|x=0} \qquad (-\infty<t<\infty),$$

so that

$$(8.8)\qquad V \equiv v \quad \text{on} \quad [0,1]\times(-\infty,\infty)$$

by Proposition 1 of §5.

Let

$$(8.9)\qquad w = \tilde{U} - u \in C^0([0,1]\times(-\infty,\infty)).$$

Then, $w=w(x,t)$ satisfies

$$(8.10.1)\quad \partial_t^{\,2} w + (-\partial_x^{\,2} + p(x))w = 0$$

$$(8.10.2)\quad (-\partial_x + h)w_{|x=0} = 0,$$

as well as

$$(8.10.3)\quad w_{|x=0} = \tilde{U}_{|x=0} - u_{|x=0}$$
$$= v_{|x=0} - u_{|x=0} = \varepsilon_0(t) \qquad (-T\leqq t\leqq T).$$

Re-exaiming the proof of Proposition 1, we see that the relation (8.10) yields

<u>Claim 1</u>. The estimate

$$(8.11)\qquad ||w(x,\cdot)||_{H_t^{\alpha+2}(-T+x,T-x)} + ||w_x(x,\cdot)||_{H_t^{\alpha+1}(-T+x,T-x)}$$
$$\leqq C||\varepsilon_0||_{H_t^{\alpha+2}(-T,T)}$$

holds for each x in $0\leqq x\leqq 1$. The constant C depends only on p,h and T.

Next, we note the equality

$$(8.12)\qquad \varepsilon_1(t) \equiv v_{|x=1} - u_{|x=1}$$
$$= V_{|x=1} - u_{|x=1} = (V-\tilde{U})_{|x=1} - (\tilde{U}-u)_{|x=1}$$
$$= \int_0^1 L(1,y)\tilde{U}(y,t)dy - w_{|x=1}$$

$$= \int_0^1 L(1,y)u(y,t)dy + \int_0^1 L(1,y)w(t,y)dy - w(1,t),$$

derived from (8.5). By means of the proof of Lemma 1 of §5, the estimate

$$(8.13) \qquad ||L(1,\cdot)||_{L_y^2(0,1)} \leqq C(\kappa)$$

is shown, provided

$$(8.14) \qquad ||q||_{L^2(0,1)} + |j| \leqq \kappa.$$

Therefore, the function

$$(8.15) \qquad f(t) = \int_0^1 L(1,y)u(y,t)dy$$

satisfies, for $T_1 = T-1$,

$$||f||_{H_t^{\alpha+2}(-T_1,T_1)} \leqq ||w(1,\cdot)||_{H_t^{\alpha+2}(-T_1,T_1)}$$
$$+ ||\varepsilon_1||_{H_t^{\alpha+2}(-T_1,T_1)} + ||L(1,\cdot)||_{L_y^2(0,1)}$$
$$\times \sup_{0\leqq y\leqq 1} ||w(y,\cdot)||_{H_t^{\alpha+2}(-T_1,T_1)}.$$

Hence we obtain

$$(8.16) \qquad ||f||_{H_t^{\alpha+2}(-T_1,T_1)} \leqq C(\kappa)\{||\varepsilon_0||_{H_t^{\alpha+2}(-T,T)}$$
$$+ ||\varepsilon_1||_{H_t^{\alpha+2}(-T,T)}\}$$

by (8.11) and (8.13).

Furthermore, from the relation

$$(8.17) \qquad v(x,t) = \tilde{U}(x,t) + \int_0^x L(x,y)\tilde{U}(y,t)dy$$

$$= w(x,t) + \int_0^x L(x,y)w(y,t)dy + u(x,t) + \int_0^x L(x,y)u(y,t)dy,$$

we get in the same way as in (7.11) that

Claim 2. The equality

$$(8.18) \qquad g(t) = -w_x(1,t) - (L(1,1)+J)w(1,t) - Jf(t) - \int_0^1 \{L_x(1,y)+JL(1,y)\}w(y,t)dy$$

holds, where

$$(8.19) \qquad g(t) = \int_0^1 L_x(1,y)u(y,t)dy + (L(1,1)-J+H)u(1,t).$$

Since the estimate

$$(8.20) \qquad ||L_x(1,\cdot)||_{L_y^2(0,1)} \leqq C(\kappa)$$

holds as in (8.13), we obtain

$$(8.21) \qquad ||g||_{H_t^{\alpha+1}(-T_1,T_1)} \leqq C(\kappa)\{||\varepsilon_0||_{H_t^{\alpha+2}(-T,T)} + ||\varepsilon_1||_{H_t^{\alpha+2}(-T,T)}\},$$

provided

$$(8.22) \qquad ||q||_{L^2(0,1)} + |j| + |J| \leqq \kappa.$$

Now, we recall

$$(8.23) \qquad f(t) = \int_0^1 L(1,y)u(y,t)dy = \sum_{n=0}^{\infty} \{a_n^0 \cos \omega_n t + a_n^1 \sin \omega_n t / \omega_n\} \int_0^1 L(1,y)\phi_n(y)dy.$$

By the assumption $E(P,a) \varepsilon G_\alpha$, we get

$$(8.24)\qquad \sum_{n=0}^{\infty}(n^2+1)^{\alpha+1}\{(n^2+1)(a_n^0)^2+(a_n^1)^2\}(\int_0^1 L(1,y)\phi_n(y)dy)^2$$
$$\approx \sum_{n=0}^{\infty}(n^2+1)(\int_0^1 L(1,y)\phi_n(y)dy)^2 \approx ||L(1,\cdot)||^2_{H_y^1(0,1)}<\infty,$$

hence $f\varepsilon Y_{\alpha+1}$. So that

$$(8.25)\qquad ||L(1,\cdot)||_{H_y^1(0,1)} \approx ||f||_{H_t^{\alpha+2}(-T_1,T_1)}$$

by Lemma 3. Hence

$$(8.26)\qquad ||L(1,\cdot)||_{H_y^1(0,1)} \leqq C(\kappa)\{||\varepsilon_0||_{H_t^{\alpha+2}(-T,T)} + ||\varepsilon_1||_{H_t^{\alpha+2}(-T,T)}\}.$$

Similarly, we have

$$(8.27)\qquad \int_0^1 L_x(1,y)u(y,t)dy = \sum_{n=0}^{\infty}\{a_n^0 \cos\omega_n t + a_n^1 \sin\omega_n t/\omega_n\}$$
$$\times \int_0^1 L_x(1,y)\phi_n(y)dy \;\varepsilon\; Y_\alpha \subset H_t^{\alpha+1}(-T_1,T_1).$$

On the other hand, as is noted in (4.6), we have

$$(8.28)\qquad u(1,\cdot) \not\in H_t^{\alpha+1}(-T_1,T_1).$$

Since $g\varepsilon H_t^{\alpha+1}(-T_1,T_1)$ follows from (8.18) and ε_0, ε_1 ε $H_t^{\alpha+2}($ $-T,T)$ as we have seen, the equality

$$(8.29)\qquad L(1,1)+J-H=0$$

follows. Hence

$$(8.30)\qquad ||L_x(1,\cdot)||^2_{L_y^2(0,1)} \approx \sum_{n=0}^{\infty}(\int_0^1 L_x(1,y)\phi_n(y)dy)^2 \approx$$
$$\sum_{n=0}^{\infty}(n^2+1)^{\alpha}\{(n^2+1)(a_n^0)^2+(a_n^1)^2\}(\int_0^1 L_x(1,y)\phi_n(y)dy)^2$$

$$||g||^2_{H_t^{+1}(-T_1,T_1)}.$$

By (8.21), we have

$$(8.31) \qquad ||L_x(1,\cdot)||_{L_y^2(0,1)} \leqq C(\kappa)\{||\varepsilon_0||_{H_t^{\alpha+2}(-T,T)} + ||\varepsilon_1||_{H_t^{\alpha+2}(-T,T)}\}.$$

From the proof of Lemma 1, the estimate

$$(8.32) \qquad ||L(z,z)||_{H_z^1(0,1)} \leqq C(\kappa)\{||\varepsilon_0||_{H_t^{\alpha+2}(-T,T)} + ||\varepsilon_1||_{H_t^{\alpha+2}(-T,T)}\}$$

is shown by (5.6.1), (5.6.3), (8.26) and (8.31). Hence, Theorem 2 follows from (5.6.2), (8.29) and (8.32).

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