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The action of Hecke operators on theta series

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1984年10月初めに数理解析研究所で行われた研究集会で私は"Theta級数にHecke作用素はどうactするか"について話した．一方，10月末から11月初めにかけての城崎での"宮田武彦氏追悼シンポジウム"でほぼ同じ内容の講演をしたが，このシンポジウムのproceedingのための英文原稿の提出をもとめられた．"講究録の作成上の注意"5，6項によれば，投稿中の論文を代用することは可，とのことなので，私は上記シンポジウムのための論文原稿をもって責を塞ぐこととした．これは3章から成るが，長さを考慮して，初めの2章のみをのせることにする．これだけでも，一応は完結しているからである．

この論文を書いている過程で気づいた問題があるので以下に記しておきたい．

$S_1, \cdots, S_h \in M_N(\mathbb{Q})$ は同じgenusに属する正値対称行列のclassesの完全代表とする．正値対称行列 $T \in M_m(\mathbb{Q})$ に対して，

$$A(S_i, T) = |\{X \in M_{N,m}(\mathbb{Z}) \mid {}^tXS_iX = T\}|,$$
$$e_i = |\{X \in M_N(\mathbb{Z}) \mid {}^tXS_iX = S_i\}|$$

とおく.

Siegelの主定理は（Gesammelte Abhandlungen I, No.20）

$$\left(\sum_{i=1}^{c} A(S_i, T)/e_i\right) \Big/ \left(\sum_{i=1}^{c} e_i^{-1}\right)$$

をlocal densityの積として表わすものである.

ここで問題は、$A(S_i, T)$の定義の右辺の$X \in M_{N,m}(\mathbb{Z})$に単因子の形を指定する制限をつけても、Siegelのものと類似の公式が成立するのではないかということである. 特別な場合（$T = p^2S_i$, p : primeなど）にこれが本文中の方法により容易に証明されることは疑いない. 一般にはどうであろうか.

The action of Hecke operators on theta series

By Hiroyuki Yoshida

Introduction

Let S be a $2n \times 2n$ positive definite symmetric matrix with rational coefficients. Let $S_1, \cdots, S_c$ be a complete set of representatives of the classes in the genus of S. For a positive integer m, put

$$\vartheta_i^{(m)}(z) = \sum_{x \in M_{2n,m}(Z)} \exp(2\pi\sqrt{-1}\,\mathrm{Trace}({}^t x S_i x z)), \quad z \in H_m,$$

where H_m denotes the Siegel upper half space of degree m and $M_{2n,m}(Z)$ denotes the set of all $2n \times m$ matrices with integral coefficients. As well known, $\vartheta_i^{(m)}(z)$ defines a Siegel modular form of weight n of a certain level. To determine the action of Hecke operators on $\vartheta_i^{(m)}(z)$ is one of classical problems concerning theta series.

In the present paper, we shall treat this problem from the point of view of the Weil representation. In §1, we shall express theta series in terms of Weil representations and shall show, as Theorems 1.6 and 1.7, that it can be reduced to a local problem (1.23). In the succeeding sections, §2 and §3, we shall solve the local problems concerning $T(p)$ and $T^{(s)}(p^2)$ respectively (cf. (1.28) and (1.32) for the definitions of these operators which are generators of the Hecke ring). The main results are formulated as Theorems 2.1, 3.7, 3.8 and 3.10. The line of such method of investigation was suggested in our previous work [12] where only the case $m = n = 2$ was treated; we shall carry out the program more systematically in this paper.

Two works should be mentioned here in the relation with our results. Freitag [5] has given a simple formula for the action of $T(p)$ on $\vartheta_i^{(m)}$ (cf. Proposition 1.9). His method of proof, which employs the theory of singular Siegel modular forms, is different from ours. Also our results are sharper in the sense that not only they are not restricted to the level 1 case but also they give explicit relations with automorphic forms on the orthogonal groups. In the case $m = n$, we can give a simple formula (cf. Proposition 1.1 for the action of $T^{(s)}(p^2)$ on $\vartheta_i^{(m)}$ in a similar fashion as [5].

The paper of Rallis [7] is closely related to our results on $T^{(s)}(p^2)$ proved in §3. In fact, it seems that one is equivalent to the other modulo some explicit computation of "the Satake transform" of $T^{(s)}(p^2)$. However we should not dispence with §3 because of the following reasons. The proofs in §2 and §3 are similar in spirit and it is aesthetically unsatisfactory to restrict only to the case $T(p)$; furthermore our method of proof, which is different from [7], seems to be applicable, with rather small number of modifications, to the case where the dimension of the quadratic space is odd.

Notation. If S is an associative ring with a unit, $S^{\times}$ denotes the group of all invertible elements of S. Let R be a commutative ring with a unit. We denote by $M_{m,n}(R)$ the set of all $m \times n$-matrices with entries in R. Let I be an ideal of R and $A = (a_{ij})$, $B = (b_{ij}) \in M_{m,n}(R)$. We denote $A \equiv B \bmod I$ when $a_{ij} \equiv b_{ij} \bmod I$ for all $1 \leqq i \leqq m$, $1 \leqq j \leqq n$. We abbreviate $M_{m,m}(R)$ to $M_m(R)$ and set $GL_m(R) = M_m(R)^{\times}$. If $A \in M_m(R)$, $\sigma(A)$ stands for the trace of A. The diagonal matrix with diagonal elements $a_1, a_2, \cdots, a_m$ is denoted by $\mathrm{diag}\left[a_1, a_2, \cdots, a_m\right]$.

By $GSp(m)$ and $Sp(m)$, we denote the group of symplectic similitudes and the symplectic group of degree m respectively. We assume that the group of R-valued points $GSp(m)_R$ (resp. $Sp(m)_R$) of $GSp(m)$ (resp. $Sp(m)$) is given explicitly by $GSp(m)_R = \left\{ g \mid g \in GL_{2m}(R),\ {}^t g w g = m(g) w,\ m(g) \in R^{\times} \right\}$ (resp. $Sp(m)_R = \left\{ g \mid g \in GL_{2m}(R),\ {}^t g w g = w \right\}$). Here $w = \begin{pmatrix} 0_m & 1_m \\ -1_m & 0_m \end{pmatrix}$, 0_m and 1_m are the zero and the unit matrix of size m respectively. We usually denote $g \in GSp(m)_R$ as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $m \times m$-matrices a, b, c, d. The Siegel upper half space of degree m is denoted by H_m. We set $e(z) = \exp(2\pi\sqrt{-1}z)$ for $z \in \mathbb{C}$.

Let k be a field and G be an algebraic group defined over k. The group of all k-rational points of G is denoted by G_k. When k is an algebraic number field, G_A denotes the adelization of G.

For a place v of k, G_{k_v} may be abbreviated to G_v. When $k = Q$, ∞ denotes the infinite place.

The finite field with q elements is denoted by F_q. Among the orthogonal groups associated to 2n-dimensional regular quadratic spaces over F_q, there are two isomorphism classes over F_q. We denote by $O_1(2n,F_q)$ (resp. $O_{-1}(2n,F_q)$), the group of all F_q-rational points of the orthogonal group when the Witt index of the quadratic space is n (resp. $n-1$).

If X is a locally compact abelian group, $S(X)$ denotes the space of all Schwarz-Bruhat functions on X.

§ 1. Theta series and Hecke operators

In this section, we shall express theta series in terms of the Weil representation and consequently we shall show that the action of Hecke operators on theta series can be explicitly localized.

Let G (resp. $\widetilde{G}$) denote the symplectic group $Sp(m)$ (resp. the group of symplectic similitudes $GSp(m)$) of degree m. Let Z denote the center of $\widetilde{G}$. Let N be a positive integer and ω be a character of finite order of $Q_A^{\times}/Q^{\times}$ whose conductor divides N. For every prime number p, we define an open compact subgroup K_p (resp. $\widetilde{K}_p$) of G_{Q_p} (resp. $\widetilde{G}_{Q_p}$) and a representation M_p (resp. $\widetilde{M}_p$) of K_p (resp. $\widetilde{K}_p$) as follows.

If $p \nmid N$, we set $K_p = Sp(m)_{Z_p}$, $\widetilde{K}_p = GSp(m)_{Z_p}$; let M_p (resp. $\widetilde{M}_p$) be the trivial representation of K_p (resp. $\widetilde{K}_p$).

If $p \mid N$, we set

$$K_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m)_{Z_p} \;\middle|\; c \equiv 0 \bmod p^{\ell_p} \right\} ,$$

$$\widetilde{K}_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GSp(m)_{Z_p} \;\middle|\; c \equiv 0 \bmod p^{\ell_p} \right\} ,$$

where p^{ℓ_p} denotes the highest power of p which divides N. We set

$$M_p(k) = \omega_p(\det a) \quad \text{for} \quad k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_p,$$

$$\widetilde{M}_p(k) = \omega_p(\det a) \quad \text{for} \quad k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{K}_p .$$

Let σ be an irreducible rational representation of $GL_m(C)$. For the infinite place ∞ of Q, we set

$$K_\infty = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in Sp(m)_R \right\} ,$$

and define a representation M_∞ of K_∞ by

$$M_\infty\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right) = \sigma(a + b\sqrt{-1}).$$

We set $\widetilde{K}_\infty = K_\infty$, $\widetilde{M}_\infty = M_\infty$. We note that $K_\infty \cong U(m,C)$, the unitary group of degree m, and it is a maximal compact subgroup of $G_\infty = Sp(m)_R$ as being the stabilizer of $\sqrt{-1}\cdot 1_m \in H_m$.

We put $K = \prod_v K_v$ and define a representation M of K by $M = \bigotimes_v M_v$; similarly we set $\widetilde{K} = \prod_v \widetilde{K}_v$, $\widetilde{M} = \bigotimes_v \widetilde{M}_v$. Let W be the representation space of M and $\widetilde{M}$.

Now let $A_\sigma^{(m)}(N,\omega)$ denote the vector space consisting of all W-valued continuous functions F on G_A which satisfy the following condition (A).

(A) $F(\gamma g k) = F(g)M(k)$ for any $\gamma \in G_Q$, $g \in G_A$, $k \in K$.

Similarly let $\widetilde{A}_\sigma^{(m)}(N,\omega)$ denote the vector space consisting of all W-valued continuous functions F on $\widetilde{G}_A$ which satisfy the following conditions $(\widetilde{A})$ and $(\widetilde{B})$.

$(\widetilde{A})$ $F(\gamma g k) = F(g)\widetilde{M}(k)$ for any $\gamma \in \widetilde{G}_Q$, $g \in \widetilde{G}_A$, $k \in \widetilde{K}$.

$(\widetilde{B})$ $F(zg) = \omega^m(z)F(g)$ for any $z \in Z_A$, $g \in \widetilde{G}_A$.

Here we have regarded ω as a character of Z_A through the isomorphism $Z_A \cong Q_A^x$.

Lemma 1.1. <u>Let</u> Res <u>denote the restriction map from</u> $\widetilde{A}_\sigma^{(m)}(N,\omega)$ <u>to</u> $A_\sigma^{(m)}(N,\omega)$. <u>Then</u> Res <u>is an isomorphism</u>.

Proof. Let $Z_{\infty +} = \left\{ z\cdot 1_{2m} \mid z \in R_+^x \right\} \subset Z_\infty$. By $Q_A^x = Q^x \cdot \prod_p Z_p^x \cdot R_+^x$, we get

(1.1) $\widetilde{G}_A = \widetilde{G}_Q G_A Z_{\infty +} \prod_p \widetilde{K}_p$.

Hence Res is injective. For $F \in A_\sigma^{(m)}(N,\omega)$, put

(1.2) $\quad \widetilde{F}(\gamma g z k) = F(g)\widetilde{M}(k), \quad \gamma \in \widetilde{G}_Q,\ g \in G_A,\ z \in Z_{\infty+},\ k \in \prod_p \widetilde{K}_p .$

Then it is easy to verify that $\widetilde{F} \in \widetilde{A}_\sigma^{(m)}(N,\omega)$. Hence Res is surjective. This completes the proof.

We are going to define the classical spaces of Siegel modular forms and investigate their relation with $A_\sigma^{(m)}(N,\omega)$. We put

(1.3) $\quad j_\sigma(g,z) = \sigma(cz+d) \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{G}_{\infty+},\ z \in H_m ,$

where $\widetilde{G}_{\infty+} = \left\{ g \in \widetilde{G}_\infty \mid m(g) > 0 \right\}$. We have the condition of automorphic factor: $j_\sigma(g_1 g_2, z) = j_\sigma(g_1, g_2 z) j_\sigma(g_2, z)$, $g_1, g_2 \in \widetilde{G}_{\infty+}$, $z \in H_m$. Set

$$\Gamma_0^{(m)}(N) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Sp(m)_Z \;\middle|\; c \equiv 0 \bmod N \right\} .$$

For $a \in Z$, $(a,N) = 1$, put

(1.4) $\quad \omega_0(a) = \prod_{p \mid N} \omega_p(a) .$

Then ω_0 is a Dirichlet character modulo N. Let $\widetilde{G}_\sigma(\Gamma_0^{(m)}(N), \omega_0)$ denote the space of all W-valued continuous functions f on H_m which satisfy

(C) $\quad f(\gamma(z)) = \omega_0(\det a) f(z) j_\sigma(\gamma, z)^{-1}$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0^{(m)}(N)$, $z \in H_m$. Let $G_\sigma(\Gamma_0^{(m)}(N), \omega_0)$ denote the space of all functions in $\widetilde{G}_\sigma(\Gamma_0^{(m)}(N), \omega_0)$ which are holomorphic on $H_m \cup \{\text{cusps}\}$. As well known, the holomorphy at cusps is automatically satisfied if $m \geq 2$.

Take $F \in A_\sigma^{(m)}(N,\omega)$. For $g \in G_\infty$, define $\widetilde{g} \in G_A$ by $\widetilde{g}_v = 1$ if $v \neq \infty$, $\widetilde{g}_\infty = g$. Set

(1.5) $\quad f(gi) = F(\widetilde{g}) j_{\check{\sigma}}(g,i)^{-1}, \quad g \in G_\infty ,$

where $i = \sqrt{-1}\, 1_m$ and $\check{\sigma}$ is the contragredient representation of σ

$\check{\sigma}(g) = \sigma({}^t g^{-1})$ for $g \in GL_m(C)$.

Lemma 1.2. *Let* Ψ *be the correspondence* $F \longrightarrow f$ *defined by* (1.5). *Then* Ψ *is an isomorphism* $A_\sigma^{(m)}(N,\omega) \cong \overline{G}_{\check{\sigma}}(\Gamma_0^{(m)}(N),\omega_0^{-1})$.

Since this lemma should be well known, we omit the proof.

Now we are going to define Hecke operators. Assume $p \nmid N$; so we have $\widetilde{K}_p = GSp(m)_{Z_p}$. Take $a \in \widetilde{G}_{Q_p}$ and let $\widetilde{K}_p a \widetilde{K}_p = \bigcup_i g_i \widetilde{K}_p$ be a coset decomposition. For $F \in \widetilde{A}_\sigma^{(m)}(N,\omega)$, we put

(1.6) $$((\widetilde{K}_p a \widetilde{K}_p)F)(g) = \sum_i F(gg_i), \quad g \in \widetilde{G}_A.$$

Then it is clear that $(\widetilde{K}_p a \widetilde{K}_p)F \in \widetilde{A}_\sigma^{(m)}(N,\omega)$. Assume $a = \mathrm{diag}[p^{d_1},\cdots,p^{d_m},p^{e_1},\cdots,p^{e_m}]$ with non-negative integers d_i, e_i $(1 \leqq i \leqq m)$ such that $d_i + e_i = u$, $m(a) = p^u$. By the strong approximati theorem, we see easily that a coset decomposition

$$\Gamma_0^{(m)}(N) a \Gamma_0^{(m)}(N) = \bigcup_{i=1}^{d} \Gamma_0^{(m)}(N)\gamma_i$$

gives rise the coset decomposition

$$\widetilde{K}_p a^{-1} \widetilde{K}_p = \bigcup_{i=1}^{d} \gamma_i^{-1} \widetilde{K}_p \quad .$$

By rather formal manipulations(cf. [12], §6), we find:

Lemma 1.3. *The assumptions being as above, let* Ψ *be the same as in Lemma* 1.2. *In view of Lemma* 1.1, *use the same letter* Ψ *for the isomorphism* $\widetilde{A}_\sigma^{(m)}(N,\omega) \cong \overline{G}_{\check{\sigma}}(\Gamma_0^{(m)}(N),\omega_0^{-1})$ *for the sake of simplicity. Put* $f = \Psi(F)$, $f^* = \Psi((\widetilde{K}_p a^{-1} \widetilde{K}_p)F)$. *Let* k *be the integer such that* $\sigma(\alpha \cdot 1_m) = \alpha^k \cdot$*identity for* $\alpha \in C^\times$. *Then we have*

$$f^*(z) = p^{uk/2} \sum_{i=1}^{d} \omega_0(\det a_i) f(\gamma_i z) j_{\check{\sigma}}(\gamma_i, z), \quad z \in H_m,$$

<u>where</u> $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$

This Lemma shows that the Hecke operators $\widetilde{K}_p a \widetilde{K}_p$ essentially coincide with the classical Hecke operators (cf. Andrianov [1]) when interpreted in terms of the isomorphism $\widetilde{A}_{\sigma}^{(m)}(N,\omega) \cong \overline{G}_{\check{\sigma}}(\Gamma_o^{(m)}(N),\omega_o^{-1})$.

Now let us consider theta series. Let $S \in M_{2n}(Q)$ be a positive definite symmetric matrix and H (resp. $\widetilde{H}$) denote the orthogonal group (resp. the group of orthogonal similitudes). Set $V = M_{2n,1}(Q)$, $Q(x) = {}^t xSx$ for $x \in V$, $X = M_{2n,m}(Q)$ and we identify X with V^m. We choose a character ψ of Q_A/Q so that $\psi_\infty(x) = e(x)$, $x \in R$, $\psi_p(x) = e(-Fr(x))$, $x \in Q_p$ for every p, where $Fr(x)$ denotes the fractional part of x. Then, associated with S and ψ_v (resp. ψ), we have the local (resp. global) Weil representation π_v (resp. π) of G_v (resp. G_A) realized on $S(X_v)$ (resp. $S(X_A)$), where v is a place of Q (cf. [13], §2). Let ω be the character of $Q_A^\times/Q^\times$ which corresponds to $Q(\sqrt{(-1)^n \det S})$ by class field theory. Let L be an integral lattice on V and K' be the stabilizer of L in H_A. We have $K' = \prod_p K'_p \times K'_\infty$ with $K'_p = \{ h \in H_{Q_p} \mid hL_p = L_p \}$ and $K'_\infty = H_\infty$ where $L_p = L \otimes_Z Z_p$. Let τ be a finite dimensional representation of K'_∞ on the vector space W_τ such that τ is unitary with respect to an inner product $\langle\ ,\ \rangle_\tau$ on W_τ. Define a representation ρ of K' by $\rho(k) = \tau(k_\infty)$ for $k \in K'$. Let φ be a W_τ-valued function on H_A which satisfies

(1.7) $\quad \varphi(\gamma hk) = \varphi(h)\rho(k)$ for any $\gamma \in H_Q$, $h \in H_A$, $k \in K'$.

Let $P(x)$ be a $W_\tau \otimes W$-valued polynomial function on X_∞ which satisfies

(1.8) $\quad P(k^{-1}x) = P(x)(\rho(k) \otimes 1)$ for any $k \in K'_\infty$, $x \in X_\infty$,

(1.9) $\quad P(xa) = P(x)(1 \otimes (\sigma(a)\det(a)^{-n}))$ for any $x \in X_\infty$, $a \in GL_m(R)$.

We set

$f_\infty(x) = \exp(-2\pi\sigma({}^t xSx))P(x) \in S(X_\infty) \otimes W_\tau \otimes W$ for $x \in X_\infty$,

f_p = the characteristic function of $L_p^m \in S(X_p)$,

$f = \prod_v f_v \in S(X_A) \otimes W_\tau \otimes W$.

For $v_1, v_2 \in W_\tau$ and $w \in W$, put $\langle v_1 \otimes w, v_2 \rangle = \langle v_1, v_2 \rangle_\tau w$; extend this pairing to the map of $W_\tau \otimes W \times W_\tau$ to W, which is C-linear(resp. C-anti-linear) with respect to the first(resp. second) argument. We put

$$(1.10) \qquad \Phi_f^\varphi(g) = \int_{H_Q \backslash H_A} \langle \sum_{x \in X_Q} (\pi(g)f)(h^{-1}x), \varphi(h) \rangle \, dh.$$

Here dh denotes the invariant measure on $H_Q\backslash H_A$ which is derived by Weil's relation from the Haar measure $\widetilde{dh}$ on H_A such that $\int_{K'} \widetilde{dh} = 1$. Let p be a prime. Put $B(x,y) = Q(x+y) - Q(x) - Q(y)$ $x,y \in V$, $\check{L}_p = \{ x \in V_{Q_p} \mid B(x,y) \in Z_p \text{ for all } y \in L_p \}$. Let $(p^{-\ell_p})$ be the Z_p-module generated by $\{ Q(x) \mid x \in \check{L}_p \}$. We see that ℓ_p is a non-negative integer and is equal to zero for almost all p. We set $N = \prod_p p^{\ell_p}$.

Lemma 1.4.(cf.[13], §2) <u>If</u> f_∞ <u>satisfies</u>[1)]

$$(1.11) \qquad \pi_\infty\left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right)f_\infty = f_\infty \cdot \sigma(a + b\sqrt{-1})$$

<u>for any</u> $\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in K_\infty$, <u>then we have</u> $\Phi_f^\varphi \in A_\sigma^{(m)}(N,\omega)$.

Hereafter we shall assume (1.11). Let $\widetilde{\Phi}_f^\varphi$ be the extension of Φ_f^φ to $\widetilde{A}_\sigma^{(m)}(N,\omega)$ guaranteed by Lemma 1.1. When there is no fear

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1) When P is a constant function on X_∞, this condition is satisfied with $\sigma(g) = \det(g)^n$, $g \in GL_m(C)$ (cf.[13], Lemma 2.2).

of confusion, we shall abbreviate Φ_f^{φ}(resp. $\widetilde{\Phi}_f^{\varphi}$) to Φ_f(resp. $\widetilde{\Phi}_f$). We shall analyse the action of Hecke operators $\widetilde{K}_p a \widetilde{K}_p$, $a \in \widetilde{G}_{Q_p} \cap M(2m, Z_p)$ for $p \nmid N$ on $\widetilde{\Phi}_f$. We may assume $m(a) = p^u$ with a non-negative integer u. Let $\widetilde{K}_p a \widetilde{K}_p = \bigcup_i g_i \widetilde{K}_p$ be a coset decomposition such that $m(g_i) = p^u$ for every i.

First let us consider the case where u is even. Put $u = 2t$ and set $z = p^{-t} \cdot 1_{2m} \in \widetilde{G}_Q$, $z_p = p^{-t} \cdot 1_{2m} \in \widetilde{G}_{Q_p}$. Then we have

$$\widetilde{\Phi}_f(gg_i) = \widetilde{\Phi}_f(zgg_i) = (\omega_o(p^{-t}))^m \widetilde{\Phi}_f(z_p g g_i)$$

$$= \omega_o(p^{mt}) \int_{H_Q \backslash H_A} \left\langle \sum_{x \in X_Q} (\pi(g z_p g_i) f)(h^{-1} x),\ \varphi(h) \right\rangle dh .$$

Hence if we put

$$(1.12) \qquad f'_p = \sum_i \pi_p(z_p g_i) f_p ,$$

$$(1.13) \qquad f' = \prod_{v \neq p} f_v \times f'_p ,$$

then we get

$$((\widetilde{K}_p a \widetilde{K}_p) \widetilde{\Phi}_f)(g) = \omega_o(p^{mt}) \Phi_{f'}(g) \quad \text{for} \quad g \in G_A .$$

Since $\pi_p(k) f'_p = f'_p$ for $k \in K_p$, we see that $\Phi_{f'} \in A_{\sigma}^{(m)}(N, \omega)$. Therefore we get

$$(1.14) \qquad ((\widetilde{K}_p a \widetilde{K}_p) \widetilde{\Phi}_f)(g) = \omega_o(p^{mt}) \widetilde{\Phi}_{f'}(g) \quad \text{for} \quad g \in \widetilde{G}_A ,$$

where $\widetilde{\Phi}_{f'}$ is the unique extension of $\Phi_{f'}$ to $\widetilde{A}_{\sigma}^{(m)}(N, \omega)$ guaranteed by Lemma 1.1.

Next let us consider the case where u is odd. Put $u = 2t + 1$,

$$\nu = \begin{pmatrix} p^{-t} \cdot 1_m & 0 \\ 0 & p^{-t-1} \cdot 1_m \end{pmatrix} \in \widetilde{G}_Q , \quad \nu_p = \begin{pmatrix} p^{-t} \cdot 1_m & 0 \\ 0 & p^{-t-1} \cdot 1_m \end{pmatrix} \in \widetilde{G}_{Q_p} .$$

Assume $g \in \widetilde{G}_A$ satisfies $g_f = 1$ and $\begin{pmatrix} 1_m & 0 \\ 0 & p^{-1} \cdot 1_m \end{pmatrix}_\infty g_\infty \in G_\infty$, where g_f denotes the finite part of g. Then we get (use $(\widetilde{A})$ and $(\widetilde{B})$)

$$\widetilde{\Phi}_f(gg_i) = \widetilde{\Phi}_f(\nu gg_i)$$

$$= \omega_o(p^{mt})\, \Phi_f(\cdots, 1, \cdots, 1, \nu_p g_i, 1, \cdots, \begin{pmatrix} 1_m & 0 \\ 0 & p^{-1} \cdot 1_m \end{pmatrix}_\infty g_\infty)$$

$$= \omega_o(p^{mt}) \int_{H_Q \backslash H_A} \Big\langle \sum_{x \in X_Q} \Big\{ \prod_{v \neq p, \infty} (\pi_v(1) f_v) \times \pi_p(\nu_p g_i) f_p$$

$$\times\, \pi_\infty\Big(\begin{pmatrix} 1_m & 0 \\ 0 & p^{-1} \cdot 1_m \end{pmatrix}_\infty g_\infty \Big) f_\infty \Big\} (h^{-1}x),\ \varphi(h) \Big\rangle\, dh .$$

Therefore if we put

$$(1.15) \qquad f'_p = \sum_i \pi_p(\nu_p g_i) f_p ,$$

$$(1.16) \qquad f' = \prod_{v \neq p} f_v \times f'_p ,$$

then we obtain

$$(1.17) \qquad (\widetilde{K}_p a \widetilde{K}_p)\widetilde{\Phi}_f)(g) = \omega_o(p^{mt})\, \Phi_{f'}\Big(\begin{pmatrix} 1_m & 0 \\ 0 & p^{-1} \cdot 1_m \end{pmatrix}_\infty g\Big),$$

for $g \in \widetilde{G}_A$ such that $g_f = 1$, $m(g_\infty) = p$.

We have $\pi_p(k) f'_p = f'_p$ for $k \in \nu_p K_p \nu_p^{-1}$. Let M_p^* (resp. $\widetilde{M}_p^*$) be the trivial representation of $\nu_p K_p \nu_p^{-1}$ (resp. $\nu_p \widetilde{K}_p \nu_p^{-1}$) and set

$$K^* = \prod_{v \neq p} K_v \times \nu_p K_p \nu_p^{-1}, \quad \widetilde{K}^* = \prod_{v \neq p} \widetilde{K}_v \times \nu_p \widetilde{K}_p \nu_p^{-1},$$

$$M^* = \bigotimes_{v \neq p} M_v \otimes M_p^*, \quad \widetilde{M}^* = \bigotimes_{v \neq p} \widetilde{M}_v \otimes \widetilde{M}_p^* .$$

Then we get

$$\Phi_{f'}(\gamma g k) = \Phi_{f'}(g) M^*(k) \quad \text{for any } \gamma \in G_Q,\ g \in G_A,\ k \in K^*.$$

By modifying Lemma 1.1 in the obvious manner, we see that $\Phi_{f'}$ can be extended to the unique function $\widetilde{\Phi}_{f'}$ on $\widetilde{G}_A$ which satisfies

$\widetilde{\Phi}_{f'}(\gamma gk) = \widetilde{\Phi}_{f'}(g)\widetilde{M}^*(k)$ for any $\gamma \in \widetilde{G}_Q$, $g \in \widetilde{G}_A$, $k \in \widetilde{K}^*$,

$\widetilde{\Phi}_{f'}(zg) = \omega^m(z)\widetilde{\Phi}_{f'}(g)$ for any $z \in Z_A$, $g \in \widetilde{G}_A$.

We obtain

$$(1.18) \quad ((\widetilde{K}_p a \widetilde{K}_p)\widetilde{\Phi}_f)(g) = \omega_o(p^{mt})\widetilde{\Phi}_{f'}(\begin{pmatrix} 1_m & 0 \\ 0 & p^{-1}\cdot 1_m \end{pmatrix}_\infty g)$$

for $g \in \widetilde{G}_A$, $g_f = 1$.

For $b \in H_{Q_p}$, let $K'_p b K'_p = \bigcup_j K'_p h_j = \bigcup_j h_j K'_p$ be coset decompositions[2]. Put

$$(1.18) \quad ((K'_p b K'_p) f_p)(x) = \sum_j f_p(h_j x), \; x \in X_p,$$

$$(1.19) \quad ((K'_p b K'_p)\varphi)(h) = \sum_j \varphi(hh_j), \; h \in H_A.$$

Then $(K'_p b K'_p)\varphi$ also satisfies (1.7). We put $\widetilde{K}'_p = \{ h \in \widetilde{H}_{Q_p} \mid hL_p = L_p \}$.

For $b \in \widetilde{H}_{Q_p}$, let $\widetilde{K}'_p b \widetilde{K}'_p = \bigcup_j \widetilde{K}'_p h_j = \bigcup_j h_j \widetilde{K}'_p$ be coset decompositions.

Put

$$(1.20) \quad ((\widetilde{K}'_p b \widetilde{K}'_p) f_p)(x) = \sum_j f_p(h_j x), \; x \in X_p.$$

Let $\widetilde{K}'$ be the stabilizer of L in $\widetilde{H}_A$, i.e. $\widetilde{K}' = \prod_p \widetilde{K}'_p \times \widetilde{H}_\infty$. Let $\widetilde{\tau}$ be a representation of $\widetilde{H}_\infty$ such that $\widetilde{\tau}|H_\infty = \tau$ and put $\widetilde{\rho}(k) = \widetilde{\tau}(k_\infty)$ for $k \in \widetilde{K}'$. Identifying the center of $\widetilde{H}_\infty$ with R^x, we have the canonical direct product decomposition $\widetilde{H}_\infty = R^x_+ \times H_\infty$. We assume that $\widetilde{\tau}$ is trivial on R^x_+-part. If $\widetilde{\varphi}$ is a function on $\widetilde{H}_A$ which satisfies

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2) Since H_{Q_p} is a unimodular group, we can take $\{h_j\}$ so that it forms a complete set of representatives for both left cosets and right cosets.

(1.21) $\widetilde{\varphi}(\gamma hk) = \widetilde{\varphi}(h)\widetilde{\rho}(k)$ for any $\gamma \in \widetilde{H}_Q$, $h \in \widetilde{H}_A$, $k \in \widetilde{K}'$,

we put

(1.22) $((\widetilde{K}_p' b \widetilde{K}_p')\widetilde{\varphi})(h) = \sum_j \widetilde{\varphi}(hh_j)$, $h \in \widetilde{H}_A$.

Remark 1.5. We can verify without difficulty that the map $m: \widetilde{K}_p' \cap b^{-1}\widetilde{K}_p' b \longrightarrow Z_p^{\times}$ is surjective for any $b \in \widetilde{H}_{Q_p}$ since $p \nmid N$. Hence, if $b \in H_{Q_p}$, the coset decompositions $K_p' b K_p' = \bigcup_j K_p' h_j = \bigcup_j h_j K_p'$ give rise coset decompositions $\widetilde{K}_p' b \widetilde{K}_p' = \bigcup_j \widetilde{K}_p' h_j = \bigcup_j h_j \widetilde{K}_p'$. Therefore we get $(\widetilde{K}_p' b \widetilde{K}_p') f_p = (K_p' b K_p') f_p$ and $((\widetilde{K}_p' b \widetilde{K}_p')\widetilde{\varphi})|H_A = (K_p' b K_p')\varphi$ if φ is the restriction of $\widetilde{\varphi}$ which satisfies (1.21).

In the succeeding sections, we shall prove the local relations of the following type.

(1.23) $f_p' = \sum_\ell c_\ell (\widetilde{K}_p' b_\ell \widetilde{K}_p') f_p$, $c_\ell \in C$, $b_\ell \in \widetilde{H}_{Q_p}$.

Theorem 1.6. *Let the notation and the assumptions be the same as above. Assume that u is even and that (1.23) holds with* $b_\ell \in H_{Q_p}$. *Then we have*

$$(\widetilde{K}_p' a \widetilde{K}_p')\widetilde{\Phi}_f^{\varphi} = \omega_0(p^{mu/2})\widetilde{\Phi}_f^{\varphi'} ,$$

where $\varphi' = \sum_\ell c_\ell (K_p' b_\ell K_p')\varphi$.

Proof. Let $K_p' b_\ell K_p' = \bigcup_k K_p' h_{\ell k} = \bigcup_k h_{\ell k} K_p'$ be coset decomposition

Put

$$\Psi_\ell(g) = \sum_k \int_{H_Q \backslash H_A} \left\langle \sum_{x \in X_Q} (\pi(g)f)(h_{\ell k} h^{-1} x),\ \varphi(h) \right\rangle dh,\quad g \in G_A.$$

By (1.17) and (1.23), we get

$$((\widetilde{K}_p a \widetilde{K}_p)\widetilde{\Phi}_f^{\varphi})(g) = \omega_0(p^{mu/2})\widetilde{\Phi}_{f'}^{\varphi}(g) = \omega_0(p^{mu/2})\sum_\ell c_\ell \Psi_\ell(g) \quad \text{for } g \in G_A$$

ince

$$\Psi_\ell(g) = \int_{H_Q\backslash H_A} \left\langle \sum_{x\in X_Q} (\pi(g)f)(h^{-1}x),\ \sum_k \varphi(hh_{\ell k})\right\rangle dh,$$

e obtain

$$((\tilde K_p a \tilde K_p)\Phi_f^{\varphi})(g) = \omega_o(p^{mu/2})\Phi_f^{\varphi'}(g) \quad \text{for} \quad g\in G_A.$$

s $\Phi_f^{\varphi'} \in A_\sigma^{(m)}(N,\omega)$, we get the conclusion by Lemma 1.1.

Theorem 1.7. Let the notation be the same as above. Assume that is odd and that (1.23) holds with $b_\ell \in \tilde H_{Q_p}$ such that $m(b_\ell) = p$. ssume furthermore that there exists a $\gamma \in \tilde H_Q$ such that $m(\gamma) = p$ nd that the map $m\colon \tilde K'_\ell \longrightarrow Z_\ell^\times$ is surjective for every prime ℓ. hen, if φ is the restriction of a function $\tilde\varphi$ which satisfies 1.21), we have

$$(\tilde K_p a \tilde K_p)\tilde\Phi_f^{\varphi} = p^{mn/2}\ \omega_o(p^{m(u-1)/2})\tilde\Phi_f^{\varphi'}$$

here $\tilde\varphi' = \sum_\ell c_\ell(\tilde K'_p b_\ell \tilde K'_p)\tilde\varphi$, $\varphi' = \tilde\varphi'|H_A$.

Proof. Let $\tilde K'_p b_\ell \tilde K'_p = \bigcup_k \tilde K'_p h_{\ell k} = \bigcup_k h_{\ell k}\tilde K'_p$ be coset decompositions ith $m(h_{\ell k}) = p$ for every k. Take any $g\in G_A$ such that $g_f = 1$ nd put $g' = \begin{pmatrix} p^{1/2}\cdot 1_m & 0 \\ 0 & p^{-1/2}\cdot 1_m\end{pmatrix}_\infty g$. By (1.18) and (1.23), we get

$$(\tilde K_p a \tilde K_p)\tilde\Phi_f^{\varphi})(g) = \omega_o(p^{m(u-1)/2})\,\Phi_f^{\varphi}(g') = \omega_o(p^{m(u-1)/2})\sum_\ell c_\ell\,\Psi_\ell(g'),$$

here

$$\Psi_\ell(g') = \int_{H_Q\backslash H_A} \sum_k \left\langle \sum_{x\in X_Q} (\pi(g')f)(h_{\ell k}h^{-1}x),\ \varphi(h)\right\rangle dh,$$

e have

$$\Psi_\ell(g') = \sum_k \int_{H_Q\backslash H_A} \left\langle \sum_{x\in X_Q} (\pi(g')f)(h_{\ell k}h^{-1}\gamma^{-1}x),\ \varphi(h)\right\rangle dh$$

$$= \sum_k \int_{H_Q\backslash \gamma H_A h_{\ell k}^{-1}} \left\langle \sum_{x\in X_Q} (\pi(g')f)(h^{-1}x),\ \varphi(\gamma^{-1}hh_{\ell k})\right\rangle dh.$$

Take $\delta \in \widetilde{H}_A$ so that $m(\delta)_v = p$ if $v \neq p$, $m(\delta)_p = 1$. Then we have $\gamma H_A h_{\ell k}^{-1} = H_A\delta$ for any ℓ and k. As

$$\Psi_\ell(g') = \int_{H_Q\backslash H_A\delta} \left\langle \sum_{x\in X_Q} (\pi(g')f)(h^{-1}x), \sum_k \widetilde{\varphi}(hh_{\ell k})\right\rangle dh,$$

we get

$$((\widetilde{K}_p a\widetilde{K}_p)\widetilde{\Phi}_f^{\varphi})(g) = \omega_o(p^{m(u-1)/2}) \int_{H_Q\backslash H_A\delta} \left\langle \sum_{x\in X_Q} (\pi(g')f)(h^{-1}x), \widetilde{\varphi}'(h)\right\rangle dh$$

By our assumption, we may assume $\delta_v \in \widetilde{K}'_v$ if v is a finite place of Q and $\delta_\infty = p^{1/2}\cdot 1_{2n}$. Then changing the variable h to $h\delta$, we see

$$((\widetilde{K}_p a\widetilde{K}_p)\widetilde{\Phi}_f^{\varphi})(g)$$
$$= \omega_o(p^{m(u-1)/2}) \int_{H_Q\backslash H_A} \left\langle \sum_{x\in X_Q} (\pi(g')f)(\delta_\infty^{-1}h^{-1}x), \varphi'(h)\right\rangle dh.$$

We have

$$(\pi_\infty(g'_\infty)f_\infty)(\delta_\infty^{-1}h_\infty^{-1}x) = \left(\pi_\infty\left(\begin{pmatrix} p^{1/2}\cdot 1_m & 0 \\ 0 & p^{-1/2}\cdot 1_m\end{pmatrix}_\infty g_\infty\right)f_\infty\right)(y)$$
$$= p^{mn/2}(\pi_\infty(g_\infty)f_\infty)(p^{1/2}y) = p^{mn/2}(\pi_\infty(g_\infty)f_\infty)(x),$$

where we put $y = p^{-1/2}x$. Therefore we obtain

$$((\widetilde{K}_p a\widetilde{K}_p)\widetilde{\Phi}_f^{\varphi})(g)$$
$$= p^{mn/2}\ \omega_o(p^{m(u-1)/2}) \int_{H_Q\backslash H_A} \left\langle \sum_{x\in X_Q} (\pi(g)f)(h^{-1}x),\ \varphi'(h)\right\rangle dh$$
$$= p^{mn/2}\ \omega_o(p^{m(u-1)/2})\Phi_f^{\varphi'}(g)$$

if $g\in G_A$, $g_f = 1$. Since $\Phi_f^{\varphi} \in A_\sigma^{(m)}(N,\omega)$, we get the formula of Theorem 1.7 by using $G_A = G_Q G_\infty K$ and Lemma 1.1.

Now we shall clarify the implication of the above Theorems for heta series in classical context. Let

$$(1.24) \qquad H_A = \bigcup_{i=1}^{c} H_Q h_i K'$$

e a double coset decomposition. We set $L_i = h_i L$, $e_i = |H_Q \cap h_i K' h_i^{-1}|$. et Ψ be as in Lemma 1.2 and put $f = \Psi(\Phi_f^{\varphi})$. Then $f \in \mathcal{G}(\Gamma_o^{(m)}(N), \omega_o)$ and we get (cf. [13], § 2)

$$(1.25) \quad f(z) = \sum_{i=1}^{c} \langle \sum_{x \in L_i^m} P(x)e(\sigma({}^t xSxz)), \ \varphi(h_i)/e_i \rangle .$$

Remark 1.8. (i) If $2 \mid n$ and $\det S \in (Q^x)^2$, then the map $: \widetilde{H}_Q \longrightarrow Q_+^x$ is surjective (cf. Eichler[4], Satz 23.6). In this ase, the γ required in Theorem 1.6 always exists.

ii) If L is a maximal lattice, then $m: \widetilde{K}_\ell' \longrightarrow Z_\ell^x$ is surjective or every ℓ (cf. Eichler[4], Satz 11.2).

iii) If the two conditions on L stated above hold, then the anonical map $H_Q \backslash H_A / K' \longrightarrow \widetilde{H}_Q \backslash \widetilde{H}_A / \widetilde{K}'$ is bijective (cf. [13], Lemma 3.5). n this case, φ always extends to $\widetilde{\varphi}$ as assumed in Theorem 1.7.

We assume that L satisfies two conditions stated in Remark 1.8. main theorem of Freitag[5](4.5) can be proved by Theorem 1.7 ombined with Theorem 2.1 which shall be proved in the next section. 'or the sake of simplicity, we shall consider the case of theta eries without spherical functions. Note that the lattices L_i $1 \leq i \leq c$) defined above make a complete set of representatives of lasses of lattices on V in the genus of L. Let $\phi_i: L_i \cong Z^{2n}$ e an isomorphism and define the symmetric matrix S_i by[3)]

3) We consider an element of Z^{2n} as a column vector.

(1.26) $\quad {}^t\phi_i(v)S_i\phi_i(v) = Q(v), \ v \in L_i.$

We put

(1.27) $$\vartheta_i^{(m)}(z) = \sum_{x \in L_i^m} e(\sigma({}^t xSxz)) = \sum_{x \in M_{2n,m}(Z)} e(\sigma({}^t xS_i xz)).$$

Let p be a prime number and set

(1.28) $$T(p)\vartheta_i^{(m)}(z) = \sum_{\ell=1}^{d} \vartheta_i^{(m)}(\gamma_\ell z)\det(c_\ell z + d_\ell)^{-n}, \quad z \in H_m,$$

where $\Gamma_o^{(m)}(N)\mathrm{diag}[1,\cdots,1,p,\cdots,p]\Gamma_o^{(m)}(N) = \bigcup_{\ell=1}^{d} \Gamma_o^{(m)}(N)\gamma_\ell$,

$\gamma_\ell = \begin{pmatrix} * & * \\ c_\ell & d_\ell \end{pmatrix}$.

Proposition 1.9. *The notation and the assumptions being as above, let c_m be the constant as in Theorem 2.1 with $q = p$. Assume $p \nmid N$. Then we have*

$$T(p)\vartheta_i^{(m)} = c_m^{-1} \sum_{j=1}^{c} (\alpha_{ij}(p)/e_j)\vartheta_j^{(m)},$$

with $\alpha_{ij}(p) = \left|\left\{X \in M_{2n}(Z) \mid {}^t XS_i X = pS_j\right\}\right|$.

Proof. Let Ψ be the map as in Lemmas 1.2 and 1.3 and φ be the characteristic function of $H_Q h_i K'$. Take $b \in \widetilde{H}_{Q_p} \cap M_{2n}(Z_p)$ such that $m(b) = p$; put $\widetilde{\varphi}' = (\widetilde{K}'_p b \widetilde{K}'_p)\widetilde{\varphi}$. We have (cf.(1.25)) $\Psi(\Phi_f^\varphi) = \vartheta_i^{(m)}/e_i$. Since $\omega = 1$, the action of the center of $\widetilde{G}_A$ on $\widetilde{\Phi}_f^\varphi$ is trivial. Take $a \in \widetilde{G}_{Q_p} \cap M_{2m}(Z_p)$ such that $m(a) = p$. Then, by Lemma 1.3, Theorems 1.7 and 2.1, we have

$$\Psi((\widetilde{K}_p a \widetilde{K}_p)\widetilde{\Phi}_f^\varphi) = p^{mn/2}T(p)\vartheta_i^{(m)}/e_i = p^{mn/2}c_m^{-1}\Psi(\widetilde{\Phi}_f^{\varphi'}).$$

Hence we get

(1.29) $$T(p)\vartheta_i^{(m)} = c_m^{-1}e_i \sum_{j=1}^{c} (\widetilde{\varphi}'(h_j)/e_j)\vartheta_j^{(m)}.$$

Therefore it is sufficient to show

(1.30) $\alpha_{ij}(p) = e_i \widetilde{\varphi}'(h_j)$.

Let $\widetilde{K}'_p b \widetilde{K}'_p = \bigcup_\ell b_\ell \widetilde{K}'_p$ be a coset decomposition. Put

$$B_{ij} = \{ b_\ell \mid h_j b_\ell \in \widetilde{H}_Q h_i \widetilde{K}' \} .$$

Then we have $\widetilde{\varphi}'(h_j) = |B_{ij}|$. Put

$$S_{ij} = \{ X \in M_{2n}(Z) \mid {}^t X S_i X = p S_j \} .$$

If $X \in S_{ij}$, we see easily that only 1 and p can appear among elementary divisors of X. Hence $S_{ij} \ni X \longrightarrow pX^{-1} \in S_{ji}$ is a bijection, i.e. $|S_{ij}| = |S_{ji}|$.

Now we are going to define a correspondence from B_{ij} to S_{ij}. Take $b_\ell \in B_{ij}$. Then $h_j b_\ell = \gamma h_i k$ with $\gamma \in \widetilde{H}_Q$, $k \in \widetilde{K}'$. Since h_i, $h_j \in H_A$ and $m(k) \in \prod_p Z_p^\times \times R_+^\times$, we get $m(\gamma) = p$. We have $h_j b_\ell L = \gamma L_i$, $h_j b_\ell L \subseteqq h_j L = L_j$. Let ι denote the inclusion map of $h_j b_\ell L$ into L_j. Then the map $v \longrightarrow \phi_j(\iota(\gamma \phi_i^{-1}(v)))$, $v \in Z^{2n}$ is a Z-linear map from Z^{2n} to Z^{2n}. We define $X \in M_{2n}(Z)$ by

(1.31) $Xv = \phi_j(\iota(\gamma\phi_i^{-1}(v)))$, $v \in Z^{2n}$.

For $v \in L_i$, we have $X\phi_i(v) = \phi_j(\iota(\gamma v)) = \phi_j(\gamma \iota(v))$. By (1.26), we get

$$ {}^t\phi_i(v) S_i \phi_i(v) = Q(v) = Q(\iota(v)) = p^{-1}Q(\iota(\gamma v))$$

$$= {}^t\phi_j(\iota(\gamma v)) p^{-1} S_j \phi_j(\iota(\gamma v)) = {}^t\phi_i(v) {}^t X p^{-1} S_j X \phi_i(v), \quad v \in L_i.$$

Therefore we get ${}^t X S_j X = p S_i$, i.e. $X \in S_{ji}$. If γ is given, X is uniquely determined by (1.31). When $b_\ell \in B_{ij}$ is given, we see that there are e_i choices of $\gamma \in \widetilde{H}_Q$ which satisfies $h_j b_\ell = \gamma h_i k$ with

$k \in \widetilde{K}'$. It is clear that if $h_j b_\ell = \gamma h_i k$, $h_j b_{\ell'} = \gamma h_i k'$ with $\gamma \in \widetilde{H}_Q$, $k, k' \in \widetilde{K}'$, then we have $b_\ell \widetilde{K}'_p = b_{\ell'} \widetilde{K}'_p$, i.e. $\ell = \ell'$. Thus the image of the "1 to e_i correspondence" $B_{ij} \ni b_\ell \longrightarrow X \in S_{ji}$ has the order $e_i |B_{ij}|$. We shall show that any $X \in S_{ji}$ can be obtained in this manner. Let $X \in S_{ji}$ and take $\gamma_1 \in \widetilde{H}_Q$ so that $m(\gamma_1) = p$. Then $\phi_j^{-1}(XZ^{2n})$ and $\gamma_1 L_i$ are isometric as lattices. Hence we can find $\gamma_2 \in H_Q$ such that $\gamma_2 \gamma_1 L_i = \phi_j^{-1}(XZ^{2n})$; put $\gamma = \gamma_2 \gamma_1$. For a prime number $\ell \neq p$, we have $(\gamma L_i)_\ell = (L_j)_\ell$; we have $(\gamma L_i)_p \subsetneqq (L_j)_p$. Hence we see that $\gamma h_i k = h_j b_\ell$ holds for some $k \in \widetilde{K}'$ and b_ℓ . Tracing back the definition, this shows that X belongs to the image of the correspondence defined above. Thus we have

$$e_i \widetilde{\varphi}'(h_j) = e_i |B_{ij}| = |S_{ji}| = |S_{ij}| = \alpha_{ij}(p).$$

This completes the proof.

Let $0 \leqq s \leqq m$ and set

$$(1.32) \qquad T^{(s)}(p^2)\vartheta_i^{(m)}(z) = \sum_{\ell=1}^{d} \vartheta_i^{(m)}(\gamma_\ell z)\det(c_\ell z + d_\ell)^{-n}, \quad z \in H_m,$$

where $\Gamma_o^{(m)}(N)\mathrm{diag}[\underbrace{1,\cdots,1}_{m-s},\underbrace{p,\cdots,p}_{s},\underbrace{p^2,\cdots,p^2}_{m-s},\underbrace{p,\cdots,p}_{s}]\Gamma_o^{(m)}(N)$

$$= \bigcup_{\ell=1}^{d} \Gamma_o^{(m)}(N)\gamma_\ell, \quad \gamma_\ell = \begin{pmatrix} * & * \\ c_\ell & d_\ell \end{pmatrix}.$$

Assume $m = n$. If we combine Theorem 1.6 with Theorem 3.7, we get the following generalization of Freitag's theorem for the Hecke operator $T^{(s)}(p^2)$.

Proposition 1.10. <u>Let the notation and the assumptions be the same as in Proposition</u> 1.9. <u>Set</u>

$$\alpha_{ij}^{(s)}(p^2) = \left|\left\{ X \in M_{2n}(Z) \;\middle|\; {}^tXS_iX = p^2S_j, \text{ the elementary divisors of } X \text{ are } \underbrace{1,\cdots,1}_{m-s},\underbrace{p,\cdots,p}_{2s},\underbrace{p^2,\cdots,p^2}_{m-s} \right\}\right|.$$

If $m = n$, *we have*

$$T^{(s)}(p^2)\vartheta_i^{(m)} = p^{-m^2}\sum_{j=1}^{c} \frac{p^{m-s}\alpha_{ij}^{(s)}(p^2) + p^{m-s-1}(p^{s+1}-1)\alpha_{ij}^{(s+1)}(p^2)}{e_j}\vartheta_j^{(m)}.$$

The proof is omitted since it is quite similar to that of Proposition 1.9.

Remark 1.11. A similar result also holds when $m \neq n$. To explain this, we use the notation of §3; there we shall prove the local relation (1.23) written in the form

$$A_m^{(i)}(x) = \sum_{\ell} c_{i\ell}(m,n)B_n^{(\ell)}(x), \quad x \in X,$$

where the coefficients $c_{i\ell}(m,n)$'s are explicitly computable. Then we have

$$T^{(s)}(p^2)\vartheta_i^{(m)} = p^{-mn}\sum_{j=1}^{c} \Big(\sum_{\ell} c_{i\ell}(m,n)\alpha_{ij}^{(\ell)}(p^2)\Big)(\vartheta_j^{(m)}/e_j) .$$

Numerical example 1.12. Let D be a defite quaternion algebra over Q which does not ramify except at 3 and ∞. Let R be a maximal order of D. When a suitable isomorphism $\phi : R \cong Z^4$ is fixed, we have

$$N(x) = {}^t\phi(x)S\phi(x) \text{ for } x \in R \text{ with } S = \begin{pmatrix} 1 & 0 & 3/2 & 0 \\ 0 & 1 & 0 & 3/2 \\ 3/2 & 0 & 3 & 0 \\ 0 & 3/2 & 0 & 3 \end{pmatrix},$$

where $N(x)$ denotes the reduced norm of $x \in R$. For this symmetric matrix S, we have $c = 1$. Put

$$\vartheta(z) = \sum_{x \in M_{4,2}(Z)} e(\sigma({}^t xSxz)), \quad z \in H_2 ,$$

$$e = \left| \left\{ X \in M_4(Z) \;\middle|\; {}^t XSX = S \right\} \right| ,$$

$$\alpha(p) = \left| \left\{ X \in M_4(Z) \;\middle|\; {}^t XSX = pS \right\} \right| ,$$

$$\alpha^{(s)}(p^2) = \left| \left\{ X \in M_4(Z) \;\middle|\; {}^t XSX = p^2 S, \text{ the elementary divisors of } X \text{ are } \underbrace{1,\cdots,1}_{2-s},\underbrace{p,\cdots,p}_{2s},\underbrace{p^2,\cdots,p^2}_{2-s} \right\} \right| ,$$

where p is a prime and $0 \leqq s \leqq 2$.

Since $|R^x| = 12$, we easily get $e = 2 \times 12^2 = 288$. We have, when $p \neq 3$,

$$T(p)\vartheta = 2p^{-1}(p+1)\vartheta ,$$

$$T^{(1)}(p^2)\vartheta = p^{-4}\left\{(p^2-1) + p(p+1)^2\right\}\vartheta ,$$

$$T^{(2)}(p^2)\vartheta = p^{-4}\left\{2p^3(p+1) + (p+1)^2 p(p-1)\right\}\vartheta .$$

(For the proof of these facts, use Lemma 1.3, Theorems 1.6, 1.7, 2.1 and 3.7. The first two of the formulas, in the case p is odd, are nothing but Theorem 6.1 of [12]). Hence, by Propositions 1.9 and 1.10, we get

$$\alpha(p)/288 = 2(p+1),$$

$$\alpha^{(2)}(p^2)/288 = 1,$$

$$\alpha^{(1)}(p^2)/288 = (p+1)^2,$$

$$\alpha^{(0)}(p^2)/288 = 2p(p+1),$$

for a prime number $p \neq 3$.

§ 2. Local relations for T(p)

Let k be a non-archimedean local field whose characteristic is not 2, O be the ring of integers and ϖ be a prime element of k. Put $q = |O/\varpi O|$. Let $G = Sp(m)$, $\tilde{G} = GSp(m)$, and H (resp. $\tilde{H}$) denote the orthogonal group (resp. the group of orthogonal similitudes) with respect to a symmetric matrix $S \in M_{2n}(k)$, det $S \neq 0$. Set $V = M_{2n,1}(k)$ and define a quadratic form Q on V by $Q(x) = {}^t xSx$, $x \in V$. Let $| \ |$ denote the normalized absolute value of k and ψ be a non-trivial additive character of k. Let π be the Weil representation of G_k realized on $S(M_{2n,m}(k))$. For the sake of completeness, let us recall that π is characterized by the following properties.

$$(2.1)\quad \pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\Phi(x) = \psi(\sigma(b\,{}^t xSx))\Phi(x),$$

$$(2.2)\quad \pi\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}\right)\Phi(x) = \omega(\det a)\,|\det a|^n\,\Phi(xa),$$

$$(2.3)\quad \pi\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\Phi(x) = \gamma\,\Phi^*(x)\ .$$

Here $\Phi \in S(M_{2n,m}(k))$; ω is the trivial character if $(-1)^n \det S \in (k^\times)^2$ and is the character of $k^\times$ which corresponds to the quadratic extension $k(\sqrt{(-1)^n \det S})$ if $(-1)^n \det S \notin (k^\times)^2$; γ is a complex number of absolute value 1 which depends only on S, ψ and m; Φ^* is the Fourier transform of Φ with respect to the self dual measure on $M_{2n,m}(k)$.

Hereafter we assume that $S \in M_{2n}(O)$, $\det S \in O^\times$ and that ψ is trivial on O and is non-trivial on $\varpi^{-1}O$. Furthermore we assume that the residual characteristic of k is not 2 (cf. Remark 2.7 when q is even). Put

$$S_1 = \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix}, \quad S_\gamma = \begin{pmatrix} 0_{n-1} & 0 & 1_{n-1} \\ 0 & \begin{matrix} 1 & 0 \\ 0 & -\gamma \end{matrix} & 0 \\ 1_{n-1} & 0 & 0 \end{pmatrix}$$

with a fixed $\gamma \in O^x - (O^x)^2$. As well known, we can find $X \in GL_{2n}(O)$ such that ${}^tXSX = S_1$ or S_γ. For our purpose, we shall lose no generality by assuming $S = S_1$ or S_γ. If $S = S_1$, we shall call H "split type"; otherwise "non-split type". We put $\varepsilon = \omega(\varpi)$. Then we have $\varepsilon = 1$(resp. -1) if H is of split type(resp. non-split type).

We identify $M_{2n,m}(k)$ with V^m and denote it by X. Set $L = M_{2n,1}(O)$, which is a lattice on V. For $g \in \widetilde{G}_k$(resp. $h \in \widetilde{H}_k$), let $m(g)$(resp. $m(h)$) denote the multiplicator of g(resp. h). Put

$$\widetilde{G}_O = \left\{ g \in \widetilde{G}_k \cap M_{2m}(O) \mid m(g) \in O^x \right\} \quad (= GSp(m)_O),$$

$$\widetilde{H}_O = \left\{ h \in \widetilde{H}_k \cap M_{2n}(O) \mid m(h) \in O^x \right\} .$$

Clearly $\widetilde{H}_O$ is the stabilizer of L in $\widetilde{H}_k$. Put $G_O = G_k \cap \widetilde{G}_O$ $(= Sp(m)_O)$ $H_O = H_k \cap \widetilde{H}_O$. For a positive integer N, put

$$T_{\widetilde{G}}(\varpi^N) = \left\{ g \in \widetilde{G}_k \cap M_{2m}(O) \mid m(g) \in \varpi^N O^x \right\} ,$$

$$T_{\widetilde{H}}(\varpi^N) = \left\{ h \in \widetilde{H}_k \cap M_{2n}(O) \mid m(h) \in \varpi^N O^x \right\} .$$

Let $T_{\widetilde{G}}(\varpi) = \bigcup_i g_i \widetilde{G}_O$, $m(g_i) = \varpi$ be a right coset decomposition and set

(2.4) $\qquad A_m(x) = \sum_i (\pi(\nu g_i)f)(x), \quad x \in X,$

where $\nu = \begin{pmatrix} 1_m & 0 \\ 0 & \varpi^{-1} \cdot 1_m \end{pmatrix} \in \widetilde{G}_k$. Similarly let $T_{\widetilde{H}}(\varpi) = \bigcup_j \widetilde{H}_O h_j$ be a left coset decomposition and set

$$(2.5) \qquad B_m(x) = \sum_j f(h_j x), \; x \in X.$$ [4]

Our purpose in this section is to prove the following Theorem which gives the local relation (1.23) for the double coset $T_{\widetilde{G}}(\varpi)$.

Theorem 2.1. (a) _If_ H _is of split type, we have_ $B_m(x) = c_m A_m(x)$, $x \in X$ _with_

$$c_m = \begin{cases} 2q^{(-m^2-m+2mn)/2} \prod_{\ell=1}^{n-m-1} (q^\ell + 1) & \text{if } m+1 < n, \\ 2q^{(n^2-n)/2} & \text{if } m+1 = n, \\ q^{(n^2-n)/2} & \text{if } m = n, \\ q^{(n^2-n)/2} \prod_{\ell=1}^{m-n} (q^\ell + 1)^{-1} & \text{if } m > n. \end{cases}$$

(b) _If_ H _is of non-split type, we have_ $A_m(x) = 0$, $x \in X$ _for_ $n \leq m$.

Before proceeding to the proof, we shall make several preliminary considerations on the nature of A_m and B_m. First we shall give an explicit expression of A_m. By the Iwasawa decomposition, representatives $\{g_i\}$ of right $\widetilde{G}_0$-cosets in $T_{\widetilde{G}}(\varpi^N)$ can be taken in the form $\begin{pmatrix} A & B \\ 0 & \varpi^N \, {}^tA^{-1} \end{pmatrix}$. Then it is easy to see that we can choose $\{g_i\}$ explicitly in the following way. For $A \in GL_m(k)$ and $B, B' \in M_m(O)$, we write $B \equiv B' \bmod A$ if and only if $A^{-1}(B - B') \in M_m(O)$. For non-negative integers $\alpha_1, \cdots, \alpha_m$, let $R(\alpha_1, \cdots, \alpha_m)$

4) When H is of non-split type, $T_{\widetilde{H}}(\varpi) = \phi$ and B_m is not defined.

(resp. $L(\alpha_1,\cdots,\alpha_m)$) denote a complete set of representatives of right(resp. left) $GL_m(O)$-cosets in

$$GL_m(O)\ \mathrm{diag}[\varpi^{\alpha_1}, \varpi^{\alpha_1+\alpha_2}, \ldots, \varpi^{\alpha_1+\alpha_2+\cdots+\alpha_m}]\ GL_m(O).$$

If we let A run over $R(\alpha_1,\cdots,\alpha_m)$, $\alpha_1+\alpha_2+\cdots+\alpha_m \leqq N$ and let B run over a complete set of representatives of $B\in M_m(O)$ such that $A^{-1}B$ is symmetric under the equivalence relation $\equiv \bmod A$, then $\left\{\begin{pmatrix} A & B \\ 0 & \varpi^N\, {}^tA^{-1}\end{pmatrix}\right\}$ give the desired set of representatives.

Therefore, by (2.1) and (2.2), we get

$$(2.6)\qquad A_m(x) = \sum_{A} \sum_{B \bmod A} \psi(\sigma(B\,{}^tA\,{}^txSx)) \times f(xA) \times \omega(\det A) \times |\det A|^n ,$$

where A extends over $R(\alpha_1,\cdots,\alpha_m)$, $\alpha_i \geqq 0 (1\leqq i\leqq m)$, $\alpha_1+\alpha_2+\cdots+\alpha_m \leqq 1$.

Lemma 2.2. $A_m(x)$ <u>and</u> $B_m(x)$ <u>are invariant under the transformation</u> $x \longrightarrow hxk + t$, <u>where</u> $h\in H_O$, $k\in GL_m(O)$, $t\in L^m$.

Proof. The assertion for $B_m(x)$ is obvious by the definition (2.5); we shall prove the assertion for $A_m(x)$. Take any $\Phi \in S(X)$, $g\in G_k$, $h\in H_k$ and put $\Psi(x) = \Phi(hx)$, $x\in X$. Then we get easily $(\pi(g)\Phi)(hx) = (\pi(g)\Psi)(x)$, $x\in X$ by using (2.1) $\sim$ (2.3) (i.e. the actions of H_k and of G_k on $S(X)$ commute). Hence $A_m(x)$ is invariant under $x \longrightarrow hx$, $h\in H_O$. By (2.1) $\sim$ (2.3), we see easily that $\pi(g)f = f$ for $g\in G_O$ (cf.[13], Lemma 2.1). Hence we have

$$(2.7)\qquad \pi(g)A_m = A_m \quad \text{for any } g\in \nu G_O \nu^{-1}$$

by (2.4). Taking $g = \begin{pmatrix} k & 0 \\ 0 & {}^tk^{-1}\end{pmatrix}$, $k\in GL_m(O)$, we get $A_m(xk) = A_m(x)$

$x \in X$. To show the invariance of A_m under $x \longrightarrow x + t$ for $t \in L^m$, we use (2.6). It suffices to prove $\psi(\sigma(B^t A^t xSx))f(xA)$ is invariant under $x \longrightarrow x + t$. We may assume $xA \in L^m$. Then we have $\sigma(B^t A^t(x + t)S(x + t)) - \sigma(B^t A^t xSx) \in O$; hence we get $\psi(\sigma(B^t A^t(x + t)S(x + t)) = \psi(\sigma(B^t A^t xSx))$; it is clear that $f(xA) = f((x + t)A)$. This completes the proof.

Lemma 2.3. (a) <u>If</u> $x \notin (\varpi^{-1}L)^m$, <u>we have</u> $A_m(x) = B_m(x) = 0$. (b) <u>Assume</u> $x \in (\varpi^{-1}L)^m$. <u>If</u> ${}^t(\varpi x)S(\varpi x) \not\equiv 0 \bmod \varpi$, <u>we have</u> $A_m(x) = B_m(x) = 0$.

Proof. Assume $B_m(x) \neq 0$. Then, for some $b \in \widetilde{H}_k \cap M_{2n}(O)$ such that $m(b) = \varpi$, we have $bx \in L^m$. Since $b^{-1} = \varpi^{-1} S^{-1} {}^t bS \in \varpi^{-1} M_{2n}(O)$, we get $x \in (\varpi^{-1}L)^m$. Since we have ${}^t(b\varpi x)S(b\varpi x) = \varpi({}^t\varpi x)S(\varpi x) \equiv 0 \bmod \varpi^2$, we get ${}^t(\varpi x)S(\varpi x) \equiv 0 \bmod \varpi$.

Next assume $A_m(x) \neq 0$. By (2.6), we have $xA \in L^m$ for some A. Then we can easily get $x \in (\varpi^{-1}L)^m$. By (2.1) and (2.7), we have

$$\psi(\sigma(\varpi b^t xSx))A_m(x) = A_m(x)$$

for any $b \in M_m(O)$ which is symmetric. Hence we get ${}^t xSx \in \varpi^{-1} M_m(O)$, i.e. ${}^t(\varpi x)S(\varpi x) \equiv 0 \bmod \varpi$. This completes the proof.

Hereafter we shall write $x = (x_1, \cdots, x_m)$ with $x_i \in V (1 \leqq i \leqq m)$.

Lemma 2.4. <u>We assume</u> $x_m = 0$, $m \geqq 2$ <u>and put</u> $x' = (x_1, \cdots, x_{m-1})$. <u>Then we have</u> $A_m(x) = (1 + \varepsilon q^{m-n})A_{m-1}(x')$.

Proof. By the Iwasawa decomposition, we may set $A = \begin{pmatrix} A_1 & 0 \\ c_1 & a \end{pmatrix}$ with $A_1 \in M_{m-1}(O)$, $a \in O$. We have $A^{-1} = \begin{pmatrix} A_1^{-1} & 0 \\ -a^{-1}c_1A_1^{-1} & a^{-1} \end{pmatrix}$,

$$\begin{pmatrix} A_1 & 0 \\ c_1 & a \end{pmatrix}^{-1} \begin{pmatrix} A_1 & 0 \\ c_1' & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-1}(c_1'-c_1) & 1 \end{pmatrix} .$$

Since $\varpi A^{-1} \in M_m(O)$, we may take either $a = 1$ or $a = \varpi$. If $a = 1$, we may assume $c_1 = 0$. If $a = \varpi$, there are $q^{\operatorname{rank}(A_1 \bmod \varpi)}$ choices of c_1. We put $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$ with $B_{11} \in M_{m-1}(O)$.

By (2.6), we get

$$(2.8) \qquad A_m(x) = \sum_{A} \sum_{B \bmod A} \psi(\sigma(B_{11}{}^tA_1{}^tx'Sx')) \times f((x'A_1,0)) \times \omega(\det A_1) \times |\det A_1|^n \times \omega(a) \times |a|^n .$$

First assume $a = 1$. Then we see immediately that $B \equiv \begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix} \bmod A$

Therefore there is a one to one correspondence between equivalence classes $B_{11} \bmod A_1$ and $B \bmod A$. Next assume $a = \varpi$. We have

$$A^{-1}B = \begin{pmatrix} A_1^{-1}B_{11} & A_1^{-1}B_{12} \\ -\varpi^{-1}c_1A_1^{-1}B_{11} + \varpi^{-1}B_{21} & -\varpi^{-1}c_1A_1^{-1}B_{12} + \varpi^{-1}B_{22} \end{pmatrix} .$$

We must have $\varpi A_1^{-1}B_{12} = {}^t(-c_1A_1^{-1}B_{11} + B_{21})$. Put $x = -c_1A_1^{-1}B_{11} + B_{21}$. Then ${}^tx$ must belong to $A_1^{-1}\varpi M_{m-1,1}(O)$. Hence there are $q^{m-1-\operatorname{rank}(A_1 \bmod \varpi)}$ choices for B_{21}. Clearly there are q choices for B_{22}. Summing up, when $B_{11} \bmod A_1$ is given, there are $q^{\operatorname{rank}(A_1 \bmod \varpi)} \times q^{m-1-\operatorname{rank}(A_1 \bmod \varpi)} \times q = q^m$ possibilities for $B \bmod A$. Therefore, by (2.8), we get

$$A_m(x) = \sum_{A_1} \sum_{B_{11} \bmod A_1} \psi(\sigma(B_{11}{}^tA_1{}^tx'Sx')) \times f((x'A_1,0)) \times \omega(\det A_1) \times |\det A_1|^n \times (1 + \varepsilon q^{m-n}),$$

where A_1 extends over $R(\alpha_1, \cdots, \alpha_{m-1})$, $\alpha_i \geqq 0 (1 \leqq i \leqq m-1)$, $\alpha_1 + \cdots + \alpha_{m-1} \leqq 1$. This proves our Lemma.

Lemma 2.5. *We assume* $x \in (\varpi^{-1}L)^m$ *and* ${}^t(\varpi x)S(\varpi x) \equiv 0 \bmod \varpi$. *For* $k \in GL_m(O)$, *put* $xk = (y_1, \cdots, y_m)$, $y_i \in V (1 \leqq i \leqq m)$ *and assume* $y_m \notin L$ *for any* k. *Then* $A_m(x) = \varepsilon^m q^{(m^2+m-2mn)/2}$.

Proof. We use (2.6). Suppose that $xA \in L^m$ for some $A \in R(\alpha_1, \cdots, \alpha_m)$, $\alpha_i \geqq 0 (1 \leqq i \leqq m)$, $\alpha_1 + \cdots + \alpha_m \leqq 1$. Put $A = (a_{ij})$. We have $x_1 a_{i1} + \cdots + x_m a_{im} \in L$. If $\alpha_1 = 0$, there are some i, j such that $a_{ij} \not\equiv 0 \bmod \varpi$; so we can find $k = (k_{ij}) \in GL_m(O)$ such that $k_{m\ell} = a_{i\ell}$, $1 \leqq \ell \leqq m$. Then, for $y = (y_1, \cdots, y_m) = xk$, we have $y_m \in L$. This is a contradiction. Therefore $xA \in L^m$ if and only if $A \in R(1,0, \quad ,0)$. In this case, we may assume $A = \mathrm{diag}[\varpi, \cdots, \varpi]$. We have

$$A_m(x) = \varepsilon^m q^{-mn} \sum_{B \bmod A} \psi(\sigma(B\,{}^tA\,{}^txSx)).$$

Since ${}^txSx \in \varpi^{-1}M_m(O)$, we have $\psi(\sigma(B\,{}^tA\,{}^txSx)) = 1$. By definition, there are $q^{m(m+1)/2}$-equivalence classes for B mod A. This completes the proof.

Lemma 2.6. *Let the assumptions on* x *be the same as in Lemma 2.5. We assume further that* H *is of split type. Then*

$$B_m(x) = \begin{cases} 1 & \text{if } m = n, \\ 2 & \text{if } m+1 = n, \\ 2 \prod_{\ell=1}^{n-m-1} (q^\ell + 1) & \text{if } m+1 < n. \end{cases}$$

Proof. Put $\overline{V} = L/\varpi L$ and $\overline{Q}(x \bmod \varpi L) = Q(x) \bmod \varpi$ for $x \in L$. Then $(\overline{V}, \overline{Q})$ defines a regular quadratic space over $F_q = O/\varpi O$. Let $\overline{H}$ denote the group of F_q-rational points of the orthogonal group associated with $(\overline{V}, \overline{Q})$. With a suitable basis of $\overline{V}$, we may assume $\overline{Q}(x) = {}^t x\overline{S}x$, $x \in \overline{V}$, $\overline{S} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in M_{2n}(F_q)$. Put $\overline{x}_i = \varpi x_i \bmod \varpi L \in \overline{V}$, $\overline{x} = (\overline{x}_1, \cdots, \overline{x}_m)$. Our assumptions on x imply that $\overline{x}_1, \cdots, \overline{x}_m$ are linearly independent over F_q and that ${}^t\overline{x}\overline{S}\overline{x} = ($ Note that $m \leq n$ must be satisfied for the existence of such $\overline{x}$.

Let $v \in \overline{V}$ be a non-zero isotropic vector. Then it can easily be shown that $\overline{h}v = {}^t(1 \ 0 \cdots 0)$ for some $\overline{h} \in \overline{H}$. By induction on m, we can find $\overline{h} \in H$ and $\overline{k} \in GL_m(F_q)$ so that $\overline{y} = (\overline{y}_1, \cdots, \overline{y}_m) = \overline{h}\overline{x}\overline{k}$ satisfies $\overline{y}_1 = {}^t(1 \ 0 \cdots 0)$, $\overline{y}_2 = {}^t(0 \ 1 \ 0 \cdots 0)$, $\cdots$, $\overline{y}_m =$ ${}^t(0 \cdots 0 \ 1 \ 0 \cdots 0)$; put $z_1 = {}^t(\varpi^{-1} \ 0 \cdots 0)$, $z_2 = {}^t(0 \ \varpi^{-1} \ 0 \cdots 0)$, $\cdots$, $z_m = {}^t(0 \cdots 0 \ \varpi^{-1} \ 0 \cdots 0)$. Here the j-th coordinate of y_i(resp. z_i) is 1(resp. ϖ^{-1}) if $j = i$ and 0 if $j \neq i$. Since the reduction map

$$\varphi : H_0 \longrightarrow \overline{H}, \quad \varphi(h) = h \bmod \varpi$$

is surjective, there exist $h \in H_0$, $k \in GL_m(O)$ and $t \in L^m$ such that $z = hxk + t$, $z = (z_1, \cdots, z_m)$. Hence, by Lemma 2.2, we may assume $x_i = z_i (1 \leq i \leq m)$.

Put $\xi = \begin{pmatrix} 1_n & 0 \\ 0 & \varpi \cdot 1_n \end{pmatrix}$. As is well known, $T_{\tilde{H}}(\varpi) = \tilde{H}_0 \xi \tilde{H}_0$. A left coset decomposition $\tilde{H}_0 = \bigcup_j (\tilde{H}_0 \cap \xi^{-1}\tilde{H}_0 \xi)\alpha_j$ gives rise the coset decomposition $\widetilde{H}_0 \xi \widetilde{H}_0 = \bigcup_j \widetilde{H}_0 \xi \alpha_j$. The canonical map $H_0 \cap \xi^{-1} H_0 \xi \backslash H_0 \longrightarrow \widetilde{H}_0 \cap \xi^{-1} \widetilde{H}_0 \xi \backslash \widetilde{H}_0$ is bijective. Put

$$B = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \overline{H} \;\middle|\; a, b, c, d \in M_n(F_q),\ b = 0 \right\}.$$

Then the reduction map φ induces a bijection $H_0 \cap \xi^{-1} H_0 \xi \backslash H_0 \longrightarrow B \backslash \overline{H}$. By definition, we see the following: For $\alpha = (\alpha_{ij}) \in H_0$, $\xi \alpha x \in L^m$ if and only if $\alpha_{ij} \equiv 0 \bmod \varpi$ for $1 \leqq i \leqq n$, $1 \leqq j \leqq m$. Therefore we have

(2.9) $\quad B_m(x) = |F| / |B|$, where

(2.10) $\quad F = \left\{ g = (g_{ij}) \in \overline{H} \;\middle|\; g_{ij} = 0,\ 1 \leqq i \leqq n,\ 1 \leqq j \leqq m \right\}.$

Now we are going to compute $|F|$. For $x, y \in \overline{V}$, put $\langle x, y \rangle = {}^t\overline{x} S y$. We write $g \in F$ as $g = (x_1, \cdots, x_{2n})$ with column vectors $x_i \in V$. Then $g \in \overline{H}$ if and only if $\langle x_i, x_j \rangle = 0$ for $|i - j| \neq n$, $\langle x_i, x_{i+n} \rangle = 1$ for $1 \leqq i \leqq n$. First we choose $x_1, \cdots, x_m$ so that they are linearly independent and that all the first n-coordinates of $x_1, \cdots, x_m$ are 0. The number of choices of such vectors is equal to $(q^n - 1)(q^n - q) \cdots (q^n - q^{m-1})$. Let F be the number of vectors $x_{1+n}, \cdots, x_{m+n}$ which satisfy

$$(2.11) \qquad \begin{aligned} \langle x_{i+n}, x_j \rangle &= \delta_{ij}, \quad 1 \leqq i, j \leqq m, \\ \langle x_{i+n}, x_{j+n} \rangle &= 0, \quad 1 \leqq i, j \leqq m. \end{aligned}$$

We have

(2.12) $\quad |F| = (q^n - 1)(q^n - q) \cdots (q^n - q^{m-1})\, F \times |O_1(2n-2m, F_q)|$,

assuming F is independent of the choice of $x_1, \cdots, x_m$. To compute F, choose a basis $u_1, \cdots, u_{2n}$ of $\overline{V}$ so that $u_i = x_i (1 \leqq i \leqq m)$,

$\langle u_i, u_j \rangle = 0$ if $|i - j| \neq n$, $\langle u_i, u_{i+n} \rangle = 1$ $(1 \leqq i \leqq n)$. Let $\overline{W}$ be the subspace of $\overline{V}$ spanned by $u_{m+1}, \cdots, u_n, u_{n+m+1}, \cdots, u_{2n}$. Put

$x_{1+n} = \alpha_1 u_1 + \beta_1 u_{1+n} + \alpha_2 u_2 + \beta_2 u_{2+n} + \cdots + \alpha_m u_m + \beta_m u_{m+n} + w, w \in \overline{W}$

From $\langle x_{1+n}, u_i \rangle = 0$ for $2 \leqq i \leqq m$, we get $\beta_2 = \beta_3 = \cdots = \beta_m = 0$; from $\langle x_{1+n}, u_1 \rangle = 1$, we get $\beta_1 = 1$; from $\langle x_{1+n}, x_{1+n} \rangle = 0$, we get $\alpha_1 + \overline{Q}(w) = 0$. Thus

$x_{1+n} = \alpha_1 u_1 + u_{1+n} + \alpha_2 u_2 + \cdots + \alpha_m u_m + w$, $w \in \overline{W}$, $\alpha_1 + \overline{Q}(w) = 0$,

is the condition posed on x_{1+n} by (2.11). Therefore the number of choices of x_{1+n} is $q^{2n-2m} \times q^{m-1}$. Repeating this procedure, we get

$$F = (q^{2n-2m} \times q^{m-1}) \times (q^{2n-2m} \times q^{m-2}) \times \cdots \times (q^{2n-2m} \times q^{0}).$$

Therefore we obtain

$$(2.13) \quad |F| = (q^n - 1)(q^{n-1} - 1) \cdots (q^{n-m+1} - 1)\, q^{m(2n-m-1)} \times |O_1(2n-2m, F_q)| .$$

Since $|B| = q^{n(n-1)}(q^n - 1)(q^{n-1} - 1) \cdots (q - 1)$ and $|O_1(2n-2m, F_q)| = 2(q^{n-m} - 1)(q^{2n-2m-2} - 1) \cdots (q^2 - 1)\, q^{(n-m)(n-m-1)}$ (we understand $|O_1(2n-2m, F_q)| = 1$ if $m = n$), we obtain the formula of our Lemma by (2.9) and (2.10).

Proof of Theorem 2.1. By Lemma 2.3, we may assume $x \in (\varpi^{-1}L)^m$, ${}^t(\varpi x)S(\varpi x) \equiv 0 \mod \varpi$. Put $xk = (y_1, \cdots, y_m)$ for $k \in GL_m(O)$. Only two cases can occur.

Case (I) $y_m \notin L$ for any $k \in GL_m(O)$.

Case (II) $y_m \in L$ for some $k \in GL_m(O)$.

Proof of (a). For Case (I), $m \leqq n$ must hold and we get $B_m(x) = c_m A_m(x)$ by Lemmas 2.5 and 2.6. Suppose that we are in Case (II). By Lemma 2.2, we may assume $x_m = 0$. When $m = 1$, we get $A_1(0) = 1 + q^{1-n}$ by (2.6). By similar considerations as in the proof of Lemma 2.6, we immediately get $B_1(0) = |O_1(2n, F_q)| / |B|$. Thus we obtain $B_1(0) = c_1 A_1(0)$. Now we shall proceed by induction on m. By lemma 2.4, we have $A_m(x) = (1 + q^{m-n}) A_{m-1}(x')$; $B_m(x) = B_{m-1}(x')$ is clear by definition. Therefore the assertion for the case m follows from the fact $c_{m-1}/c_m = 1 + q^{m-n}$ and the inductive hypothesis for the case $m - 1$.

Proof of (b). Since we have assumed $m \geqq n$, the vectors $\overline{x_1} = \varpi x_1 \bmod \varpi L, \cdots, \overline{x_m} = \varpi x_m \bmod \varpi L$ are linearly dependent over F_q. Thus Case (I) cannot occur. Suppose $n \geqq 2$. Then, by Lemma 2.4, we get $A_m(x) = 0$ for $m = n$. Therefore we obtain $A_m(x) = 0$ for $m > n$ again by Lemma 2.4. Suppose $n = 1$. If $m = 1$, we immediately get $A_m(x) = 0$ by (2.6). Then $A_m(x) = 0$ for $m > 1$ follows from Lemma 2.4.

Remark 2.7. Let us consider the case where the residual characteristic of k is 2. We assume

(A1) $L = O^{2n}$ is an integral lattice.

Put

$$B(x,y) = Q(x + y) - Q(x) - Q(y) = 2\,{}^t x S y, \quad x, y \in V,$$

$$\check{L} = \left\{ x \in V \mid B(x,y) \in O \text{ for all } y \in L \right\} .$$

We assume

(A2) $\check{L} = L$.

By (A1) and (A2), we have $\pi(g)f = f$ for $g \in G_O$, where f is the characteristic function of L^m (cf. [13], Lemma 2.1). In particular,

$k(\sqrt{(-1)^n \det S})$ is unramified over k. Put $S = (s_{ij})$; we see that $s_{ii} \in O$ and $2s_{ij} \in O$ when $i \neq j$. We can also show $\det 2S \in O^{\times}$.

Now Theorem 2.1 holds true without any change. The proof follows along the same line as before though we must be more careful in this case; we shall therefore indicate briefly the places where the proof must be modified.

Lemma 2.3 (b) must be changed to "If $A_m(x) \neq 0$ or $B_m(x) \neq 0$ then

(2.14) the diagonal (resp. 2 x non-diagonal) components of

$${}^t(\varpi x)S(\varpi x) \in \varpi O \text{ ."}$$

For the assumption in Lemmas 2.5 and 2.6, we assume (2.14) instead of ${}^t(\varpi x)S(\varpi x) \equiv 0 \bmod \varpi$.

Concerning the proof of Lemma 2.6, we must be very careful since we are dealing with the quadratic space $(\overline{V}, \overline{Q})$ over a finite field of characteristic 2. We shall only indicate the following point. By choosing suitable basis of $\overline{V}$, we may assume that $\overline{Q}$ has one of the following forms which are distinguished each other by the Arf invariant (cf. Dieudonné [3], p.34).

(I) $\overline{Q}(\xi) = \xi_1\xi_{1+n} + \xi_2\xi_{2+n} + \cdots + \xi_n\xi_{2n}$,

(II) $\overline{Q}(\xi) = \xi_1\xi_{1+n} + \xi_2\xi_{2+n} + \cdots + \xi_{n-1}\xi_{2n-1} + (\alpha\xi_n^2 + \xi_n\xi_{2n} + \alpha\xi_{2n}^2)$,

where $\xi = (\xi_1, \cdots, \xi_{2n}) \in \overline{V}$ and $\alpha \in F_q$ does not belong to the image of the map $x \longrightarrow x^2 - x$ of F_q into F_q. Since $\omega = 1$ (i.e. $(-1)^n \det 2S \in (O^{\times})^2$ which is the definition of H to be split type), we see that the case (II) cannot occur. To prove this, assume

$$\begin{cases} s_{ii} \equiv 0 \mod \varpi \ (1 \leqq i \leqq 2n-2), \\ s_{n,n} \equiv s_{2n,2n} \equiv \tilde{\alpha} \mod \varpi , \\ 2s_{i,i+n} \equiv 1 \mod \varpi \ (1 \leqq i \leqq n), \\ 2s_{ij} \equiv 0 \mod \varpi, \ |i-j| \neq n, \end{cases}$$

where $\tilde{\alpha} \in O$ satisfies $\tilde{\alpha} \mod \varpi = \alpha$. When $n = 1$, we can show $-\det 2S \notin (O^{x})^2$ by noting

(a) $1 + 4\varpi O \subseteqq (O^{x})^2$,

(b) $|1 + 4O/(1 + 4O \cap (O^{x})^2)| = 2$ and $1 + 4\tilde{\alpha} \notin (O^{x})^2$,

(c) $(1 + \varpi O)^2 - 4\tilde{\alpha} \cap (O^{x})^2 = \phi$.

In general case, we get easily $(-1)^n \det 2S \notin (O^{x})^2$ by induction on n. Thus $\overline{Q}$ must be of the form (I)[5] and the rest of the proof of Lemma 2.6 can be done quite similarly.

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5) In the same way as above, we can show $\overline{Q}$ is of the form (II) if $\omega \neq 1$.

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