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FINITE DETERMINACY OF EQUIVARIANT MAP GERMS

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Equivariant Contact Equivalence

Let k denote either $\mathbb{R}$ or $\mathbb{C}$ and "smooth" mean C^∞ or analytic according to context. Let V and W be finite dimensional k-representations of a group G which is a compact Lie group if $k = \mathbb{R}$ and a reductive complex Lie group if $k = \mathbb{C}$. The space of (resp. equivariant) smooth map germs from (V,0) to (W,0) is denoted by $\mathcal{E}(V,W)$ (resp. $\mathcal{E}_G(V,W)$).

Definitions

(1) Let $\mathcal{K}_G$ denote the group of germs, H, of equivariant diffeomorphisms of $(V \times W, 0)$ for which there exist germs of equivariant diffeomorphisms, h, of (V,0) such that the following diagram commutes:-

$$\begin{array}{ccccc} (V,0) & \longrightarrow & (V\times W,0) & \longrightarrow & (V,0) \\ \downarrow h & & \downarrow H & & \downarrow h \\ (V,0) & \longrightarrow & (V\times W,0) & \longrightarrow & (V,0) \end{array} .$$

(2) Two germs f,g in $\mathcal{E}_G(V,W)$ are $\mathcal{K}_G$ - equivalent if there exists (H,h) in $\mathcal{K}_G$ such that $H(h(x), f(h(x))) = (x, g(x))$, where x is a co-ordinate system for V.

(3) A germ, f, in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if there exists a positive integer, s, such that any g in $\mathcal{E}_G(V,W)$ with $j^s g(0) = j^s f(0)$ is $\mathcal{K}_G$ - equivalent to f.

If the actions of G on V and W are trivial then we have the usual definitions of $\mathcal{K}$ - equivalence and finite determinacy

due to Mather. The above definition of $\mathcal{K}_G$ seems to be the most natural generalisation to equivariant map germs; there are analogous definitions of $\mathcal{R}_G$, $\mathcal{L}_G$, $\mathcal{A}_G$ and $\mathcal{C}_G$, but, for simplicity, only $\mathcal{K}_G$ finite determinacy will be considered in the present paper.

Criteria for Finite Determinacy

For f in $\mathcal{E}(V,W)$ denote by tf the map $\mathcal{E}(V,V) \to \mathcal{E}(V,W)$; $\alpha \longmapsto Df.\alpha$

if f belongs to $\mathcal{E}_G(V,W)$ then tf restricts to a map $\mathcal{E}_G(V,V) \to \mathcal{E}_G(V,W)$. Mather defined the (extended) $\mathcal{K}$-tangent space of f to be the subspace of $\mathcal{E}_G(V,W)$:-

$$T\mathcal{K}(f) := tf\mathcal{E}(V,V) + f^*m(W).\mathcal{E}(V,W)$$

where $m(W)$ is the maximal ideal of $\mathcal{E}(W)$, the ring of germs of smooth functions on $(W,0)$. If f belongs to $\mathcal{E}_G(V,W)$ the $\mathcal{K}_G$-tangent space is defined analogously by:-

$$T\mathcal{K}_G(f) := tf\mathcal{E}_G(V,V) + (f^*m(W).\mathcal{E}(V,W))^G$$

where the notation $(\ .\ )^G$ is used to denote the fixed point set of the natural action of G. Clearly we have

$$T\mathcal{K}_G(f) = (T\mathcal{K}(f))^G.$$

Theorem 1 [2], [7]

A germ f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$-finitely determined if and only if $\dim._k \mathcal{E}_G(V,W) / T\mathcal{K}_G(f) < \infty$. □

An immediate corollary of this is that if f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}$-finitely determined then it is also $\mathcal{K}_G$-finitely determined. In fact Wall [11] proved that if $j^s f(0)$ is $\mathcal{K}$-sufficient, then it is also $\mathcal{K}_G$-sufficient.

A second characterisation of finitely determined germs states that a germ is finitely determined if and only if it has a representative which is "stable" in a punctured neighbourhood of 0 (see Wall [12] for the non-equivariant case).

Definition

A germ f in $\mathcal{E}_G(V,W)$ is (infinitesimally) $\mathcal{K}_G$ - stable at 0 if $T\mathcal{K}_G(f) = \mathcal{E}_G(V,W)$. Stability of germs of equivariant maps $f : V \to W$ at points $x \neq 0$ can be defined similarly after restricting f to a slice transversal to the orbit $G.x$ at x, provided the orbit is closed in V.

By a closed orbit of the action of G on V we mean an orbit which is closed as a subset of V with its usual topology. If $|G|$ is finite or $k = \mathbb{R}$ and G is compact then every orbit of G on V is closed. However this is not true in the general complex case (consider, for example, the action of $G = \mathbb{C}^*$ on $\mathbb{C}^2$ given by $t.(x,y) = (t^{-1}x, ty)$ where t is an element of $\mathbb{C}^*$ and (x,y) of $\mathbb{C}^2$). In fact if $k = \mathbb{C}$ and G is a reductive complex Lie group with $\dim.G \geqslant 1$ then there always exist non-closed orbits. The necessity of the restriction to closed orbits in the definition is due to the fact that non-closed orbits do not, in general, possess suitable slices. Fortunately we only need a definition of stability at points on closed orbits.

Theorem 2 [7]

Let $k = \mathbb{C}$. Then a germ f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if and only if there exists a G - invariant neighbourhood U of 0 in V and a G - equivariant representative

of f on U which is $\mathcal{K}_G$ - stable at every point in $U \setminus \{0\}$ for which the orbit $G.x$ is closed. □

If the actions of G on V and W are trivial then the theorem reduces to the usual statement that f is $\mathcal{K}$ - finitely determined if and only if a representative of f is $\mathcal{K}$- stable on $U \setminus \{0\}$ or, equivalently, if and only if it is transversal to 0 on $U \setminus \{0\}$.

Since a real analytic germ f in $\mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if and only if its complexification $f_{\mathbb{C}}$ is $\mathcal{K}_{G_{\mathbb{C}}}$ - finitely determined (where $G_{\mathbb{C}}$ is the reductive complex Lie group obtained by complexifying the compact group G), this theorem also characterises $\mathcal{K}_G$ - finitely determined analytic germs when $k = \mathbb{R}$. Results analogous to Theorems 1 and 2 also hold for $\mathcal{R}_G, \mathcal{L}_G, \mathcal{A}_G$ and $\mathcal{C}_G$ finite determinacy.

Invariant Maps

Suppose the action of G on W is trivial; then the following lemma is not difficult to prove.

Lemma 3. An invariant map $f : V \longrightarrow W$ is $\mathcal{K}_G$- stable at a point x such that $G.x$ is closed if and only if it is $\mathcal{K}$-stable, and hence transversal to 0, at x. □

Corollary 4. If G is finite, then $f \in \mathcal{E}_G(V,W)$ is $\mathcal{K}_G$ - finitely determined if and only if it is $\mathcal{K}$- finitely determined.

The proof of the corollary is by the lemma, Theorem 2 and the fact that all the orbits of G in V are closed. □

If $\dim.G \geqslant 1$, and if $\dim.V > \dim.W > 1$, then Wall has shown [11] that no invariant map germ $f : V \to W$ with $Df(0) = 0$ can be $\mathcal{K}$ - finitely determined (Slodowy [10] proved a similar result for the case $\dim.W = 1$, but the situation when $\dim.V \leqslant \dim.W$ is unclear). However, although most invariant map germs are not $\mathcal{K}$ -finitely determined if $\dim.G \geqslant 1$, they are $\mathcal{K}_G$ -finitely determined. More precisely we have the following result .

<u>Theorem 5</u> [8]

For any G, if G acts trivially on W, then the germs in $\mathcal{E}_G(V,W)$ which are not $\mathcal{K}_G$ - finitely determined lie in a subset of infinite codimension. □

In a rather different direction we have the following proposition about real analytic germs.

<u>Proposition 6</u>

If $k = \mathbb{R}$ and f is a $\mathcal{K}_G$ - finitely determined real analytic invariant map germ then it is also C^k - $\mathcal{K}$ - finitely determined for all k such that $0 \leqslant k < \infty$.

<u>Proof</u>. If f is $\mathcal{K}_G$ - finitely determined there is a suitable representative $\tilde{f}$ of f such that $\tilde{f}_{\mathbb{C}}$ is transversal to 0 at all points with closed $G_{\mathbb{C}}$ - orbits in $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. If x is any point in V then $G_{\mathbb{C}}.x$ is closed (Schwarz [9] p.59) and so f is transversal to 0 at all points in a punctured neighbourhood of 0 in V. The result now follows from the geometric criterion for C^k - finite determinacy (see Wall [12] Theorem 6.1). □

Z_p - Equivariant Functions

If the action of G on W is non-trivial then we may lose the genericity, and even the existence, of $\mathcal{K}_G$ - finitely determined germs.

Let $G = Z_p$, identified with the p-th roots of unity; let t denote a generator of Z_p. Define an action of G on $V = \mathbb{C}^{m+n}$ by

$$t.(x_1,\ldots,x_m,y_1,\ldots,y_n) = (tx_1,\ldots,tx_m,y_1,\ldots,y_n)$$

and on $W = \mathbb{C}$ by

$$t.z = t^q z$$

for some q satisfying $0 \leqslant q < p$.

Theorem 7 [8]

(1) There exist $\mathcal{K}_G$ - finitely determined germs in $\mathcal{E}_G(V,W)$ if and only if $n = 0$ or $n \geqslant \frac{(q+m-1)!}{q!(m-1)!} - m^2$

(2) Finite determinacy with respect to $\mathcal{K}_G$ holds outside a subset of infinite codimension if and only if one (or more) of the following conditions holds:-

(a) $n = 0$ (b) $q \leqslant 2$

(c) $m \leqslant 1$ (c) $(m,q) = (2,3),(2,4)$ or $(3,3)$.

We give a sketch of a proof of the theorem.

Any f in $\mathcal{E}_G(V,W)$ can be written

$$f(x,y) = \sum_j \bar{f}_j(x,y)\,\tau_j(x)$$

where $\{\tau_j\}$ is the set of monomials of degree q in $x_1,\ldots,x_m$ and the $\bar{f}_j$ are invariant functions on V. So f is determined by a (non-unique) invariant map $\bar{f} : V \longrightarrow M$, where M is the space of homogeneous forms of degree q in m variables.

Using arguments similar to some which are standard in singularity theory it can be shown that

f is K_G - stable at x in $V \setminus V^G$ if and only if it is transversal to 0 at x,

and

f is K_G - stable at x in V^G if and only if $\bar{f}|_{V^G} : V^G \longrightarrow M$ is transversal to the orbits of the natural action of $GL(m)$ on M.

Thus, by Theorem 2, K_G - finite determinacy is reduced to a set of transversality conditions on a punctured neighbourhood of 0.

The number of these transversality conditions is finite if and only if $n = 0$ or the number of $GL(m)$ orbits in M is finite; this latter condition is satisfied if and only if $m \leq 1$ or $q \leq 2$ or $(m,q) = (2,3)$. In these cases K_G finite determinacy is certainly generic.

More generally we can stratify M so that each stratum is foliated by $GL(m)$ orbits. If the strata of codimension $< n$ are foliated by orbits of codimension ≤ 1 in the stratum then the points where $\bar{f}$ is not transversal to the $GL(m)$ orbits generically form a set of dimension 0 and so they can be excluded from a sufficiently small punctured neighbourhood of 0. Conversely, if there is a stratum of codimension $< n$ which is foliated by orbits of codimension > 1, then we can construct $\bar{f}$ (and hence f) with points of non-transversality in any neighbourhood of 0 which can not be removed by perturbing $\bar{f}$ slightly. This means that no extension of a suitable jet

of f can be $\mathcal{K}_G$ - finitely determined, and so $\mathcal{K}_G$ - finite determinacy can not be generic. There exists a stratification of M such that the strata of codimension $< n$ are foliated by orbits of codimension $\leqslant 1$ if and only if we are in one of the cases of the previous paragraph or $(m,q) = (3,3)$ or $(2,4)$. This completes the proof of (2).

The necessity of the condition $n = 0$ or $n \geqslant \dim.M - m^2 = \frac{(q+m-1)!}{q!(m-1)!} - m^2$ in (1) is clear from the above since $\dim.M - m^2$ is the codimension of the orbits of maximal dimension in M. The sufficiency of this condition follows by more or less explicit construction of $\mathcal{K}_G$ - finitely determined germs. □

Theorems 5 and 7 follow from a more general result [8] which gives necessary and sufficient conditions for $\mathcal{R}_G$, $\mathcal{C}_G$ and $\mathcal{K}_G$ finite determinacy to be generic in $\mathcal{E}_G(V,W)$ for any reductive complex Lie group G and complex representations V and W. There appears to be a serious problem with proving the necessity of the conditions when $k = \mathbb{R}$.

Stable Unfoldings

In complete analogy with the non-equivariant case Damon [2] proved that an equivariant map germ has a $\mathcal{K}_G$ - versal unfolding if and only if it is $\mathcal{K}_G$ - finitely determined (corresponding results hold for $\mathcal{R}_G$, $\mathcal{L}_G$, $\mathcal{A}_G$ and $\mathcal{C}_G$). However there is one aspect of unfolding theory which doesn't generalise in such a straightforward manner. Recall that for

ordinary map germs an unfolding is $\mathcal{A}$ stable if and only if it is $\mathcal{K}$ - versal and so the property of possessing an $\mathcal{A}$ stable unfolding is generic; this latter fact is important in the proof of the topological stability theorem [4] and also in that of the genericity of topological $\mathcal{A}$ - finite determinacy [6].

However for equivariant map germs this is no longer true, instead we need to replace $\mathcal{K}_G$ by a subgroup, denoted $\mathcal{K}_G^*$ and defined in [13]. Then it is true that an equivariant unfolding is $\mathcal{A}_G$ - stable if and only if it is $\mathcal{K}_G^*$ - versal and so an equivariant germ has an $\mathcal{A}_G$ - stable unfolding if and only if it is $\mathcal{K}_G^*$ - finitely determined. If the action of G on W is trivial then $\mathcal{K}_G^* = \mathcal{K}_G$, but in general $\mathcal{K}_G^*$ finite determinacy is a stronger property than $\mathcal{K}_G$ - finite determinacy and $\mathcal{K}_G^*$ finite determinacy may not be generic even when $\mathcal{K}_G$ - finite determinacy is.

<u>Example.</u> Let $G = Z_2$ with generator t and V be the m+n dimensional representation defined by:-

$$t.(x_1,\ldots,x_m,y_1,\ldots,y_n) = (tx_1,\ldots,tx_m,y_1,\ldots,y_n)$$

and W the similarly defined p+q dimensional representation.

<u>Theorem 8</u> [8]

(1) Finite determinacy with respect to $\mathcal{K}_G$ is generic in $\mathcal{E}_G(V,W)$ for all m,n,p and q.

(2) Finite determinacy with respect to $\mathcal{K}_G^*$ is generic in $\mathcal{E}_G(V,W)$ if and only if one of the following holds:-

(a) $m = 0$ (b) $p \leq 2$

(c) $n \leq \begin{cases} m-p+q+1 & m \geq p \\ q & m < p \end{cases}$.

The non-genericity of K_G^* finite determinacy casts doubts on the possibility of an equivariant version of the topological stability theorem and Nakai [5] has found examples of pairs (M,N) of Z_2 - manifolds for which equivariant topologically stable maps are not dense and, locally, topological $\mathcal{A}_G$ - finite determinacy is not generic. It is perhaps interesting to note that in these examples the pairs of slice representations (V,W) of pairs of points (x,y) in $M^G \times N^G$ lie in the range $n = q+1$, $m = p-1$ and so K_G^* finite determinacy is not generic in $\mathcal{E}_G(V,W)$. It may be possible to prove an equivariant version of the topological stability theorem if K_G^* finite determinacy is generic for all possible pairs of slice representations; we might also conjecture that if K_G^* - finite determinacy is generic in $\mathcal{E}_G(V,W)$, then so is topological $\mathcal{A}_G$ - finite determinacy.

A Conjecture

So far we have no equivalence relation for which finite determinacy is generic for all G,V and W. However the following looks promising.

Definition

Two germs f,g in $\mathcal{E}_G(V,W)$ are C^0 - V_G - equivalent if there exists a germ of an equivariant homeomorphism of (V,0) taking $f^{-1}(0)$ to $g^{-1}(0)$.

Conjecture

The set of germs in $\mathcal{E}_G(V,W)$ which are not C^0 - V_G - finitely determined has infinite codimension.

A proof of this conjecture could perhaps be constructed using Thom's "blowing-up" idea (e.g. [6]) together with the equivariant transversality theory and isotopy theorem of Bierstone [1] and Field [3].

Finally we note that some related work, on the infinite determinacy of equivariant germs, has been done by Wall [14].

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