<table>
<thead>
<tr>
<th>Title</th>
<th>On Morin Singularities (C$^\infty$-maps and their Singularities)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Ando, Yoshifumi</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1985), 550: 126-130</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1985-02</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/98883">http://hdl.handle.net/2433/98883</a></td>
</tr>
</tbody>
</table>

**Departmental Bulletin Paper**

**Textversion**

Kyoto University
On Morin Singularities

by

Yoshifumi Ando

Introduction

In this paper we study the problem of finding a smooth map between smooth manifolds with nice Morin singularities in a given homotopy class. A geometric interpretation of Morin singularities of a smooth map $f: N \to P$ is as follows. Let $S^i_1(f)$ denote the set of all points $x$ of $N$ such that the kernel rank of $df_x$ is $i$. For a certain map $f$, $S^i_1(f)$ becomes a submanifold of $N$ and we may define $S^i,j_1(f)$ as the set $S^j_1(f|S^i_1(f))$ for $f|S^i_1(f): S^i_1(f) \to P$ similarly. Let $n = \dim N$, $p = \dim P$ and $i = \max (1, n-p+1)$. Let $I^i_r$ be the $r$-sequence $(i, 1, \ldots, 1)$. Then we may continue to define $S^i_1(f)$ as $S^i_1(f|S^{i-1}_r)$ inductively. A point of $S^i_1,0(f)$ or $I^i_r(f)$ is called a Morin singularity of symbol $(i, 0)$ or $I^i_r$ respectively. However this approach does not make it clear for what part of smooth maps $f$, $S^i_1(f)$ can be defined. For this we review the following important observation due to Boardman[2].

There exist a submanifold $E^{i,0}_1(N, P)$ and a series of submanifolds: $E^{i,1}_1(N, P) \supset E^{i,2}_1(N, P) \supset \cdots \supset E^{i}_1(N, P) \supset \cdots$ in the infinite jet space $J^\infty(N, P)$. The codimension of $E^{i,0}_1(N, P)$ is $i(p-n+1)$ and that of $E^{i}_1(N, P)$ is $n-p+r$ for $n \geq p$ and
r(p-n+1) for n < p. He has shown that if a jet map $j^\omega f: N \to J^\omega(N,P)$ of f is transverse to all submanifolds $\Sigma^{i,0}_r(N,P)$ and $\Sigma^r_I(N,P)$, then $S^{i,0}_r(f)$ and $S^r_I(f)$ coincide with $(j^\omega f)^{-1}(\Sigma^{i,0}_r(N,P))$ and $(j^\omega f)^{-1}(\Sigma^r_I(N,P))$ respectively. Therefore for generic maps f we may consider $S^{i,0}_r(f)$ and $S^r_I(f)$.

For any integer $r \geq 1$ we define a subset $\Omega_r(N,P)$ of $J^\omega(N,P)$ as the set of all jets z such that either z is of maximal rank or a point of $\Sigma^{i,0}_r(N,P)$ or $\Sigma^2_I(N,P) \setminus \Sigma^{r+1}_I(N,P)$. Then $\Omega_r(N,P)$ becomes an open subbundle of the fibre bundle $J^\omega(N,P)$ over N.

The first result of this paper is the following

**Theorem 1.** Let $p \geq 2$. Then for any section s of N into $\Omega_r(N,P)$, there exists a smooth map g: N → P such that $j^\omega g$ becomes a section of N into $\Omega_r(N,P)$ homotopic to s in $\Omega_r(N,P)$.

Next we will study the problem of eliminating the Morin singularities $S^r_I(f)$ with codim $S^r_I(f) = n$ from f admitting only Morin singularities. Theorem 1 reduces it to a problem of finding a section of N into $\Omega_{r-1}(N,P)$ homotopic to $j^\omega f$. We will show that if $j^\omega f$ is transverse to $\Sigma^r_I(N,P)$ for a connected and closed manifold N, then the number of points of $S^r_I(f)$ modulo 2 is the unique obstruction of finding the above section.

We should note that this number is just the Thom polynomial $\Sigma^r_I(N,P)$ for f (see the definition of [9]).

**Theorem 2.** Let $r \geq 2$, $p \geq 2$ and codim $\Sigma^r_I(N,P) = n$.

Let N and P be orientable manifolds. Then

1. A smooth map f with $j^\omega f(N) \subset \Omega_r(N,P)$ is homotopic to
a smooth map $g$ such that $j^\infty g(N) \subset \Omega_{r-1}^r(N,P)$ and $j^\infty f$ and $j^\infty g$
are homotopic as sections of $N$ into $\Omega_r(N,P)$ if and only if the
Thom polynomial of $\Sigma^f r(N,P)$ for $f$ vanishes.

(2) In particular $f$ is homotopic to such a smooth map $g$
in the following cases:

i) $n > p$ and $r \equiv 1 \pmod{4}$

ii) $n > p$, $r \equiv 2,3 \text{ or } 4 \pmod{4}$ and $n-p \equiv 1 \pmod{2}$ and

iii) $n \leq p$ and $n+p+r+\frac{1}{2}r(r+1) \equiv 0 \pmod{2}$.

It will be shown by the Morse inequalities that the similar
statement of Theorem 1 for $p = 1$ is not true. If $N$ is an
open manifold, then Theorem 1 is a direct consequence of
Gromov[7, Theorem 4.1.1] and if $n < p$, it is also a special
case of [4, Theorem B]. So the rest cases will be treated
in this paper. The case $r = 2$ of Theorem 1 should be compared
with [6, Theorem 1.3] which will play an important role in a
proof of Theorem 1 (Sections 2 and 3).

The case $n \geq p$ and $p = 2$ of Theorem 2 has been proved by
Levine[11, Theorems 1 and 2] for $n > 2$ and by Eliashberg[5,
Corollary of Theorem 4.9] for $n = 2$. 

References

241-268.

[2] J. M. Boardman, Singularities of differentiable maps,

[3] J. Damon, Thom polynomials for contact class singularities,

[4] A. A. du Plessis, Maps without certain singularities,


mappings, Math. USSR Izv., 6(1972), 1302-1326.

[7] M. L. Gromov, Stable mappings of foliations into manifolds,


singularités des applications différentiables, Séminaire

[10] A. Lascoux, Calcul de certains polynômes de Thom, C.R.

263-296.


