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Criteria & Problems on Truncation of Taylor Expansions

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Consider an analytic, or C^∞, function germ $f:(\mathbb{R}^n,0)\to(\mathbb{R},0)$. Let $T^\infty(f)$ denote its Taylor expansion

$$T^\infty(f) = \Sigma a_i x_i + \Sigma a_{ij} x_i x_j + \cdots, \quad a_i = \frac{\partial f}{\partial x_i}(0), \quad \text{etc.}$$

Recall that if some $a_i \neq 0$, then f is equivalent to x_i under a coordinate transformation (Implicit Function Theorem), and if all $a_i=0$ but $\det(a_{ij}) \neq 0$, then f is equivalent to $\sum_{i=1}^n \pm x_i^2$ (Morse Lemma). In case all $a_i=0$ and $\det(a_{ij})=0$, we like to consider the problem of finding the smallest integer r such that all terms of degree $\geq r+1$ in $T^\infty(f)$ can be omitted without effecting the local behavior of f. An answer to this problem is needed in calculus (Maximam & Minima), Differential Geometry (local theory), and for Singularity Theory (see Thom's Bombay Lecture [6]).

We recall some results and state some problems.

Call a polynomial $z(x_1,\cdots,x_n)$, of degree r with $z(0)=0$, an r-jet. We say z is topologically determined (C^0-sufficient) in C^r if for any C^r-function f with $T^r(f)=z$, there exists a local homeomorphism $h:(\mathbb{R}^n,0)\approx(\mathbb{R}^n,0)$ such that $f\circ h=z$. Here $T^r(f)$ denote the Taylor expansion of f up to degree r.

<u>Theorem 1</u> ([1][3][4]) An r-jet z is topologically determined in C^r if and only if $\exists\ \varepsilon>0$,

$$|\mathrm{Grad}\, z| \geqq \varepsilon |x|^{r-1}, \quad x \text{ near } 0.$$

<u>Corollary</u>. If $z = H_r(x_1,\cdots,x_n)$ is a non-degenerate r-form, then $|\mathrm{Grad}\, z| \geq \varepsilon|x|^{r-1}$ and z is topologically determined as an r-jet.

Call an r-jet z C^1-determined if for any f with $T^r(f) = z$, there exists a C^1-local diffeomorphism h, $f \circ h = z$.

Problem 1. Consider the Koike-Kucharz function ([2])

$$k(x,y) = x^3 - 3xy^s, \quad s \geqq 3.$$

Find the smallest r such that $T^r(k)$, as an r-jet, is C^1-determined.

We believe a solution of this problem will lead to interesting criteria for C^1-determinancy.

Theorem 1 is an analytic criterion. Next we recall a geometric criterion. Given an r-jet $z(x_1,\cdots, x_n)$. Consider

$$F(x;\lambda) = z(x) + \Sigma_{|\alpha|=r} \lambda_\alpha x^\alpha , \qquad \lambda_\alpha \text{ indeterminants},$$

$\alpha=(\alpha_1,\cdots,\alpha_n)$, $\alpha_i \in \mathbb{Z}^+$, $\alpha_i \geqq 0$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let $V_F = \{(x,\lambda) \mid F(x;\lambda) = 0\}$. This is an algebraic variety in $\mathbb{R}^n \times \Lambda$, Λ the space of parameters $\lambda=(\lambda_\alpha)$. It is easy to see that the singular subvariety of V_F is $\{0\} \times \Lambda$.

Theorem 2 ([5]) z is topologically determined in C^r if and only if V_F is Whitney (a,b)-regular over Λ at 0.

Problem 2. Translate the above two theorems into algebra, i.e. find algebraic criteria for topological determinancy.

Probelm 3. Find geometric and analytic criteria for z to be blow-analytically determined.

References

[1] J. Bochnak and S. Lojasiewicz, A converse of the Kuiper-Kuo Theorem. Proc. Liverpool Singularities Symposium 1, Lecture Notes in Math. No. 192. Springer-Verlag, 1971.

[2] S. Koike and W. Kucharz, Sur les realisations de jets non-suffisants, C.R. Acad. Sc. Paris, 288, 457-459 (1979).

[3] N.H. Kuiper, C^1-equivalence of Functions near Isolated Critical Points, Symp. Inf. Dim. Top., Ann. Math. Studies 69, Princeton University Press 1972.

[4] T.C. Kuo, On C^0-sufficiency of jets of potential functions, Topology 8 (1969), 167-171.

[5] T.C. Kuo and Y.C. Lu, Sufficiency of jets via stratification theory, Inventions Math, Vol 57, (1980), 219-226.

[6] R. Thom, Local topological properties of differentiable mappings, Differential Analysis, Bombay Colloquium 1964, Oxford University Press.