数理解析研究所講究録
554巻 1985年 184-199

Ecotons of a Nonlinear Diffusion Equation

広大理　三村　昌泰　(Masayasu Mimura)
宮崎医大　薩摩　順吉　(Junkichi Satsuma)

Abstract

The present note is devoted to a short introduction of "ecoton" (ecological soliton). The ecoton which we will show is a sort of localized solitary wave produced by the interplay of two forces, diffusion processes and aggregation ones. The equation governing such solutions is a nonlinear degenerate diffusion equation including an aggregative advection of long range interaction. We give an analytical explanation of ecotons and then show some numerical simulations of initial value problems for the equation.

§1 Introduction

In this note, we study solutions of the following nonlinear diffusion equation including an aggregative advection of long range interaction;

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(u^m) - \int K(x-y)u(y)dy\cdot u(x)\right], \tag{1}$$

where u is the population density, m is a real number and K(x) is a certain integral kernel. This type of equations has been proposed by Nagai and Mimura[1] to describe spatially aggregating pattern of biological individuals.

The first term on the right-hand side of (1) is a very simple density dependent diffusion process if $m > 1$. The diffusion rate is given by mu^{m-1}. If $m = 1$ (linear diffusion), an initial distribution of populations in a bounded region spreads out at an infinite speed. However, in the case of $m > 1$, the situation is quite different. It is known that the initial distribution in a bounded region spreads out at a finite speed, since the diffusion rate degenerates at the point where $u(x) = 0$[1].

The second term of (1) exhibits an aggregation effect. As an oversimple example, we may consider

$$K_1(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x > 0 \end{cases} \tag{2}$$

For this kernel, the integral term is expressed as

$$\frac{\partial}{\partial x} [(\int_x^{\infty} u(y,t)dy - \int_{-\infty}^{x} u(y,t)dy)u(x)], \tag{3}$$

which provides the mechanism that u(x,t) moves in the right direction if

$$\int_{-\infty}^{x} u(y,t)dy < \int_x^{\infty} u(y,t)dy, \tag{4}$$

and in the left direction when the inequality is reversed. Hence it is phenomenologically understood to be a kind of aggregation process of populations.

In the case of linear diffusion ($m = 1$), there are several examples where (1) can be solved explicitly. When $K(x)$ is specified as (2), (1) becomes equivalent to Burgers' equation, and therefore is exactly linearized by using Hopf-Cole transformation. Another example is the equation with the integral kernel,

$$K_2(x) = \mathcal{P} (-\frac{1}{2\delta}) \coth \frac{\pi x}{2\delta} , \tag{5}$$

where $\mathcal{P}$ means the principal value with a positive parameter δ. By introducing a splitting of real functions and using a dependent variable transformation, the equation can also be exactly linearized.[2),3)]

In the case of density dependent diffusion ($m > 1$), any exact result has not been reported to our knowledge. Recently, however, Ikeda[4)] has shown an explicit equilibrium solution with $m = 2$ when $K(x)$ is chosen as

$$K_3(x;r) = \begin{cases} -1 & \text{for } 0 < x < r \\ +1 & \text{for } -r < x < 0 \ , \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

where r is a positive parameter denoting the parceptible distance. Of course $K_3(x;\infty) = K_1(x)$. According to his result, stationary localized solitary pulse solutions with compact support can be explicitly written down if $r > \pi$. His result can be easily extended to the equation with an unsymmetric kernel,

$$K_4(x;r) = \begin{cases} -(1+\theta) & \text{for } 0 < x < r \\ 1 & \text{for } -r < x < 0 \ , \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

where θ is a positive parameter. The addition of θ in the kernel implies that a kind of flow is introduced into the system. Hence one can expect that the solitary pulse moves at a certain constant speed. In the following section, we show an explicit form of travelling wave solutions. The solutions play an important role in the development of the initial distribution. In §3, we show several numerical simulations which give time-dependent behavior of interaction between travelling waves.

§2 Travelling wave solutions

The equation we are now conserned is

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(u^2) - \{(1+\theta)\int_x^{x+r} u(y)\,dy - \int_{x-r}^{x} u(y)\,dy\}u(x)\right]. \tag{8}$$

It is easy to see that the total population density

$$I = \int_{-\infty}^{\infty} u(x)\,dx \tag{9}$$

is a conserved quantity of (8). In the following, we assume $I < \infty$. Let us introduce a "potential" function $w(x)$ by

$$w(x) = \int_{-\infty}^{x} u(y)\,dy. \tag{10}$$

Substituting (10) into (8), we have a differential-difference equation,

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 - \{w(x-r) - (2+\theta)w(x) + (1+\theta)w(x+r)\}\frac{\partial w}{\partial x}. \tag{11}$$

The boundary conditions on (11) are

$$w(-\infty) = 0 \tag{12}$$

and

$$w(\infty) = I. \tag{13}$$

Let us take a limit of $r \to \infty$ in (11). Then it is reduced to a differential equation,

$$\frac{\partial w}{\partial t} = \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2 + \{(2+\theta)w - (1+\theta)I\}\frac{\partial w}{\partial x}. \tag{14}$$

Introducing $\xi = x-ct-x_0$ (c and x_0 are constant), and noticing (12) and (13), we obtain the equation governing a travelling wave solution,

$$\left(\frac{\partial w}{\partial \xi}\right)^2 + \left(1 + \frac{1}{2}\theta\right)(w^2 - Iw) = 0. \tag{15}$$

The speed of the travelling wave is determined from the boundary condition as

$$c = \frac{1}{2}\,\theta I. \tag{16}$$

Applying an undetermined coefficients method, we find that

$$w(\xi) = I \sin^2 \frac{1}{2}\sqrt{1+\frac{1}{2}\theta}\;\xi \tag{17}$$

is a formal solution of (15). The boundary conditions (12) and (13) also demand that $w(\xi)$ in (17) should be between $n\pi$ and $\frac{1}{2}\pi + n\pi$ where n is an integer. Without loss of generality, we may choose $n = 0$. Then from (10) we have an explicit form of travelling wave solution,

$$u(x,t) = \begin{cases} \frac{1}{2}\sqrt{1+\frac{\theta}{2}}\, I \sin\sqrt{1+\frac{\theta}{2}}\,(x - \frac{1}{2}\theta It - x_0) \\ \qquad \text{for } 0 < x - \frac{1}{2}\theta It - x_0 < \dfrac{\pi}{\sqrt{1+\theta/2}} \\ 0 \qquad \text{otherwise.} \end{cases} \tag{18}$$

This solution describes a localized solitary travelling wave with the speed $\frac{1}{2}\theta I$, the amplitude $\frac{1}{2}\sqrt{1+\theta/2}\, I$ and the width $\pi/\sqrt{1+\theta/2}$. Of course if $\theta = 0$ is taken, this agrees with the Ikeda's equilibrium solution. We would like to emphasize that the speed and the amplitude of travelling wave (18) depend on the total density I but the width is independent of it.

We have obtained the travelling wave solution from the differential equation (14). However, it can be the solution of (8) for $r > \pi/\sqrt{1+\theta/2}$, since the support of the wave is compact.

§3 Numerical results

The travelling wave solution obtained in the previous section is a special solution of (8). It is an interesting problem to see the role of the travelling waves in the initial value problem of (8). We have done several numerical computations for the purpose. Figs. 1 and 2 show the time development of initial distributions with rectangular shape. The total densities $\int_R u(x,0)dx$ of both examples are the same but the hight and width are different.

Fig. 1.1 shows the initial stage of the time development. One can see that the rectangular pulse is deformed to make three solitary waves. The first travelling wave with compact support is observed in Fig. 1.2. The second pulse appears in Fig. 1.3 and the third in Fig. 1.4. Finally in Fig. 1.5, we see that these three pulses move to the right at their corresponding constant speeds. We find that each pulse has the form of the wave analytically obtained in §2. Since the speed ($= \frac{1}{2}\theta I$) of the waves is proportional to the amplitude ($= \frac{1}{2}\sqrt{1+\theta I/2}$), these pulses are separated each other in the course of time. This kind of behavior is similar to that of solitons in nonlinear dispersive systems. For this reason, we may call these pulses "ecotons", i.e., ecological solitons. It is noted, however, that the ripple-like waves do not appear in this nonlinear diffusive system.

In Fig. 2.1-2.4 we again observe the appearance of three ecotons. In this case the second ecoton is larger than the first, and the third is much smaller than the other two. Since the second ecoton moves to the right faster than the first, one can expect that the second overtakes the first subsequently. Figs.3.1-3.3 are a numerical simulation of such an overtaking. One large ecoton is placed on the left-hand side of a small one. Both approach each other and merge to make a single larger ecoton. The coordinate moving at the speed the smaller ecoton initially has is taken in Fig.3.1-3.3. Fig.4 shows the trajectory of peaks of two ecotons which eventually become a single ecoton.

We have only shown analytically the explicit form of one ecoton. The numerical simulations suggest that there may be a way to treat the collision of ecotons or the initial value problem of (8) analytically. It is worth while investigating the qualitative properties from various point of view. Unfortunately, we have not been able to discuss these properties, which will be a future problem for us.

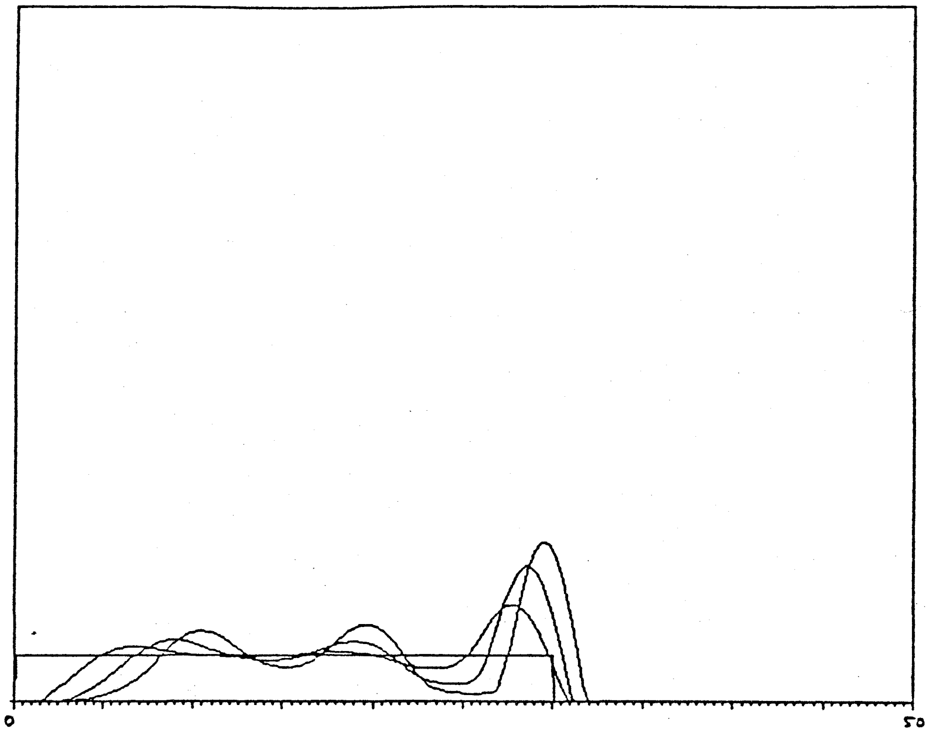

Fig.1.1 Initial value problem of (8).

Rectangular pulse 1.1.

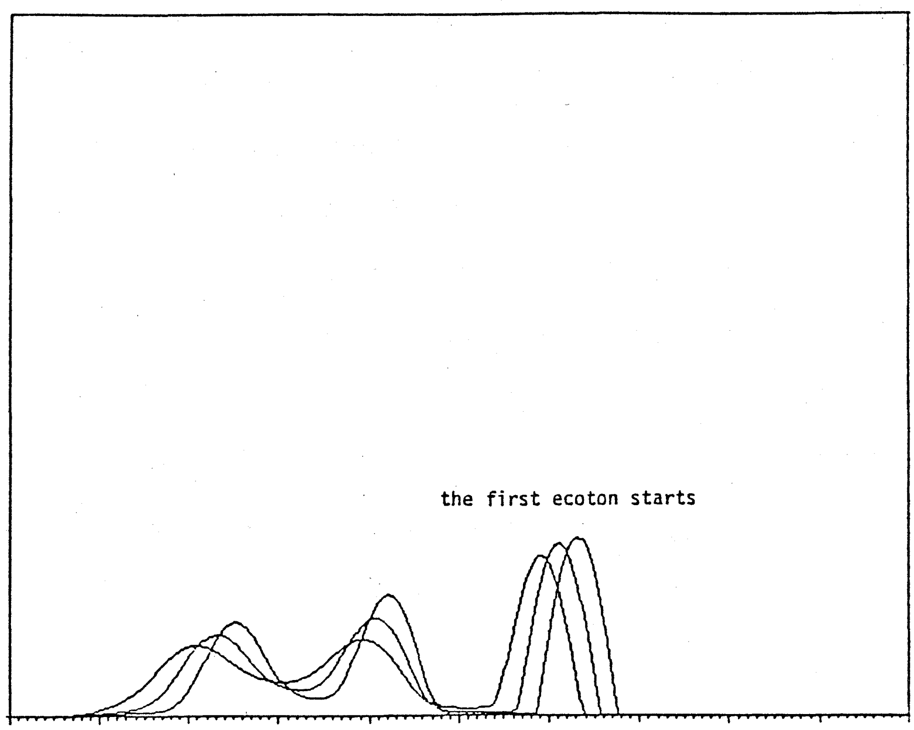

Fig.1.2 Initial value problem of (8).

Rectangular pulse 1.2.

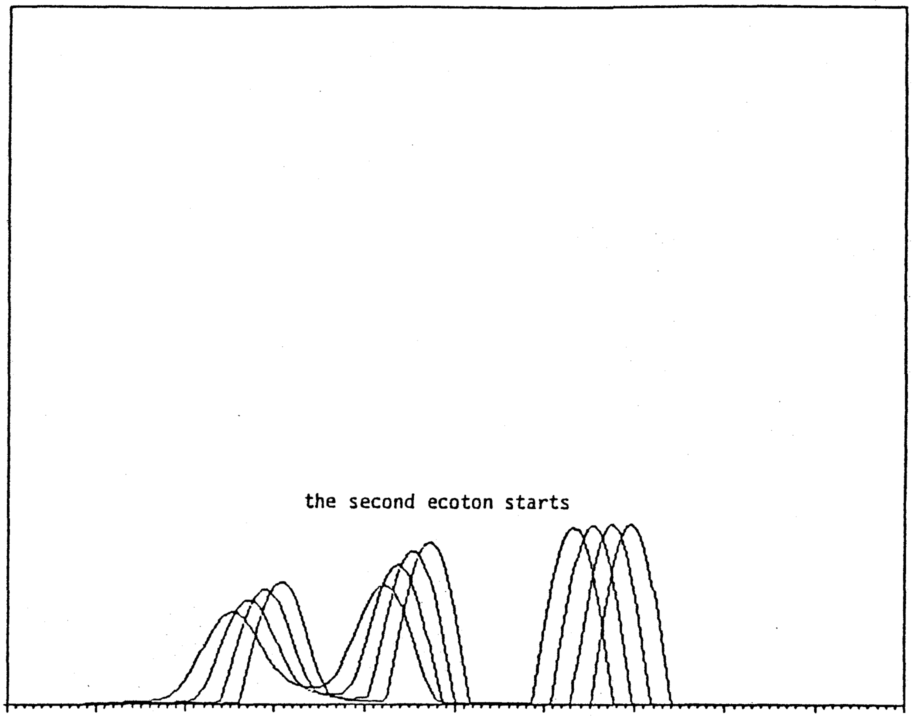

Fig. 1.3 Initial value problem of (8).

Rectangular pulse 1.3.

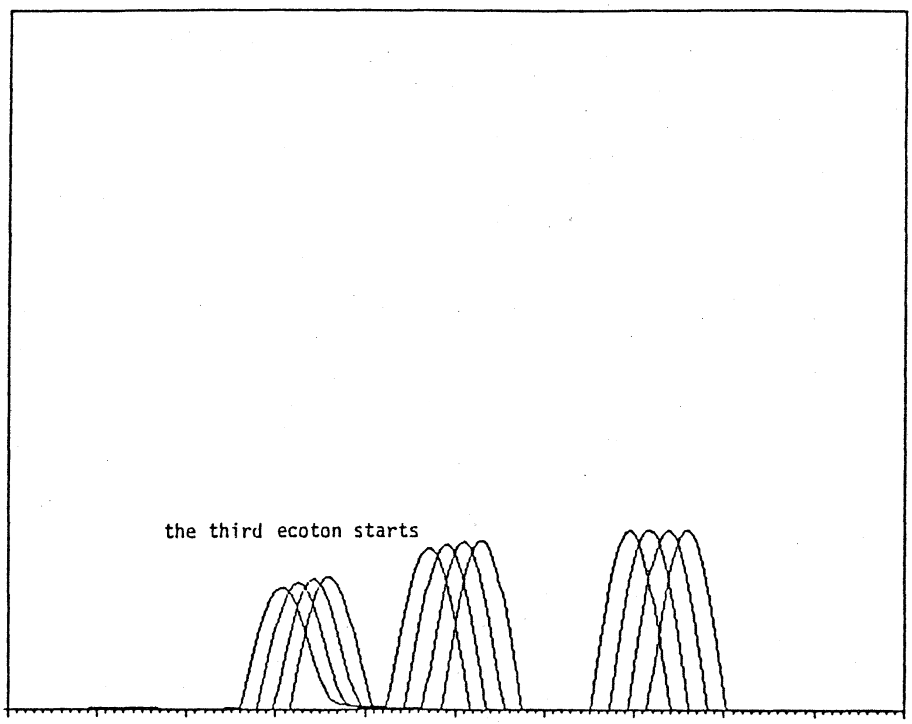

Fig. 1.4. Initial value problem of (8).

Rectangular pulse 1.4.

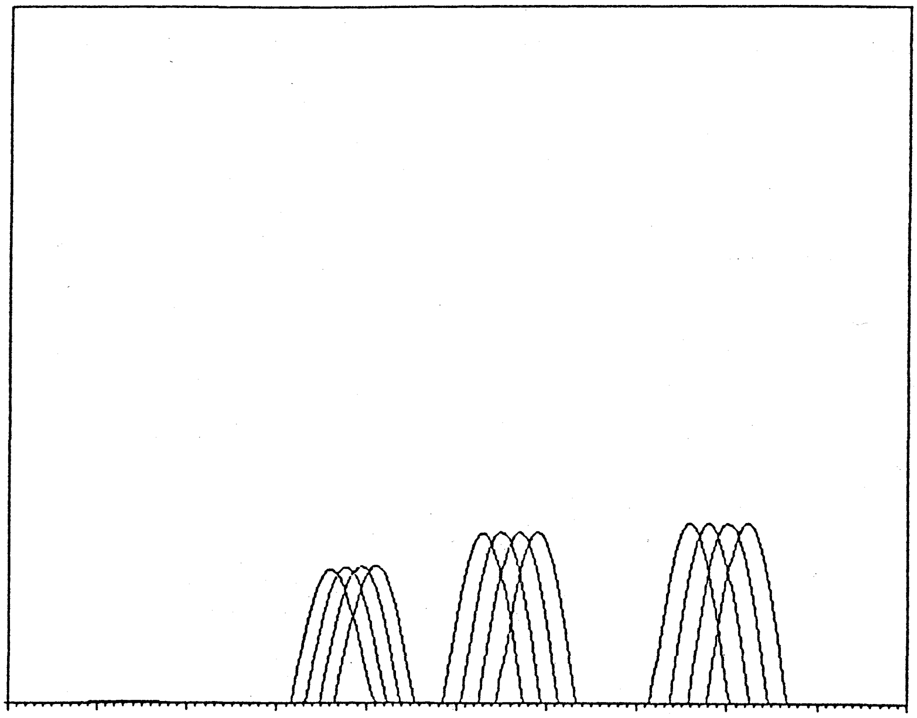

Fig. 1.5. Initial value problem of (8).

Rectangular pulse 1.5.

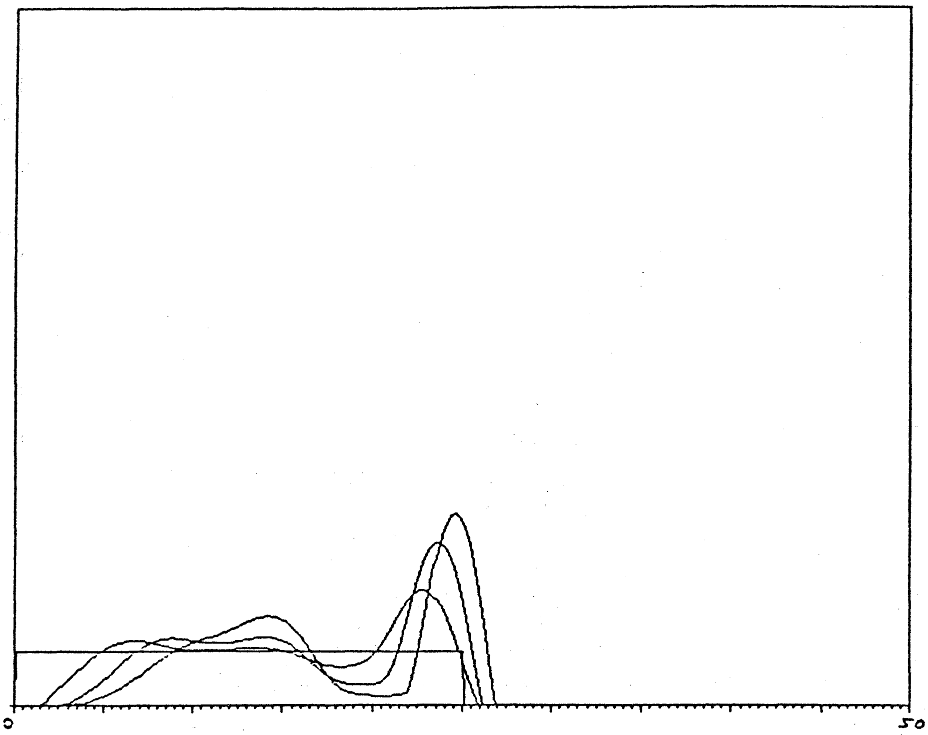

Fig. 2.1. Initial value problem of (8).

Rectangular pulse 2.1.

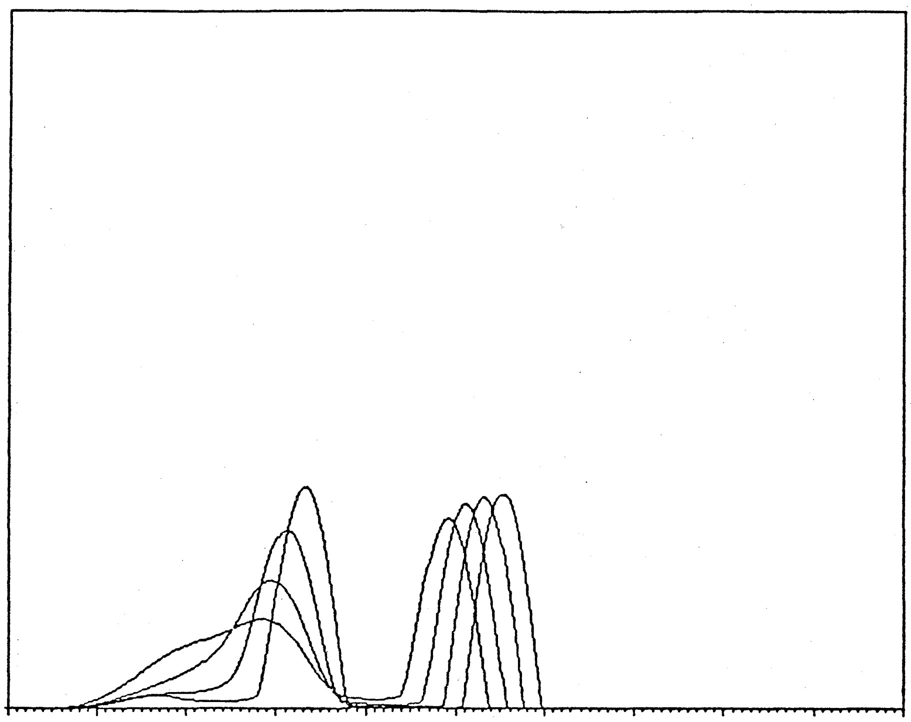

Fig. 2.2. Initial value problem of (8).

Rectangular pulse 2.2.

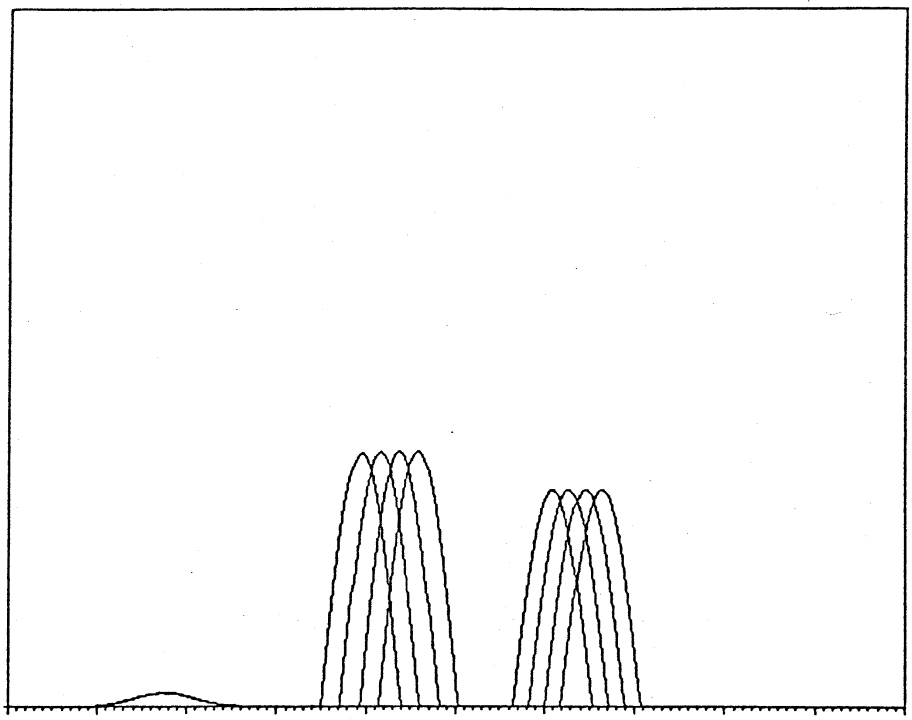

Fig. 2.3. Initial value problem of (8).

Rectangular pulse 2.3.

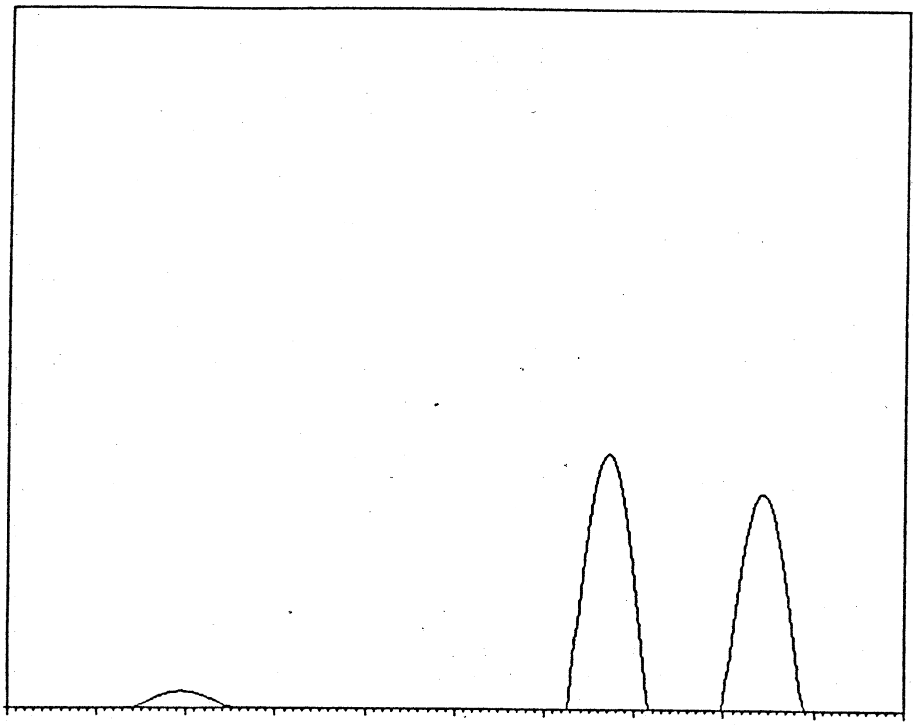

Fig. 2.4. Initial value problem of (8).

Rectangular pulse 2.4.

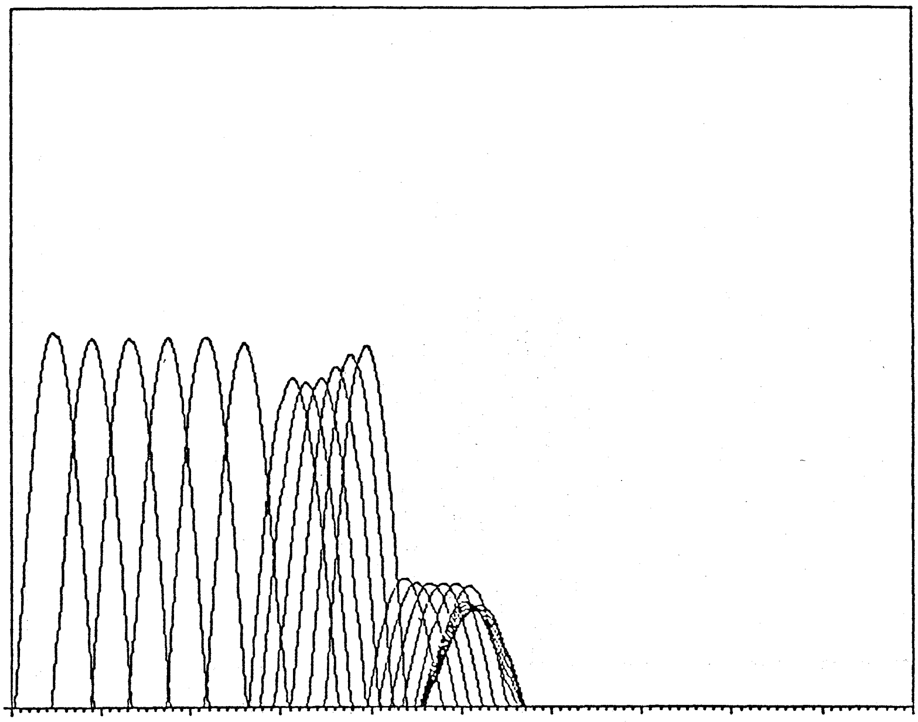

Fig. 3.1. Collision of two ecotons 1.

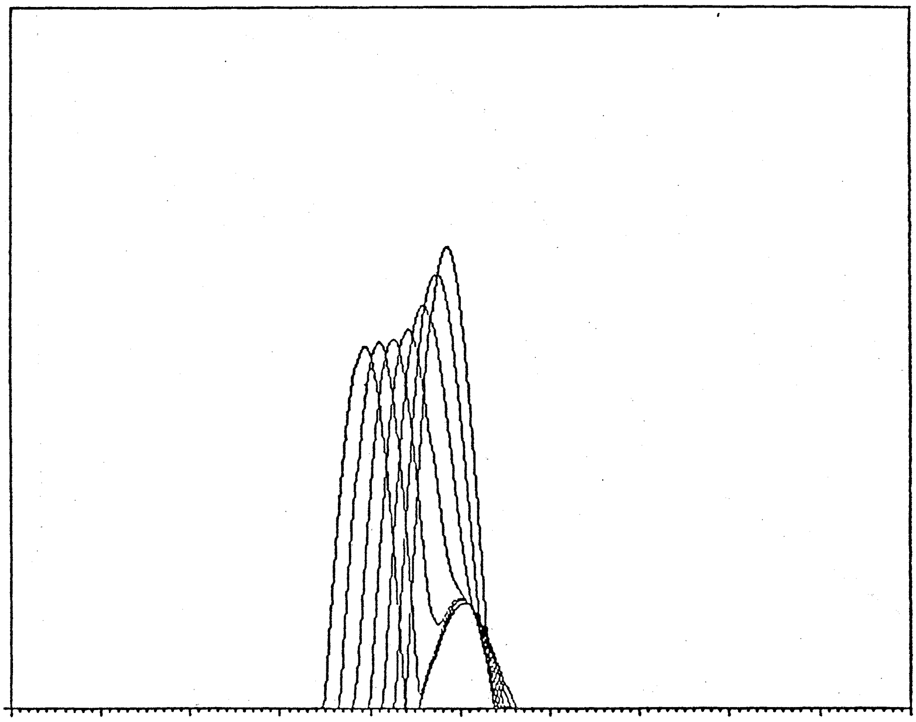

Fig. 3.2. Collision of two ecotons 2.

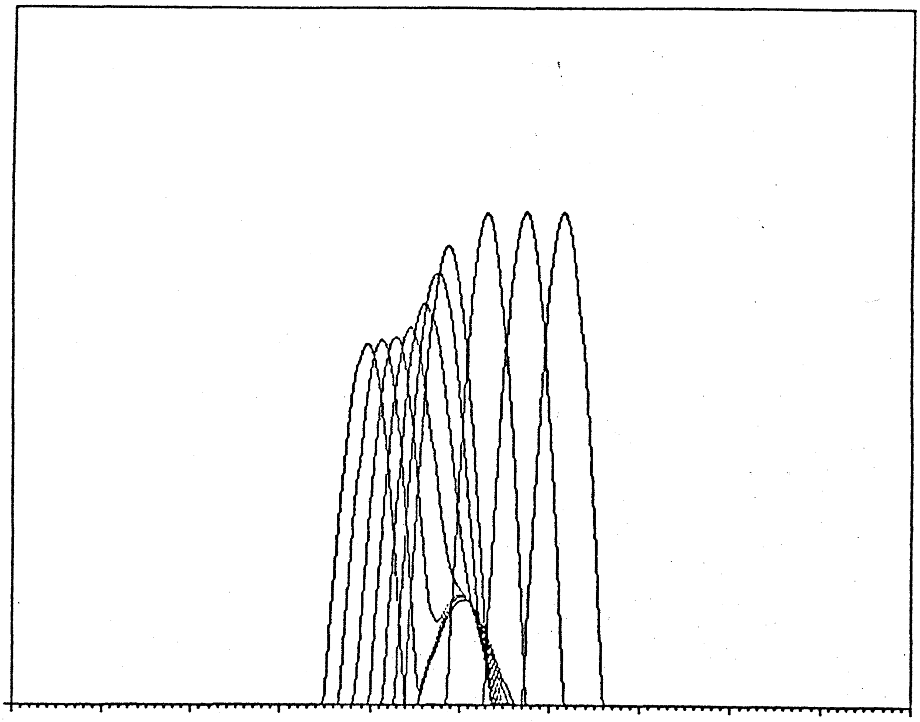

Fig. 3.3. Collision of two ecotons 3.

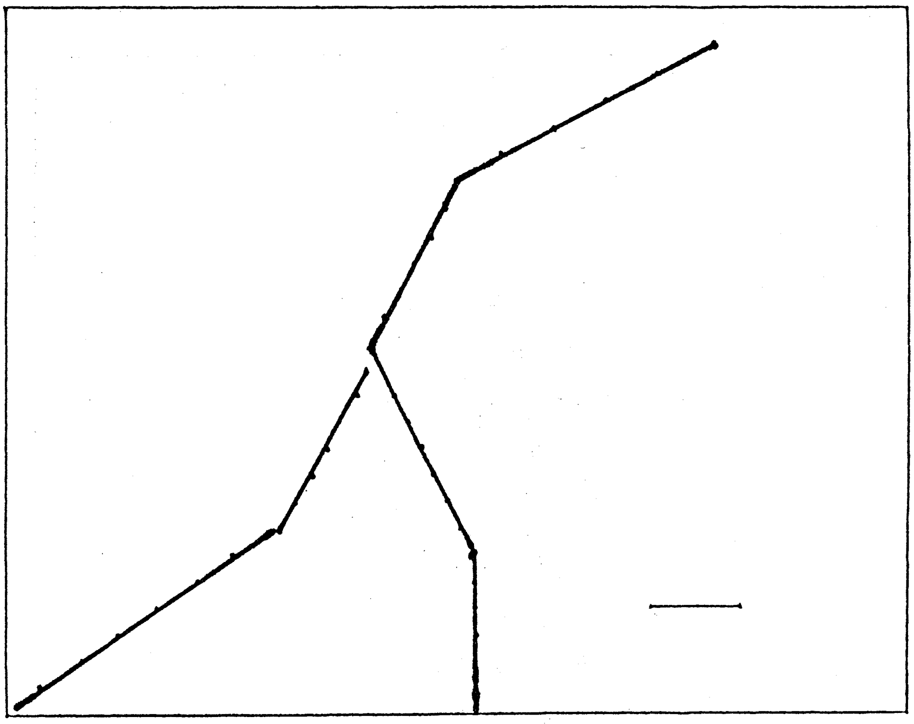

Fig. 4. Trajectory of peaks of ecotons in Fig.3.

References

1) T.Nagai and M.Mimura: J. Math. Soc. Japan 35 (1983) 539.

2) J.Satsuma: J. Phys. Soc. Jpn. 50 (1981) 1423.

3) J.Satsuma and M.Mimura: J. Phys. Soc. Jpn. 54 (1985) No.3.

4) T.Ikeda: Standing Pulse-like Solutions of a Spatially Aggregating Population Model, to appear in Japan J. Appl. Math. (1985).