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Solving the star-triangle relation

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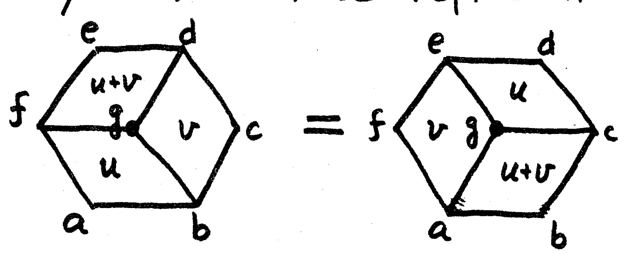
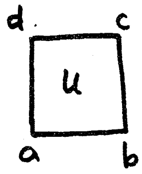
1°) Let X be a finite set with q elements, e.g. $X=\{0,1,\cdots,q-1\}$. By the star-triangle (ST) relation we mean the following set of functional equations for q^4 unknown functions $w(a,b,c,d;u)$ $(a,b,c,d\in X,\ u\in\mathbb{C})$:

$$(1)\qquad \sum_{g\in X} w(a,b,g,f;u)\,w(f,g,d,e;u+v)\,w(g,b,c,d;v)$$
$$= \sum_{g\in X} w(g,c,d,e;u)\,w(a,b,c,g;u+v)\,w(f,a,g,e;v)$$
for any $a,b,\cdots,f\in X$.

Graphically (1) can be represented as



where each face stands for the function $w(a,b,c,d;u)$:



$$= w(a,b,c,d;u) .$$

Equations (1) are the condition for 'exact solvability' of the corresponding statistical mechanics model in 2 dimensions (so-called the interaction-round-the-face model); $w(a,b,c,d;u)$ are the Boltzmann weights. For details see ref. 1).

A number of solutions to (1) are already known (most often in a slightly different formulation). For these see the references quoted in 2). However complete classification of solutions is still an entirely open problem.

2°) It is easy to see that (1) becomes a tautology for $u=0$ provided

$$(2) \qquad w(a,b,c,d;0) = \delta_{ac} .$$

Therefore it is natural to assume the expansion

$$w(a,b,c,d;u) = \delta_{ac} + \sum_{j\geq 1} W_j(a,b,c,d)u^j .$$

Since (1) is a highly overdetermined system (

q^6 equations for q^4 unknowns), the coefficients $W_j(a,b,c,d)$ are also subject to overdetermined relations. It can be shown that up to the 2^{nd} order in u and v, (1) determine $W_2(a,b,c,d)$ uniquely in terms of $W_1(a,b,c,d)$, leaving no extra constraints on the latter. To the 3^{rd} order one gets a non-trivial system of homogeneous cubic equations for $W_1(a,b,c,d)$. We call them the <u>tangential star-triangle</u> (TST) relation. They are a necessary (but a priori far from being sufficient) condition for the existence of solutions to (1) satisfying (2).

3°) Suppose that the weights $w(a,b,c,d)$ satisfy further the following $\mathbb{Z}_q$-symmetry conditions:

$$(3)\qquad \begin{matrix} d & & c \\ & \square & \\ a & & b \end{matrix} \;=\; \begin{matrix} d & & c+n \\ & \square & \\ a+n & & b \end{matrix} \;=\; \begin{matrix} d+n & & c \\ & \square & \\ a & & b+n \end{matrix} \qquad \text{for any } n$$

where the indices $a, \cdots, d, n$ are read modulo q. (i.e. take $X = \mathbb{Z}_q$). This reduces the number of unknowns and equations to q^2 and $\frac{1}{2}q^2(q^2-1)$, respectively. Put

$$w_{\alpha_1\alpha_2} = \frac{1}{q}\sum_{\nu=0}^{q-1} \zeta^{-\alpha_1\nu} \begin{array}{ccc} \nu & & \alpha_2 \\ & \boxed{} & \\ 0 & & 0 \end{array} \quad , \quad \zeta = e^{2\pi i/q} \quad .$$

Then the star-triangle relation (1) reads as

$$(4) \qquad w_\lambda(u+v)\sum_{\alpha+\beta=\mu} \zeta^{+\alpha\cdot\lambda} w_\alpha(u) w_\beta(v) = (\lambda \leftrightarrow \mu)$$

for any $\lambda, \mu \in \mathbb{Z}_q^2$

where $\alpha\cdot\lambda = \alpha_1\lambda_2 + \alpha_2\lambda_1$ for $\alpha = (\alpha_1, \alpha_2)$, $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}_q^2$.
Differentiating (4) with respect to v and setting $v = 0$, we have

$$(5)' \qquad \frac{d}{du}\bar{w}_\lambda = \sum_\alpha W_\alpha \bar{w}_{\lambda-\alpha} - \bar{w}_\lambda \sum_\alpha \zeta^{-\lambda\cdot\alpha} W_\alpha \bar{w}_{-\alpha}$$

$$\begin{aligned}(5)'' \qquad & w_\lambda w_\mu \sum_\alpha W_\alpha w_{-\alpha} (\zeta^{-\lambda\cdot\alpha} - \zeta^{-\mu\cdot\alpha}) \\ & + w_\mu w_0 \sum_\alpha W_\alpha w_{\lambda-\alpha} (\zeta^{-\mu\cdot\alpha} - 1) \\ & + w_0 w_\lambda \sum_\alpha W_\alpha w_{\mu-\alpha} (1 - \zeta^{-\lambda\cdot\alpha}) = 0 \quad .\end{aligned}$$

Here $\bar{w}_\lambda = w_\lambda / w_0$ and $W_\alpha = \frac{d}{du}\bar{w}_\alpha \big|_{u=0}$.
Finally the TST relation can be written as

$$(6) \qquad W_\lambda \sum_{\alpha+\beta=\mu} (\zeta^{\alpha\cdot\lambda} - 1)(\zeta^{\beta\cdot\lambda} - 1) W_\alpha W_\beta = (\lambda \leftrightarrow \mu)$$

for any $\lambda, \mu \neq (0,0)$.

Since (5)-(6) are necessary conditions, they can be used to classify all the solutions satisfying (2).

Actual working involves the following steps.

$$\begin{tikzcd} \boxed{\text{solve TST (6)}} \arrow[d, "\{\overline{W}_\lambda\}"] \\ \boxed{\text{solve } (5)'' + (5)'} \arrow[d, "\{W_\lambda\}"] \\ \boxed{\text{check ST (4)}} \end{tikzcd}$$

We have gone through these steps in the cases $q = 3$ and 4 (for $q = 2$, (6) is void and the solution to (4) is Baxter's 8-Vertex solution) with the aid of algebraic manipulation system (MACSYMA). The resulting classification of solutions is given in the tables of ref. 2). All the solutions are parametrized by elliptic, trigonometric or rational functions. In the course we observed the following facts by explicit computation:

(i) All the irreducible components of TST (6) are <u>linear</u> manifolds.

(ii) To each solution of TST, there always exists an actual (unique) solution of (4)

(It is the property (i) that makes the actual working to be carried through to the end ; otherwise such brute force calculation would have been impossible.)

Thus we are led to

Conjecture. The statements (i) and (ii) are valid in the most general case (not assuming the $\mathbb{Z}_q$-symmetry).

References

1) R. J. Baxter, Exactly solved models in statistical mechanics , Academic press , New York , 1982.

2) M. Jimbo and T. Miwa, Classification of solutions to the star-triangle relation for a class of 3- and 4- state IRF models, preprint NSF-ITP-84-154, UC Santa Barbara, 1984.

__________, in preparation.