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Nonlinear Evolution Equations
and
Inverse Scattering in Multidimensions

by

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In this lecture I will review some recent work done by my colleagues and me at Clarkson. I will concentrate on the basic underlying ideas and refer interested readers to suitable references for complete details; specifically background material can be found in various texts on this subject (e.g. [1] by Ablowitz and Segur) and more recent references will be given as necessary. The outline of the lecture is as follows.

1. Introductory Remarks.
2. A discussion of two separate but related issues. Namely, (a) solving certain nonlinear evolution equations in infinite space and, (b) Inverse Scattering. These are important problems having many physical applications. Moreover, they are related to each other by what I refer to as the Inverse Scattering Transform (IST). It turns out that the relationship between these problems depend crucially on the number of dimensions involved. The explanation of this phenomenon will occupy most of the lecture.
3. At the end of the lecture I will make some remarks on the possibility of solving nonlinear evolution equations in high dimensions (i.e. equations with more than two spatial and one time variable) by using the IST method as we now understand it.

The prototype nonlinear evolution equations for our purposes will be the Korteweg deVries (KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0 \tag{1}$$

in one spatial dimension, and the Kadomtsev-Petviashvili (KP) equation

$$(u_t - 6uu_x + u_{xxx})_x = -3\sigma^2 u_{yy} \tag{2}$$

in two spatial dimensions. It turns out that the sign of σ^2 is critical: there being two cases labeled by KP_I: $\sigma^2 = -1$; $KP_{II} = \sigma^2 = 1$.

Historically speaking, the KdV equation was the first equation solved (on the infinite line) by use of Inverse Scattering. Subsequently numerous other equations of physical interest in one spatial dimension were solved e.g. Nonlinear Schrödinger, sine-Gordon, Three wave interaction, Modified KdV, Boussinesq,$\cdots$. These equations are all partial differential equations. In fact, there are other equations which are discrete in space - continuous in time (differential-difference) and equations discrete in both space and time which also may be solved by IST. One other class of equations in one spatial and one time dimension fits into this scheme, namely nonlinear singular integro-differential equations; with the prototype being the so-called Intermediate Long Wave equation [2a]:

$$u_t + \frac{1}{\delta}u_x + 2uu_x + (Tu)_{xx} = 0; \tag{3}$$

$$Tu = \frac{1}{2\delta}\int_{-\infty}^{\infty} \coth \frac{\pi}{2\delta}(\xi - x)u(\xi)d\xi .$$

As $\delta \to 0$ (3) tends to the KdV equation (with appropriate coefficients) and as $\delta \to \infty$ to the so-called Benjamin-Ono equation

$$u_t + 2uu_x + (Hu)_{xx} = 0 \tag{4}$$

$$H_u = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{u(\xi)}{\xi - x}\, d\xi$$

The method to solve (4) was recently found and it's method of solution has certain features in common with some two-dimensional problems - specifically KP_I (see [2b]).

It should also be remarked that some ode's can also be solved by similar methods; specifically the classical equations of Painlevé (see [3]). I will not dwell on this aspect any further in this lecture.

In two spatial one time dimension the KP equation is only one of the equations that can be solved in infinite space. However an effective method was not realized until a short time ago. In fact new ideas had to be introduced for KP_{II} [4]. This paved the way for the development of the IST for a wide class of equations in 2 + 1 dimensions (a review of this and related work can be found in [5]). It should be mentioned that earlier work on KP_I had been done by Manakov [6a], more recently by Fokas and Ablowitz [6b] and on the three wave equation by Kaup [7]. KP_{II} and others like it depart significantly from previous work and its study has led us to develop a general method to do inverse scattering in n spatial dimensions as I will indicate in this lecture. However, the situation with regard to solving nonlinear evolution equations in spatial dimensions greater than two is still unclear. There are few cases known which seem to fit into the IST schema and, in fact, the inverse scattering results found in [8,9] shed some more light as to why in n dimensions it is difficult to isolate such multidimensional equations.

The method of solution by IST begins with the study of two compatible linear operators (Lax pairs) (L depends on one or more "potentials" or functions which we call u)

$$Lv = \lambda v \tag{5}$$

$$v_t = Mv \tag{6}$$

connected by the compatibility condition

$$L_t + [L,M] = 0 \tag{7}$$

when the flow is isospectral: $\lambda_t = 0$. (7) is the nonlinear evolution equation to be solved. λ is a spectral parameter, which as it turns out loses significance in spatial dimensions greater than one. L is a spatial operator only; with time acting as a parameter. The parametric dependence in time is what allows us to study the question of inverse scattering separately and then after this task is completed allows us to solve the relevant nonlinear equation (7). For KdV the operators are,

$$L = \frac{\partial^2}{\partial x^2} - u, \; M = (4\lambda+2u)\frac{\partial}{\partial_x} - u_x \; , \tag{8}$$

then the reader can now verify that (7) yields (1).

The direct scattering problem associated with L means given a potential, solve for relevant eigenfunctions and associated scattering data. The scattering data generally contains asymptotic information about u. The inverse problem for all the one dimensional problems I am familiar with (connected to the equations mentioned above) may be written in a simplified form. Namely, solve the following Riemann-Hilbert factorization problem on a specified contour C:

$$\mu_+(x,k)-\mu_-(x,k) = \mu_-(x,\alpha(k))V(x,k) \quad \text{on } C \quad (9)$$

$\alpha(k)$, $V(x,k)$ given on C, $\mu_\pm \to 1$ as $|k| \to \infty$ and $\mu_\pm(x,k)$ are to be meromorphic in $k \in \mathbb{C}$ off C with the poles of $\mu_\pm$ fixed and appropriate information given about the residues of $\mu_\pm$. μ is essentially an eigenfunction of v being connected to v by an exponential transformation: $\mu = ve^{\theta_L}$; θ_L depends on the form of the operator (5), and the parametric dependence of λ on k is explicitly given (μ may be matrix valued). Thus with this information given, the inverse problem is posed. Since intuitively speaking, the somewhat simpler case of $\mu_\pm$ sectionally analytic contains the essential ideas I shall simply assume <u>the scattering data</u>: $V(x,k)$ is given and we look for $\mu_\pm$ sectionally analytic. Hence the Riemann-Hilbert boundary value problem (RH) is well posed-without need for any extra information regarding the pole structure of $\mu_\pm$. With regard to the question of pure inverse scattering, the scattering data may not be what is referred to as the physical scattering data. Nevertheless in specific cases of interest, transformations between the physical scattering data and $V(x,k)$ can be found.

To be concrete the scattering problem for KdV, the one dimensional time independent Schrödinger equation has $\lambda = -k^2$:

$$v_{xx} + (k^2-u)v = 0, \ -\infty<x<\infty \qquad (10)$$

$v = \mu \, e^{-ikx}$

$$\mu_{xx} - 2ik\mu_x - u\mu = 0 . \qquad (11)$$

The direct problem (11) with $u \to 0$ sufficiently rapidly $\left(\int_{-\infty}^{\infty}(1+|x|)|u|dx<\infty\right)$.

has solutions with appropriate analytic properties namely for Im $k \gtrless 0$, $\mu\pm(x,k)$ are +/- functions of k. It may be shown that in this case the functions in (9) are given by (k) = -k

$$V(x,k)= r(k)e^{2ikx} \tag{12}$$

where r(k) has suitable smoothness and decay properties. The contour C is Imk = 0, r(k) is usually called the reflection coefficient and is a physically meaningful function. In the sense of the inverse problem, specifying r(k) determines the potential u(x); assuming no poles in $\mu_\pm$; i.e. no bound states. As mentioned earlier when bound states are included, the residue structure of the eigenfunction: $\mu_-(x,k_j)$ must be specified. Finally we note that the potential u(x) follows directly from $\mu_+(x,k)$ since $\mu_\pm$ are eigenfunctions associated with L. (In fact, for u the Schrödinger operator it may be shown that $u(x) = \partial/\partial_x(\frac{1}{\pi}\int r(k)e^{2ikx}\mu_-(x,-k)dk.)$ Similar formulae hold in the general case.

The solution of the initial value problem of KdV is obtained by noting that $r(k,t) = r(k,0)e^{8ik^3t}$. This follows from the second linear operator M: see (6), (8). The reconstruction of u(x,t) then follows from the inverse problem. In general, the scattering data V(x,k,t) also evolves simply in time e.g. $V(x,k,t) = V(y,k,0)e^{\omega(k)t}$ (when V,ω are scalars).

Schematically, we have:

$$u(x,0) \xrightarrow{\text{(Direct problem)}} \mu_\pm(x,k,t=0) \longrightarrow V(x,k,0) \xrightarrow{\text{(From M operator)}} V(x,k,t) \xrightarrow{\text{(From inverse problem)}} \mu_\pm(x,k,t) \longrightarrow u(x,t)$$

The method of solution is what is usually referred to as the Inverse Scattering Transform: IST. This program has been carried out for a surprisingly large number of physically interesting equations in one spatial dimension. In fact, the only equation in one spatial dimension mentioned above that does not have an associated inverse problem of the form (9) is the Benjamin-Ono equation (4). It shares with the KP_I equation an inverse problem of the nonlocal R-H form:

$$\mu_+(x,k) - \mu_-(x,k) = \int \mu_-(x,k')V(x,k',k)dk' \quad (13)$$

Next, I shall discuss the KP equation and its associated scattering operator L:

$$\sigma v_y + v_{xx} - u(x,y)v = 0, \quad (14)$$

where, for $Lv = \lambda v$, $\lambda = 0$ may be taken without loss of generality (by the scaling property of v). Since the analysis for the generalization:

$$\sigma v_y + \Delta v - u(x,y)v = 0 \quad (15)$$

where $\sigma = \sigma_R + i\sigma_I$, $\Delta = \sum_{\ell=1}^{n} \partial^2/\partial x_\ell^2$. $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, is a natural extension of that in two dimensions, I will discuss this case. Scattering parameters are put into (15) by looking for a function $\mu = \mu(x,y,k)$ where

$$v = \mu e^{ik\cdot x + k^2 y/\sigma}, \tag{16}$$

$$\sigma\mu_y + \Delta\mu + 2ik \cdot \nabla\mu - u\mu = 0. \tag{17}$$

and $k = k_R + ik_I \varepsilon C^n$. We shall consider $\sigma_R \neq 0$, $\sigma_R < 0$.

We look for a solution $\mu(x,y,k)$ bounded for all x,y and $\mu \to 1$ as $|k| \to \infty$. The latter condition is a convenient normalization. If we should consider (17) for $\sigma = \pm 1$ in analogy to the KPII scattering problem, we immediately notice that the dominant operator is the heat operator which is known to be illposed as an initial value problem. Even though we pose a boundary problem, immediately we are led to believe that in this case there there will be some type of unusual behavior. In fact in ref. [4,8] it is shown that the function μ for $\sigma_R \neq 0$ is nowhere an analytic function of k. Specifically $\mu = \mu(x,y,k_R,k_I)$. In particular μ is constructed from the following equation. Given a $u(x,y) \to 0$ sufficiently rapidly at ∞, (the direct problem),

$$\mu = 1 + \tilde{G}(u\mu),$$

$$\tilde{G}f = G*f = \iint G(x-x', y-y', k_R,k_I), f(x',y')dx'dy'. \tag{18}$$

The Green's function G is obtained from:

$$G(x,y,k_R,k_I) = \frac{1}{(2\pi)^{n+1}} \iint \frac{e^{i(x\cdot\xi+y\eta)}}{i\sigma\eta-\xi^2-2k\cdot\xi} d\xi d\eta$$

$$= \frac{\text{sign}(y)}{(2\pi)^n\sigma} \int e^{(y/\sigma)(\xi^2+2k\cdot\xi)+ix\cdot\xi} \Theta(-y\sigma_R(\xi^2+2(k_R + \frac{k_I\sigma_I}{\sigma_R})\cdot\xi) d\xi \tag{20}$$

where $\Theta(x) = \{1 \text{ for } x > 0,\ 0 \text{ for } x < 0\}$. In constructing (20) we have used the principle of boundedness, and have taken the Fourier transform in both x and y to find G.

Taking the DBAR derivative of (18) we find (assuming no homogeneous solution to (18)):

$$\frac{\partial\mu}{\partial\bar{k}_j} = \tilde{T}_j\mu = C_n\int e^{i\beta(\xi)}\delta(s(\xi))(\xi_j-k_{jR})T(k_R,k_I,\xi)\mu(x,y,\xi,k_I), \quad j=1,2,\dots,n$$

where

$$C_n = -\frac{1}{(2\pi)^n|\sigma_R|};\ \beta(\xi) = (x+2y\frac{k_I}{\sigma_R})\cdot(\xi-k_R);\ s(\xi)=(\xi+\frac{\sigma_I}{\sigma_R}k_I)^2-(k_R+\frac{\sigma_I}{\sigma_R}k_I)^2,$$

$$T(k_R,k_I,\xi) = \iint e^{-i\beta(\xi)}u(x,y)\mu(x,y,k_R,k_I)dx\,dy \qquad (21)$$

and $\delta(x)$ is the usual dirac delta function. Thus (21) expresses the fact that $\partial\mu/\partial\bar{k}_j$ depends linearly on μ. This is all within the context of direct scattering.

The inverse problem is: given $T(k_R,k_I,\xi)$ construct $u(x,y)$. However it is immediately transparent that there is a serious redundancy question. Namely $T(k_R,k_I,\xi)$ is a function of 3n parameters with one restriction (the reduction is due to $\delta(s(\xi))$ in (21): i.e. T will be given as a function of 3n-1 variables and we wish to construct a function $u(x,y)$ depending on n+1 variables. But for n=1, namely for the two spatial dimensional problem, the difficulty disappears.

In fact there are numerous reconstruction formulae for u available and serious restrictions on T must be imposed in order to obtain a function $u(x,y)$ vanishing at ∞. This is the admissability question.

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Our first inversion formula for μ is obtained from the generalized Cauchy formula:

$$\mu(x,y,k_R,k_I) = 1 + \frac{1}{\pi} \iint \frac{\frac{\partial\mu}{\partial\bar{k}_j}(x,y,k_R',k_I')}{k_j-k_j'} dk_{jR}'dk_{jI}' \qquad (22)$$

where we use the simplified notation $k_R' \equiv (k_{1R},\ldots,k_{jR}',\ldots,k_{nR})$ and similarly for k_I'. (22) is a linear integral equation for μ, using (21). The potential is constructed from:

$$u(x,y) = \frac{2i}{\pi}\frac{\partial}{\partial x_j} \iint \frac{\partial\mu}{\partial\bar{k}_j}(x,y,k_R',k_I')dk_{jR}'dk_{jI}'. \qquad (23)$$

But the issue arises as to what conditions on $T(k_R,k_I,\xi)$ will ensure that $u(x,y)$ is independent of the parameters k_{Ri},k_{Ii}, $i \neq j$.

For n=1 the above procedure is effective and in fact (21) reduces to ($k_1 \equiv k$):

$$\frac{\partial\mu}{\partial\bar{k}} = \frac{1}{2\pi|\sigma_R|} \operatorname{sgn}(k_R + \frac{\sigma_I}{\sigma_R}k_I)e^{i\beta(\xi_0)}\ T(k_R,k_I,\xi_0)\mu(x,y,\xi_0 k_I) \qquad (24)$$

where $\xi_0 = -(k_R + 2k_I\sigma_I/\sigma_R)$. For K-P the IST is carried forth by noting that T evolves simply in time $T(\cdot,t) = T(\cdot,0)e^{i\omega(k_R,k_I)t}$ (see [4]). It should also be noted that while KP_{II} is obtained directly $\sigma_I = 0$, $\sigma_R = -1$; the solution to KP_I can also be obtained by taking the limit: $\sigma_I = 1$, $\sigma_R \to 0-$, $\hat{k}_R = k_R$, $\hat{k}_I = k_I/\sigma_R$ ($\hat{k}_R$, $\hat{k}_I$ finite).

For n > 1 the compatibility condition $\frac{\partial^2\mu}{\partial\bar{k}_i\partial\bar{k}_j} = \frac{\partial^2\mu}{\partial\bar{k}_j\partial\bar{k}_i}$ (i $\neq$ j) leads to a nontrivial restriction on T; one which is nonlinear:

$$\mathcal{L}_{ij}(T) = N_{ij}(T) \qquad (25)$$

where

$$\mathcal{L}_{ij} = (\xi_j - k_{jR})(\frac{\partial}{\partial\bar{k}_i} + \frac{1}{2}\frac{\partial}{\partial\xi_i}) - (\xi_i - k_{iR})(\frac{\partial}{\partial\bar{k}_j} + \frac{1}{2}\frac{\partial}{\partial\xi_j}) \qquad (26a)$$

$$N_{ij}[T](k,\xi) = \int[(\xi'_j-k_{jR})(\xi_i-\xi'_i)-(\xi'_i-k_{iR})(\xi_j-\xi'_j)]\delta(s(\xi'))x$$

$$x\ T(k_R,k_I,\xi')T(\xi',k_I,\xi)d\xi'. \qquad (26b)$$

In fact there is a change of variables which allows (25) to be put in a simplified form. Without loss of generality we may consider the case of (25) with i=1, then introduce new variables $(\chi,w,w_0) \varepsilon\ C^{n-1} \times \mathbb{R}^n \times \mathbb{R}$ which parameterize the sphere $s(\xi)$, $(\chi = (\chi_2,\ldots,\chi_n)$

$$k_{1R} = \sum_{j=2}^{n} w_j\chi_{jR} - \frac{w_1}{2} - \frac{\sigma_I w_0 w_1}{2w^2}, \quad k_{jR} = -w_1\chi_{jR} - w_{j/2} - \frac{\sigma_I w_0 w_j}{2w^2}$$

$$k_{1I} = \sum_2^n w_j\chi_{jI} + \frac{\sigma_R w_0 w_1}{2w^2}, \quad k_{jI} = -w_1\chi_{jI} + \frac{\sigma_R w_0 w_j}{2w^2}$$

$$\xi_1 = \sum_2^n w_j\chi_{jR} + \frac{w_1}{2} - \frac{\sigma_I w_0 w_1}{2w^2}, \ \xi_j = -w_1\chi_{jR} + \frac{w_j}{2} - \frac{\sigma_I w_0 w_j}{2w^2} \quad (j \geq 2)\ .$$

(27)

Thus for $w_1 \neq 0$ there is a 1-1 map : $(k_R,k_I,\xi) \to (\chi,w,w_0)$ such that

$$w = \xi-k_R, \quad w_0 = 2k_I\cdot(\xi-k_R)/\sigma_R \qquad (28)$$

$$\frac{\partial}{\partial\bar{\chi}_j} = \mathcal{L}_{1j}$$

Thus (25) for i = 1, j = 2,...,n yields

$$\frac{\partial T}{\partial\bar{\chi}_j} = N_{1j}(T)(\chi,w,w_0); \quad j = 2,\ldots,n. \qquad (29)$$

Thus by inversion using the generalized Cauchy formula we have

$$\ell = T(\chi,w,w_0) - \frac{1}{\pi}\iint \frac{N_{1j}[T](\chi',w,w_0)}{\chi_j-\chi'_j}\, d\chi'_{jR}d\chi'_{jI} = \hat{u}(w,w_0). \qquad (30)$$

($\hat{u}(w, w_0)$ is the Fourier Transform of $u(x,y)$) (30) leads both to admissability criteria as well as reconstruction of $u(x,y)$. Namely given $T(k_R, k_I, \xi)$ compute $\mathcal{J}$. We conjecture that if $\mathcal{J}$ is independent of χ and j and has suitable decay properties for large w, w_0, then T is admissable. The potential is recovered from

$$u(x,y) = \mathcal{F}^{-1}(\hat{u}\ (w, w_0)), \tag{31}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier Transform. Moreover we see that reconstruction follows purely by quadratures given $T(k_R, k_I, \xi)$ on $s(\xi) = 0$.

It turns out that the physically interesting case of the time dependent Schrödinger equation in n dimensions fall out as special cases of the above result. I mention these facts in the following concluding remarks.

Remarks

(1) The nonlinear equation for T: (25) precludes the existence of a simple flow in time for the scattering data i.e. $T(\cdot,0) \neq T(\cdot,t)e^{i\omega t}$. This provides still another explanation for why no local nonlinear evolution equations have been associated with (15).

(2) The results for the time dependent Schrödinger equation $\sigma = i$ in (15) are obtained via the limit: $\sigma_I = 1$, $\sigma_R \to 0-$, $\hat{k}_R = k_R$, $\hat{k}_I = k_I/\sigma_R$, $\hat{k}_R$, $\hat{k}_I$ finite.

(3) The results for the time independent Schrödinger equation are obtained by specializing those of (2) to the case of $u = u(x)$. In fact we find that our results then agree with those of Faddeev 1966, 1976 for the three dimensional case. However it should be remarked that the admissability

criterion we give is somewhat different from that of Faddeev - which is given in terms of certain analyticity behavior of T. It must also be mentioned that $T(k_R,k_I,\xi)$ is not the usual scattering data. Namely the so called Scattering Amplitude $A(k_R,\xi)$ is the physically measurable data and hence the physical scattering data. Nevertheless there is a linear integral equation relating T to A which holds on the energy shell. Similar remarks can be made for the scattering theory associated with the time dependent Schrödinger problem.

(4) Although here we have discussed the analysis for the generalized Schrödinger scattering problem, the algorithm also works other operators in a straight forward way. In [9] the scattering problem

$$V_{x_0} + \sigma \sum_{\ell=1}^{n} J_\ell V_{x_\ell} = u(x_0,x)V, \tag{32}$$

with $\sigma = \sigma_R + i\sigma_I$, $x \in \mathbb{R}^n$, $x_0 \in \mathbb{R}$, u an mxm matrix, and $J_\ell = \mathrm{diag}(J_\ell^1,\ldots,J_\ell^m)$. Again results analogous to (25) generically follow; i.e. the scattering data satisfies a nonlinear constraint. (32) is one of the few operators that has a compatible time evolution operator and hence a Lax pair describing a nonlinear evolution equation in n dimensions - the so called m-wave interaction equation. However the compatible equations follow only if certain restrictions are put on J_ℓ^i: namely that the vectors $\vec{J}^i = (J_1^i,J_2^i,\ldots,J_n^i)$ are all colinear. In this case the coefficient of the nonlinear term in the equation for T vanishes - i.e. the analogy to (25) is now purely linear and allows a simple flow, in time. The m wave equation follows. Nevertheless despite outward appearances the m-wave interaction equation can be reduced to two spatial dimensions [11].

(5) As a final remark, if one is interested in perhaps very complicated nonlocal nonlinear equations which are linearizable, then (25) and its generalizations clearly provide such models.

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