On the Semantics of Infinite Computations in Logic Programs

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Abstract In the classical semantics, successful finite computations in logic programs are characterized by the least fixedpoint. We characterize infinite computations by the greatest fixedpoint on the complete Herbrand universe.

1. Logic Programs

Firstly we give some basic definitions.

Definition A term is a variable, a constant or \( f(t_1, \ldots, t_n) \), where \( f \) is an \( n \)-ary function symbol and \( t_1, \ldots, t_n \) are terms. An atom is \( p(t_1, \ldots, t_n) \), where \( p \) is an \( n \)-ary predicate symbol and \( t_1, \ldots, t_n \) are terms. A clause is a pair of sets of atoms of the form: \( A_1, \ldots, A_m \leftarrow B_1, \ldots, B_n \) \((m \geq 0, n \geq 0)\), which is to be understood as: for all \( x_1, \ldots, x_k \), \( A_1 \lor \cdots \lor A_m \lor \neg B_1 \lor \cdots \lor \neg B_n \), where \( x_1, \ldots, x_k \) are all variables in the clause. A definite clause is a clause where \( m = 1 \). A goal clause is a clause where \( m = 0 \). An empty clause is a clause where \( m = 0 \) and \( n = 0 \) which is to be understood as a contradiction. A logic program is a finite set of definite clauses.

Definition A substitution is an operation, say \( \theta \), which replaces simultaneously each occurrence of the variable in an atom \( A \) by a term. The result, denoted by \( A\theta \), is called an instance of \( A \) by \( \theta \). A substitution \( \theta \) is called a unifier for atoms \( A_1, \ldots, A_n \) if \( A_1 \theta = \ldots = A_n \theta \). A unifier \( \theta \) for given atoms is
called a **most general unifier** (m.g.u. for short) if for each unifier $\sigma$ of those atoms, there is a substitution $\gamma$ such that $\sigma = \theta \gamma$.

**Definition** Let $P$ be a logic program. The **Herbrand universe** $HU$ for $P$ is the set of all ground terms which can be formed out of the constants and functions in $P$. The **Herbrand base** $HB$ for $P$ is the set of all ground atoms which can be formed out of predicates, functions and constants in $P$.

A **Herbrand interpretation** is an interpretation whose domain is the Herbrand universe $HU$ and in which constants and functions are literally interpreted. We can identify it with a subset of the Herbrand base.

Next we introduce SLD-resolution which is a special-purpose resolution used as interpreter of logic programs. The procedural semantics of logic programs is defined using this SLD-resolution.

**Definition** A **computation rule** is a rule to choose an atom, called the **selected atom**, out of any nonempty goal.

Let $G$ be $\leftarrow A_1, \ldots, A_m, \ldots, A_k$ and $C$ be $A \leftarrow B_1, \ldots, B_q$ and $R$ be a computation rule. Then $G'$ is said to be **derived** from $G$ and $C$ with $\theta$ via $R$ if the following conditions hold:

1. $A_m$ is the selected atom by $R$.
2. $\theta$ is a m.g.u. of $A$ and $A_m$.
3. $G'$ is the goal $\leftarrow (A_1, \ldots, A_{m-1}, B_1, \ldots, B_q, A_m+1, \ldots, A_k) \theta$.

Let $P$ be a logic program, $G$ be a goal and $R$ be a computation rule. An **SLD-derivation** (simply **derivation**) of $P \cup \{G\}$ via $R$ consists of a finite or infinite sequence $G=G_0, G_1, \ldots$ of goals, a sequence $C_1, C_2, \ldots$ of variants of clauses in $P$ and a sequence $\theta_1, \theta_2, \ldots$ of m.g.u.'s, such that each $G_{i+1}$ is derived from $G_i$. 

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and $C_{i+1}$ with $\theta_{i+1}$ via R. Each variant $C_1, C_2, \ldots$ is called an input clause of the derivation.

A ground derivation is same as the derivation except that the substitutions $\theta_i$ are unifiers (not necessary to be m.g.u.) and goals $G_i$ are ground (i.e. variable-free).

**Definition** An SLD-refutation (simply refutation) of $P \cup \{G\}$ is a finite derivation of $P \cup \{G\}$ which has the empty clause $\square$ as the last goal in the derivation.

The success set of $P = \{A \in HB : P \cup \{\neg A\} \text{ has a refutation}\}$.

**Theorem 1.1 (Soundness and completeness of refutations)** ([1],[4])

Let $P$ be a logic program and $G$ a goal. Then, there exists a refutation of $P \cup \{G\}$ iff $P \cup \{G\}$ is unsatisfiable.

**Definition** Let $P$ be a logic program and $G$ a goal. We say a derivation of $P \cup \{G\}$ is finitely failed if the derivation is finite and ends with a goal where the selected atom does not unify with the head of any clause in $P$.

The finite-failure set of $P = \{A \in HB : \text{all derivations of } P \cup \{\neg A\} \text{ via a computation rule are finitely failed} \}$.

From the "negation as failure" point of view, we infer $\neg A$ if $A$ is in the finite-failure set of $P$.

**Definition (fairness)** ([5]) A derivation is fair if, for every atom $B$ in the derivation, some instantiated copy is selected within a finite number of steps. Note that a fair derivation is finite iff it is a refutation.

**Theorem 1.2** ([5]) Let $P$ be a logic program and $G$ a goal. If all
derivations of \( P^+ G \) via a computation rule are finitely failed, then \( P^+ G \) has no fair derivation.

2. Least fixedpoint semantics

In the fixedpoint semantics for conventional programs, the semantics of a recursively defined procedure is defined to be the least fixedpoint of the transformation associated with the procedure definition. Here we give a similar definition of fixedpoint semantics for logic programs.

Firstly, with a logic program \( P \) we associate a mapping \( T_P \) from the set \( 2^{HB} \) of all Herbrand interpretations of \( P \) to itself. It provides the link between the declarative and procedural semantics of \( P \).

**Definition** Let \( P \) be a logic program. The mapping \( T_P : 2^{HB} \to 2^{HB} \) is defined as

\[
T_P(I) = \{ A \in HB : \text{there is a clause } B_0 \leftarrow B_1, \ldots, B_n \ (n \geq 0) \text{ in } P \\
\text{such that } A = B_0 \theta \text{ and } B_1 \theta, \ldots, B_n \theta \in I \text{ for some ground substitution } \theta \}.
\]

**Lemma 2.1** The mapping \( T_P \) is monotonic in the sense that \( I_1 \subseteq I_2 \) implies that \( T_P(I_1) \subseteq T_P(I_2) \), for any \( I_1 \) and \( I_2 \) in \( 2^{HB} \).

**Theorem 2.2 (the Knaster-Tarski fixedpoint theorem [7])** Let \( L \) be a complete lattice and \( T:L \to L \) a mapping. Then a monotonic mapping \( T \) has a greatest fixedpoint (\( gfp(T) \), for short) and a least fixedpoint (\( lfp(T) \), for short). Furthermore,

\[
gfp(T) = \bigcup \{ I : I = T(I) \} = \bigcup \{ I : I \subseteq T(I) \},
\]

\[
lfp(T) = \bigcap \{ I : I = T(I) \} = \bigcap \{ I : T(I) \subseteq I \}.
\]

We define the following ordinal powers of \( T_P \):

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\[ T_p \downarrow 0 = \emptyset \]
\[ T_p \uparrow n = T_p(T_p \uparrow (n-1)) \quad \text{if } n \text{ is a successor ordinal}, \]
\[ = \bigcup (T_p \uparrow k: k < n) \quad \text{if } n \text{ is a limit ordinal} : \]
\[ T_p \downarrow 0 = \text{HB} \]
\[ T_p \downarrow n = T_p(T_p \downarrow (n-1)) \quad \text{if } n \text{ is a successor ordinal}, \]
\[ = \bigcap (T_p \downarrow k: k < n) \quad \text{if } n \text{ is a limit ordinal}. \]

We have that \( T_p \uparrow \omega \subseteq \text{lfp}(T_p) \subseteq \text{gfp}(T_p) \subseteq T_p \downarrow \omega \)

Lemma 2.3 The mapping \( T_p \) is continuous in the sense that for every increasing chain \( I_1 \subseteq I_2 \subseteq \ldots \) of elements of \( 2^{\text{HB}} \),
\[ T(\bigcup \{ I_i: i < \omega \}) = \bigcup \{ T(I_i): i < \omega \}. \]

Now we have a theorem which provides a fixedpoint characterization of the success set of a logic program.

Theorem 2.4 (Fixedpoint characterization of success set [4])
Let \( P \) be a logic program. Then
\[ \text{the success set of } P = \text{lfp}(T_p) = T_p \uparrow \omega. \]

As for the greatest fixedpoint, however we may not have \( \text{gfp}(T_p) = T_p \downarrow \omega \). Indeed, \( T_p \downarrow \omega \) may not be a fixedpoint of \( T_p \). The following example is such a \( T_p \) in [11].

Example Consider the logic program \( P \)
\[ P = \{ p(a) \leftarrow p(x), q(x) \}. \]
\[ p(s(x)) \leftarrow p(x). \]
\[ q(b) \leftarrow -. \]
\[ q(s(x)) \leftarrow q(x). \} . \]

For all finite \( n \) we have
\[ T_p \downarrow n = \text{HB}-(q(a), \ldots, q(s^{n-1}(a)), p(b), \ldots, p(s^{n-1}(b)) \} . \]
Hence $T_p \downarrow \omega = (p(s^n(a)) : n \in \omega) \cup (q(s^n(b)) : n \in \omega)$. Now $p(a) \notin T_p(T_p \downarrow \omega)$, therefore $T_p \downarrow \omega \neq \operatorname{gfp}(T_p)$. In fact, $T_p^n(T_p \downarrow \omega) = T_p \downarrow \omega - (p(s^i(a)) : i(n))$ for finite $n$, and we have $T_p \downarrow (\omega + \omega) = (q(s^n(b)) : n \in \omega) = \operatorname{gfp}(T_p) = \operatorname{lfp}(T_p)$.

**Theorem 2.5** ([1],[5]) Let $P$ be a logic program. The finite-failure set of $P$ is the complement in HB of $T_p \downarrow \omega$.

3. **Greatest fixedpoint semantics**

Consider the following logic program $P$ and the goal $G$:

$$P = (p(f(x)) \leftarrow \neg p(x)). \quad G = \langle \neg p(x) \rangle.$$

The SLD-resolution for this $P \cup \{G\}$ yields infinite derivation and one might expect its answer to be $(f(f(...))/x)$. However in the classical fixedpoint semantics, $\operatorname{lfp}(T_p) = \operatorname{gfp}(T_p) = \Phi$, that is, $\Phi$ is the only fixedpoint, and infinite computations are not taken into account. In this section, we extend the Herbrand universe so that it includes infinite terms, and we will characterize infinite computations by the greatest fixedpoint.

Let $P$ be a logic program, $F$ be the set of functions and constants in $P$ and $V$ be a (finite or infinite) set of variables. First we need the definitions of infinite term and infinite atom to make ready for discussions of infinite computations. We denote the set of all (possibly infinite) trees over a ranked alphabet $X$ by $M^{\infty}(X)$ (in [3]).

**Definition** An infinite term is an element of $M^{\infty}(F \cup V)$. An infinite atom $A$ is $p(t_1, \ldots, t_n)$, where $p$ is an $n$-ary predicate symbol in $P$ and $t_i \in M^{\infty}(F \cup V)$ ($1 \leq i \leq n$). The complete Herbrand universe $HU'$ for $P$ is the set of all ground infinite terms. The complete Herbrand base $HB'$ for $P$ is the set of all ground atoms.

We extend a substitution, a unifier and a m.g.u. to those
for infinite atoms. Note that for unifiable infinite atoms, a
m.g.u. always exists as in the case of finite atoms. See [3] for
details.

In the sequel we adopt the convention that "term" and "atom"
will always mean possibly infinite term and atom. If a term or an
atom is finite, this will always be explicitly stated.

A complete Herbrand interpretation for \( P \) is an
interpretation whose domain is the complete Herbrand universe
and in which constants and functions are literally interpreted.
In an analogous way to HB, the set of all complete Herbrand
interpretations for \( P \) can be identified with \( 2^{HB'} \). We also define
the mapping \( T_P : 2^{HB'} \rightarrow 2^{HB'} \) as

\[
T_P'(I) = (A \in HB' : \text{there is a clause } B_0 \preceq B_1, \ldots, B_n \ (n \geq 0) \text{ in } P \\
\text{such that } A = B_0 \theta \text{ and } B_1 \theta, \ldots, B_n \theta \in I \text{ for some} \\
ground substitution \theta).
\]

Then what we want to show is the important property of \( T_P' \)
that \( gfp(T_P') = T_P' \downarrow \omega \). For this goal, we present some properties
of \( T_P \) and \( T_P' \).

**Lemma 3.1** Let \( P \) be a logic program. For \( A \in gfp(T_P) \), \( P \cup \{\langle -A \rangle\} \) has
a (possibly infinite) ground fair derivation.

**Proof:** For \( A \in gfp(T_P) \), we construct a ground fair derivation of
\( P \cup \{\langle -A \rangle\} \), \( \langle -A \rangle \leftarrow \langle -N_0 \rangle, \langle -N_1 \rangle, \langle -N_2 \rangle, \ldots \) inductively. Since \( gfp(T_P) \) is
the fixedpoint of \( T_P \) and a set of ground atoms in HB, it is
enough for us to get \( N_n \) such that \( \langle N_n \rangle \subseteq gfp(T_P) \ (n \geq 0) \).

For \( n=0 \), since \( A \in gfp(T_P) \), clearly \( \langle N_0 \rangle \subseteq gfp(T_P) \). Now
suppose that we set \( N_{n-1} = \langle C_1, \ldots, C_m \rangle \), \( m \geq 1 \), and its selected atom
(by a fair computation rule) is \( C_1 \). By the inductive hypothesis,
\( C_1 \in gfp(T_P) \) and so \( C_1 \in T_P(gfp(T_P)) \). Then there exists a clause
\( D_0 \preceq D_1, \ldots, D_q \) in \( P \) such that \( C_1 = D_0 \theta \) and \( D_1 \theta, \ldots, D_q \theta \in gfp(T_P) \).
for some ground substitution \( \theta \). We define
\[
N_n = (C_1, \ldots, C_{i-1}, D_1, \ldots, D_q, C_{i+1}, \ldots, C_m) \theta.
\]
Then clearly \((N_n) \subseteq \text{gfp}(T_p)\). Thus \(P \cup \{\lnot A\}\) has a ground fair derivation. \(\Box\)

Lemma 3.2 Let \(P\) be a logic program. For \(A \in \text{HB}\), if \(P \cup \{\lnot A\}\) has a (possibly infinite) ground fair derivation, then \(A \in \text{gfp}(T_p)\).

Proof: For a ground fair derivation \(\lnot A = \lnot N_0, \lnot N_1, \lnot N_2, \ldots\) of \(P \cup \{\lnot A\}\), let \(J = \cup (N_i : i \leq \omega)\) (If the derivation is finite of length \(n\), then \(J = \cup (N_i : i \leq n)\)). Since the derivation is fair, it is clear that \(J \subseteq T_p(J)\). Thus \(J \subseteq \cup (I : I \subseteq T_p(I)) = \text{gfp}(T_p)\). Hence \(A \in J \subseteq \text{gfp}(T_p)\). \(\Box\)

Example Consider the following logic program \(P\)
\[
P = \{p(s(x)) \lhd p(x), q(0) \lhd p(x).\}
\]
We have the following infinite fair derivation of \(P \cup \{\lnot q(0)\}\)
\[
\lnot q(0), \lnot p(x_1), \lnot p(x_2), \lnot p(x_3), \ldots
\]
but no ground fair derivation. We only have the finitely failed ground derivations
\[
\lnot q(0), \lnot p(n), \lnot p(n-1), \ldots, \lnot p(0).
\]

For a sequence \(\theta_1, \theta_2, \theta_3, \ldots\) of substitutions in a derivation, when the sequence \(\theta_1, \theta_1 \theta_2, \theta_1 \theta_2 \theta_3, \ldots\) of substitution compositions converges to a substitution \(\theta\), we denote \(\theta\) by \(\lim_n(\theta_1 \cdots \theta_n)\).

Lemma 3.3 (m.g.u. lemma) Let \(P\) be a logic program and \(A\) an atom. Suppose that \(P \cup \{\lnot A\}\) has an (possibly infinite) ground fair derivation. Then \(P \cup \{\lnot A\}\) has a (possibly infinite) fair derivation.
Now we come to the main result of this chapter that $T_p \downarrow \omega = \text{gfp}(T_p')$. The results of lemmas 3.1, 3.2, theorems 2.5 and 1.2 can easily be extended to $T_p'$ and we use the extended versions of them in the proofs of next theorems.

Theorem 3.4 Let $P$ be a logic program. For $A \subseteq HB'$, there is a (possibly infinite) fair derivation of $P \cup \{\neg A\}$ iff $A \subseteq \text{gfp}(T_p')$.

Proof: (only-if part) Suppose that $P \cup \{\neg A\}$ has a (possibly infinite) fair derivation $\neg A = \neg N_0, \neg N_1, \neg N_2, \ldots$ with the sequence $\theta_1, \theta_2, \theta_3, \ldots$ of most general unifiers and the sequence $C_1, C_2, \ldots$ of input clauses. By lemma 3.2, it suffices to show that $P \cup \{\neg A\}$ has the (possibly infinite) ground fair derivation.

Without loss of generality, we can assume that $\lim_m (\theta_i^{i+1} \ldots \theta_i^{i+m}) (i \geq 0)$ exists. Take a ground substitution $\sigma$ and define $N_i' = N_i \lim_m (\theta_i^{i+1} \ldots \theta_i^{i+m}) \sigma$ for each $i \geq 0$. Then it is clear that $N_i' \subseteq HB'$ ($i \geq 0$). Next we confirm that $\neg N_0', \neg N_1', \neg N_2', \ldots$ is a ground fair derivation from $\neg A$ with unifiers $\lim_m (\theta_1 \ldots \theta_m) \sigma, \lim_m (\theta_2 \ldots \theta_m) \sigma, \ldots$ and input clauses $C_1, C_2, \ldots$ by induction on the derivation steps.

First note that $N_n' = N_n = A$. Suppose that we get $N_{n-1}' = \lim_m (\theta_n \ldots \theta_{n+m}) \sigma$, where $N_{n-1}' = (B_1, \ldots, B_k)$. $B_i$ is its selected atom. $C_n$ is $D_0 \prec D_1 \ldots, D_q$ and $N_n = (B_1, \ldots, B_{n-1}, D_1, \ldots, D_q, B_{n+1}, \ldots, B_k) \theta_n$ for the m.g.u. $\theta_n$ such that $B_i \theta_n = D_0 \theta_n$. Then $\lim_m (\theta_n \ldots \theta_{n+m}) \sigma$ is also a unifier of $B_i$ and $D_0$, and from $N_{n-1}' = (B_1, \ldots, B_k) (\lim_m (\theta_n \ldots \theta_{n+m}) \sigma)$, we can derive

\[
\begin{align*}
(B_1, \ldots, B_{n-1}, D_1, \ldots, D_q, B_{n+1}, \ldots, B_k) & (\lim_m (\theta_n \theta_{n+1} \ldots \theta_{n+m}) \sigma) \\
& = (B_1, \ldots, B_{n-1}, D_1, \ldots, D_q, B_{n+1}, \ldots, B_k) \theta_n (\lim_m (\theta_{n+1} \ldots \theta_{n+m}) \sigma) \\
& = N_n (\lim_m (\theta_{n+1} \ldots \theta_{n+m}) \sigma)
\end{align*}
\]
Hence we can get a (possibly infinite) ground fair derivation \(-A \leftarrow -N'_0 \leftarrow \cdots \leftarrow -N'_{i-1} \leftarrow \cdots \leftarrow -N'_{2} \leftarrow \cdots\) of \(P \cup \{\leftarrow -A\}\).

(if part) It is clear from lemma 3.1 and m.g.u. lemma. \(\square\)

**Corollary 3.5** Let \(P\) be a logic program. Then we have

\[
T_P \downarrow \omega = \text{gfp}(T_P').
\]

Proof: Straightforward by theorems 3.4, 2.5 and 1.2. \(\square\)

Corollary 3.5 is also obtained by a topological approach in [6].

4. **Answer substitutions of infinite computations**

Let \(P\) be a logic program, \(G\) a goal and \(\theta_1, \theta_2, \ldots, \theta_n, \ldots\) the sequence of m.g.u.'s in a derivation of \(P \cup \{G\}\). An **answer substitution** for \(P \cup \{G\}\) is the substitution obtained by restricting the composition \(\lim_n(\theta_1 \cdots \theta_n)\) (or \(\theta_1 \cdots \theta_n\) if the sequence is finite of length \(n\)) to the variables of \(G\) [1]. Then we have the following result, a stronger version of theorem 3.4.

**Theorem 4.1** Let \(P\) be a logic program and \(A\) an atom.

1. If \(\theta\) is the answer substitution of a fair derivation of \(P \cup \{\leftarrow -A\}\), then \(A \theta \sigma \in \text{gfp}(T_P')\) for a ground substitution \(\sigma\).
2. For every substitution \(\theta\) such that \(A \theta \in \text{gfp}(T_P')\), there is a fair derivation of \(P \cup \{\leftarrow -A\}\) with the answer substitution \(\theta'\) such that \(\theta = \theta' \sigma\) for some substitution \(\sigma\).

Proof: (1) is shown exactly as theorem 3.4.

(2) By theorem 3.4, \(P \cup \{\leftarrow -A\}\) has a fair derivation. Then, by lifting lemma (see, for example, [2]), we get the conclusion. \(\square\)
Example Consider the following logic program $P$ which computes the infinite "factorial" sequence from 1.

\[ P = \{ \text{fact}(z) \leftarrow \text{integer}(1,n), \text{fact}(n,1,z). \right. \]
\[ \left. \text{fact}(x,n,y,v,w) \leftarrow \text{mult}(x,y,v), \text{fact}(n,v,w). \right. \]
\[ \left. \text{integer}(n,n,x) \leftarrow \text{plus}(n,1,y), \text{integer}(y,x). \right\} \]

where the argument of fact is the factorial sequence.

(Note the convention that $\text{plus}(x,y,z)$ means $x+y=z$ and $\text{mult}(x,y,z)$ means $x\cdot y=z$).

For the goal $\leftarrow \text{fact}(z)$, we get the answer substitution $z=1.2.6.24.120, \ldots$ through a fair derivation. Then clearly $\text{fact}(1.2.6.24.120, \ldots) \in \text{gfp}(T'_p)$.

5. Concluding remarks

As we have seen, taking the complete Herbrand universe as the domain seems to be the simplest and most natural way of providing a semantics for infinite computations. Then $\text{gfp}(T'_p)$ is the analogue of the semantics $\text{lfp}(T_p)$ for (ordinary) terminating logic programs and a fair derivation is the analogue of a refutation.

The characterization of infinite computations remains an open question. The problem is to find the appropriate sense of an infinite computation being "useful". Then finding a satisfactory semantics for useful infinite computations is also a research problem. The greatest fixedpoint semantics gives a nonempty denotation not only to non-terminating logic programs which compute infinite terms, but also to "bad (or not useful)" non-terminating logic programs (for example, "loop" programs). It is a further work to give a good fixedpoint semantics for infinite computations.
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