数理解析研究所講究録
556巻 1985年 171-176

Similarity Relation between Automata Networks

オートマタネットワーク間の類似性について

Takashi SAITO and Hidenosuke NISHIO

斉藤 隆　　　　西尾 英之助

Department of Biophysics, Faculty of Science, Kyoto University

1. Introduction

In previous papers [2,3,4] we discussed the structure and behavior of automata network by using two kinds of equivalence relations on the vertex set of network. In this paper we discuss the "similarity" between automata networks by using the similarity relation. This similarity relation is not only an extension of structural homomorphism between automata networks [1] but also an extension of the structural equivalence relation of automata network.

2. Similarity relation between automata networks

<u>Definition 1</u> Edge-labeled directed graph G

An edge-labeled graph (simply a graph) G is defined by $G = (V', E)$, where V' is a set of finite or infinite number of vertices with conditions $V' = V \cup V_I$, $V \cap V_I = \phi$ and $V \neq \phi$, where V is called the set of inner vertices and V_I the set of input vertices. E is a set of labeled edges, i.e. $E = (E_1, E_2, \ldots, E_k)$ where $E_i \subseteqq V \times V'$ and for every vertex $u \in V$ there is at most one vertex $v \in V'$ such that $(u,v) \in E_i$ for $i = 1,2,\ldots,k$.

We also write $E_i(u) = v$ if $(u,v) \in E_i$, which means that there is an edge labeled with i from v to u.

When we consider the automata network with output, we specify the set of output vertices $V_0 \subseteqq V$. If $V_I = \phi$ (i.e. $V' = V$) then the graph G, denoted by (V,E), becomes the base of an autonomous automata network.

Definition 2 Colored network A

Suppose that a graph G is given. Then we can define a colored network $A = (G,\beta)$ by coloring the vertices.

That is, in $A = (G,\beta)$ G is an edge-labeled directed graph and β is a vertex coloring mapping, i.e. $\beta: V \rightarrow H$ where H is the set of colors. If $\beta(u) = \beta(v)$ for $u,v \in V$ then indegree of u is equal to that of v.

Note: If we treat the ordinary automata network such that automata are allocated to vertices, we define the mapping γ such that $\gamma:H \rightarrow M$ where M is the set of automata. But in this paper, we discuss the properties that are connected with to which vertices identical automata are allocated, not with what actual automata are allocated. So we don't use the mapping γ.

Definition 3 Similarity relation

Let $A = (G,\beta)$ and $\tilde{A} = (\tilde{G},\tilde{\beta})$ be two colored networks where $G = (V \cup V_I,E)$ and $\tilde{G} = (\tilde{V} \cup \tilde{V}_I,\tilde{E})$. A relation $R \subseteqq V \times \tilde{V}$ is defined to be similarity relation if and only if if $(u,\tilde{u}) \in R$ then $\beta(u) = \beta(\tilde{u})$ and $(E(u),\tilde{E}(\tilde{u})) \in R$.

Note: In above definition we use the abbreviation $(E(u),\tilde{E}(\tilde{u})) \in R$ for $(E_i(u),\tilde{E}_i(\tilde{u})) \in R$ for every i. If $E_i(u) \in V_I$ or $\tilde{E}_i(\tilde{u}) \in \tilde{V}_I$, we formally define that $(E_i(u),\tilde{E}_i(\tilde{u})) \in R$ only in the case where $E_i(u) \in V_I$ and $\tilde{E}_i(\tilde{u}) \in \tilde{V}_I$, and $(E_i(u),\tilde{E}_i(\tilde{u})) \notin R$ in the other cases. If $E_i(u)$ or $\tilde{E}_i(\tilde{u})$ is undefined, we formally define that $(E_i(u),\tilde{E}_i(\tilde{u})) \in R$ only in the case where $E_i(u)$ and $\tilde{E}_i(\tilde{u})$ are both undefined, and $(E_i(u),\tilde{E}_i(\tilde{u})) \notin R$ in the other cases.

It is clear that if $\tilde{A} = A$ then every SER(structural equivalence relation) of A is a similarity relation between A and A. Since a similarity relation needs not satisfy the reflexive, symmetric or transitive low, there exists a similarity relation R which is not an SER.

First we show that a similarity relation between A and $\tilde{A}$ is a natural extension of a structural homomorphism between A and $\tilde{A}$. Of course if h is a structural homomorphism between A and $\tilde{A}$ then the relation $R = \{(u,h(u)) | u \in V\}$ is the similarity relation. We connect a similarity relation with a mapping in a usual way.

<u>Definition 4</u> Induced function

Let $R \subseteqq V \times \tilde{V}$ be a similarity relation. We define an induced function $f_R : V \to 2^{\tilde{V}}$ ($2^{\tilde{V}}$ is the power set of $\tilde{V}$) as follows: for $u \in V$ $f_R(u) = \{\tilde{u} \in \tilde{V} | (u,\tilde{u}) \in R\}$.

<u>Definition 5</u> Induced relation

Let $f : V \to 2^{\tilde{V}}$. An induced relation $R_f \subseteqq V \times \tilde{V}$ is defined as follows: If $\tilde{u} \in f(u)$ then $(u,\tilde{u}) \in R_f$.

<u>Proposition 1</u>

If R is a similarity relation then f_R satisfies that $\tilde{E} \circ f_R \subseteqq f_R \circ E$ and $\beta(u) = \tilde{\beta}(\tilde{u})$ for $\tilde{u} \in f_R(u)$.

Note: Expression $\tilde{E} \circ f_R \subseteqq f_R \circ E$ means that for every $u \in V$ $\tilde{E}_i(f_R(u)) \subseteqq f_R(E_i(u))$ for every i. For a set $U \subseteqq V$ we define that $f_R(U) = \bigcup_{u \in U} f_R(u)$ as usual, and also define $\tilde{E}_i(\tilde{U}) = \bigcup_{\tilde{u} \in \tilde{U}} \tilde{E}_i(\tilde{u})$ [or $E_i(U) = \bigcup_{u \in U} E_i(u)$] for $\tilde{U} \subseteqq \tilde{V}$ [or $U \subseteqq V$].

<u>Proof</u> Let $\tilde{v} \in E_i(f_R(u))$ for some $u \in V$. Then we can choose $\tilde{u} \in f_R(u)$ such that $\tilde{v} = \tilde{E}_i(\tilde{u})$. So $(u,\tilde{u}) \in R$ from the definition. Then $(E_i(u),E_i(\tilde{u})) \in R$ and $\beta(u) = \tilde{\beta}(\tilde{u})$. Therefore $\tilde{v} = \tilde{E}_i(\tilde{u}) \in f_R(E_i(u))$. ∎

<u>Proposition 2</u>

Let $f : V \rightarrow 2^{\tilde{V}}$. If $\tilde{E}\circ f \subseteqq f\circ E$ and $\beta(u) = \tilde{\beta}(\tilde{u})$ for $\tilde{u} \in f(u)$ then R_f is a similarity relation.

<u>Proof</u> Suppose $(u,\tilde{u}) \in R_f$. From $\tilde{E}(f(u)) \subseteqq f(E(u))$ and $\tilde{u} \in f(u)$, we obtain $\tilde{E}(\tilde{u}) \in f(E(u))$. Then $(E(u),\tilde{E}(\tilde{u})) \in R_f$. ∎

Next we show how the structure of A correspond to that of $\tilde{A}$ by the similarity relation between A and $\tilde{A}$.

<u>Definition 6</u> Structural covering

Let $A = (G,\beta)$. A family of subset of V $\{C_\omega\}_{\omega\in\Omega}$ where Ω is a index set and $C_\omega \subseteqq V$ is a structural covering of A if and only if if $u,v \in C_\omega$ then $\beta(u) = \beta(v)$ and for every i there exists $C_{\omega'}$ such that $E_i(u) \in C_{\omega'}$ and $E_i(v) \in C_{\omega'}$.

<u>Proposition 3</u>

A similarity relation R between A and $\tilde{A}$ induces the structural covering of A and $\tilde{A}$.

<u>Proof</u> It is clear that $\{C_{\tilde{u}}\}_{\tilde{u} \in \tilde{V}}$ where $C_{\tilde{u}} = \{u \in V | (u,\tilde{u}) \in R\}$ and $\{\tilde{C}_u\}_{u \in V}$ where $\tilde{C}_u = \{\tilde{u} \in \tilde{V} | (u,\tilde{u}) \in R\}$ are structural coverings of these networks. ∎

<u>Proposition 4</u>

Let $\{C_\omega\}_{\omega\in\Omega}$ be structural covering of A. If $\bigcup_\omega C_\omega = V$ and $C_\omega = C_{\omega'}$ or $C_\omega \cap C_{\omega'} = \phi$ for every ω and ω' then this covering defines the SER of A.

Let A and $\tilde{A}$ be given. To investigate the relationship between similarity relation and SER, we assume that similarity relation satisfies the following property: Let $\{C_{\tilde{u}}\}_{\tilde{u} \in \tilde{V}}$ and $\{\tilde{C}_u\}_{u \in V}$ be structural coverings induced by similarity relation between A and $\tilde{A}$, respectively. Then $\bigcup_{\tilde{u}} C_{\tilde{u}} = V$ and $\bigcup_{u} \tilde{C}_u = \tilde{V}$.

<u>Proposition 5</u>

Let R be a similarity relation between A and $\tilde{A}$, and let $\{C_{\tilde{u}}\}_{\tilde{u} \in \tilde{V}}$ and $\{\tilde{C}_u\}_{u \in V}$ be the structural coverings induced by R. If $\{C_{\tilde{u}}\}_{\tilde{u} \in \tilde{V}}$ [or $\{\tilde{C}_u\}_{u \in V}$] defines the SER of A [or $\tilde{A}$] then $\{\tilde{C}_u\}_{u \in V}$ [or $\{C_{\tilde{u}}\}_{\tilde{u} \in \tilde{V}}$] also defines the SER of $\tilde{A}$ [or A]. Moreover in this case, the blocks of SER of A correspond to those of $\tilde{A}$ injectively.

<u>Proof</u> We express a similarity relation by using the matrix-like expression as illustration in Fig.1. So if the condition of proposition holds then this expression looks like Fig.2. Then the first part of this proposition is trivial and the correspondence between the blocks is also expressed like in Fig.2. ∎

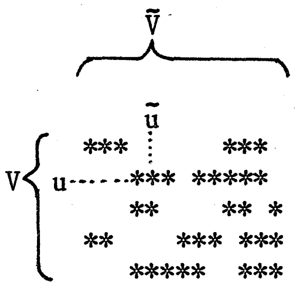

Fig.1

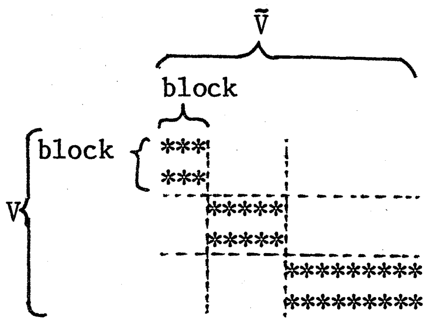

Fig.2

3. Concluding Remarks

We defined the similarity relation between automata networks and showed some of its fundamental properties. But it is just at the preliminary stage and there are many to be done in the future. It seems the structural covering may play an important role in investigating the structure of automata network like a structural equivalence relation did.

References

1) H.Yamada and S.Amoroso, "Structural and behavioral equivalence of tessellation automata" (1971),Inform. and Cont. 18,1-31

2) T.Saito and H.Nishio, "Structural and Behavioral Equivalence Relations in Automata Networks" 1984 Winter LA Symposium,Kyoto

3) T.Saito and H.Nishio, "Structural and Behavioral Equivalence Relations in Automata Networks — 2nd Report" 1984 Summer LA Symposium,Hakone

4) T.Saito and H.Nishio, "Structural and Behavioral Equivalence Relations in Automata Networks" (to appear)