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Approximate innerness of positive linear maps of
factors of type II

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We in this paper shall discuss the properties of positive linear maps which continue from the former work by the author [7].

Let $M$ be a $\sigma$-finite, semi-finite von Neumann algebra, then there exists a faithful, normal semi-finite trace $\text{Tr}$ and we can define a norm $||\cdot||_2$ on the ideal $S = \{x \in M; \text{Tr}(x^*x) < +\infty\}$. In particular, if $M$ is a finite von Neumann algebra, then $S = M$.

Let $A$ and $B$ be $C^*$-algebras. A linear map $\rho$ of $A$ to $B$ is said to be $n$-positive if the multiplicity map $\rho_n$ from the matrix algebra $M_n(A)$ over $A$ to the algebra $M_n(B)$ over $B$ defined by $\rho_n([a_{ij}]) = [\rho(a_{ij})]$ is a positive map. If $\rho$ is $n$-positive for every positive integer $n$, we call $\rho$ completely positive. Many authors (for example, [1], [5], [7], [8] and [9]) studied the completely positive linear maps of $C^*$-algebras. In particular, we have the following Stinespring's theorem [5]:

Let $A$ be a $C^*$-algebra and $\rho$ a completely positive linear map of $A$ to $B(H)$ where $B(H)$ is the von Neumann algebra of all bounded operators on a Hilbert space $H$. Then, there exists a representation $\pi$ of $A$ to a Hilbert space $K$ and a bounded operator $v$ of $H$ to $K$ such that $\rho(x) = v^*\pi(x)v$ for every
\( x \in A \). In particular, if \( \rho \) is unital (i.e. \( \rho(1) = 1 \)), \( \nu \) is an isometry. Furthermore, \( A \) is a von Neumann algebra and \( \rho \) is normal, then \( \pi \) is a normal representation. We in general cannot take the above operator \( \nu \) in \( A \). For this problem, we have the following result by Haagerup [3; Proposition 2.1]: Let \( N \) be a properly infinite von Neumann algebra and let \( F \) be a finite dimensional subfactor. Let \( \rho \) be a completely positive map from \( F \) to \( N \). Then there exists an element \( a \in N \) such that \( \rho(x) = a^{*}xa \) for every \( x \in F \). In this report, we shall consider the above problem for finite von Neumann algebras by using the approximate innerness and extend the obtained results to the semi-finite von Neumann algebras. Thus, we here introduce the notation of the approximate innerness.

**Definition 1.** Let \( M \) be a \( \sigma \)-finite, finite von Neumann algebra with a fixed faithful, normalized normal trace \( Tr \) and \( A \) a C*-subalgebra of \( M \). A positive linear map \( \rho \) of \( A \) into \( M \) is approximate inner if there exists a net \( \{a_{\lambda}\} \) (not necessarily bounded) in \( M \) satisfying \( \lim ||\rho(x) - a_{\lambda}^{*}xa_{\lambda}||_2 = 0 \) for every \( x \in A \).

If we consider the approximate innerness for positive linear maps, we can show that those positive linear maps are closely related to the *-homomorphisms. Before we denote the theorems, we shall mention the following lemma by Choi [1] (and also see [9]).
Lemma 2. Let $A$ and $B$ be unital $C^*$-algebra and $\rho$ a unital completely positive map of $A$ to $B$. If $\rho$ is a $C^*$-homomorphism (i.e., $\rho(a^2) = \rho(a)^2$ for every self-adjoint element $a$ of $A$), then $\rho$ is a $*$-homomorphism of $A$ to $B$.

Consider Lemma 2, we have the following theorem that a positive linear map with the approximate innerness is closely related to $*$-homomorphism. The following theorem is in a sense a generalization of Theorem 3 in [7].

Theorem 3. Let $M$ be a $\sigma$-finite, finite von Neumann algebra and $A$ a $C^*$-subalgebra with the unit in $M$. Let $\rho$ be a positive linear map of $A$ to $M$ and approximate inner with respect to a net $\{a_\lambda\}$ such that $||a_\lambda a_\lambda^* - e||_2 \to 0$ and $||a_\lambda a_\lambda^* - f||_2 \to 0$ for a projection $e$ of $M$ and a projection $f$ in $A$. Then, $\rho$ is a $*$-homomorphism of $fAf$ to $eMe$.

Proof. By the assumption for the net $\{a_\lambda\}$ and the approximate innerness of $\rho$ with respect to $\{a_\lambda\}$, $\rho(1) = e$ and $\rho(1 - f) = 0$. Thus, we can assume that $f a_\lambda e = a_\lambda$ for every $\lambda \in A$. By the remark before Definition 1, $\rho$ is completely positive map. So $\rho$ is a unital completely positive map of $C^*$-algebra $fAf$ to von Neumann algebra $eMe$. To show that $\rho$ is a $*$-homomorphism of $fAf$ to $eMe$, we must show by Lemma 2 that
\[ \rho(x^2) = \rho(x)^2 \] for every self-adjoint element \( x \in \mathfrak{f} \mathfrak{f} \mathfrak{A} \). Given an arbitrary self-adjoint element \( x \in \mathfrak{f} \mathfrak{f} \mathfrak{A} \). Then,

\[
\| \rho(x) - a_\lambda^*x a_\lambda \|_2^2 = \text{Tr}(\rho(x)^2) - 2\text{Tr}(\rho(x)a_\lambda^*x a_\lambda) + \text{Tr}(a_\lambda^*xA_\lambda^*x A_\lambda).
\]

Now, since

\[
|\text{Tr}(a_\lambda^*xA_\lambda^*x A_\lambda - a_\lambda^*x^2 A_\lambda)|
\]

\[
= |\text{Tr}(a_\lambda^*x(a_\lambda a_\lambda^* - f)x A_\lambda)| = |\text{Tr}((a_\lambda a_\lambda^* - f)x A_\lambda x A_\lambda)|
\]

\[
\leq \text{Tr}((a_\lambda a_\lambda^* - f)^2)^{1/2} \text{Tr}(x A_\lambda a_\lambda^*x a_\lambda a_\lambda^* x A_\lambda)^{1/2}
\]

\[
\leq |a_\lambda a_\lambda^* - f|^2 \cdot |x|^2 \cdot |\text{Tr}(x A_\lambda a_\lambda^*x a_\lambda a_\lambda^* x A_\lambda)|^{1/2}
\]

\[
\leq |x|^2 \cdot |a_\lambda a_\lambda^* - f|^2 \cdot |\text{Tr}(a_\lambda a_\lambda^* x A_\lambda)|^{1/2}
\]

\[
= |x|^2 \cdot |a_\lambda a_\lambda^*|^2 \cdot |a_\lambda a_\lambda^* - f|^2.
\]

{\( |a_\lambda a_\lambda^*|^2 \)} is bounded and \( \lim |a_\lambda a_\lambda^* - f|^2 = 0 \), we have the relation

\[
\lim \{\text{Tr}(a_\lambda^*xA_\lambda^*x A_\lambda) - \text{Tr}(a_\lambda^*x^2 A_\lambda)\} = 0.
\]

Thus, since \( \lim \text{Tr}(a_\lambda^*x^2 A_\lambda) = \text{Tr}(\rho(x^2)) \) by the assumption,
\[ \lim \text{Tr}(a_\lambda^* x_\lambda a_\lambda^* x_\lambda) = \text{Tr}(\rho(x^2)). \] Furthermore, since

\[
|\text{Tr}(\rho(x) a_\lambda^* x_\lambda) - \text{Tr}(\rho(x)^2)| = |\text{Tr}(\rho(x)(a_\lambda^* x_\lambda - \rho(x))| \\
\leq \|\rho(x)\|_2 \cdot \|\rho(x) - a_\lambda^* x_\lambda\|_2,
\]

\[ \lim \text{Tr}(\rho(x) a_\lambda^* x_\lambda) = \text{Tr}(\rho(x)^2). \] By the above considerations and the relation \( \lim \|\rho(x) - a_\lambda^* x_\lambda\|_2 = 0, \)

\[ \text{Tr}(\rho(x)^2) - 2\text{Tr}(\rho(x)^2) + \text{Tr}(\rho(x^2)) = 0. \]

So, \( \text{Tr}(\rho(x^2) - \rho(x)^2) = 0. \) Now, since \( \rho \) is a completely positive map, \( \rho(x)^2 \leq \rho(x^2) \). Therefore, \( \rho(x^2) = \rho(x)^2 \) and so, by Lemma 2, \( \rho \) is a \( \ast \)-homomorphism of \( \mathcal{A} \mathcal{F} \) to \( \mathcal{E} \mathcal{M} \). \hspace{1cm} \text{q.e.d.}

Under the definition of approximate innerness, if \( \rho \) is approximate inner, then \( \rho \) is completely positive like as [7]. Furthermore, we can replace the conditions in Theorem 3 as the following by the remark in [7]. That is, if \( \rho \) is approximately inner with respect to \( \{a_\lambda\} \) and \( \rho(1) = e \) is a projection, then the conditions in Theorem 3 is equivalent that \( A \) has a projection \( f \) satisfying \( \rho(1 - f) = 0 \) and \( \text{Tr}(e) = \text{Tr}(f) \).

By considering Theorem 3 and a Sakai's result [4], we have the following theorem.
Theorem 4. Let $M$ be an approximately finite dimensional factor of type $II_1$. Let $\rho$ be a positive linear map of $M$ into $M$ such that $\rho(1) = e$ is a projection, $\rho(1 - f) = 0$ and $\text{Tr}(e) = \text{Tr}(f)$ for a projection $f$ of $M$. Then $\rho$ is approximately inner with respect to a net $\{a_\lambda\}$ if and only if $\rho$ is a $\ast$-isomorphism of $FMf$ to $eMe$.

Proof. Necessity: By Theorem 3, $\rho$ is a $\ast$-homomorphism of $FMf$ to $eMe$, and so the kernel of $\rho$ in $FMf$ is a closed two-sided ideal. Since $M$ is a finite factor, the kernel of $\rho = \{0\}$ and so $\rho$ is a $\ast$-isomorphism of $FMf$ to $eMe$.

Sufficiency: Since $M$ is an approximately finite dimensional factor of type $II_1$, both $FMf$ and $eMe$ are so. Let $FMf = \bigcup A_n$ ($\supseteq$ means the weak closure of $\cdot$) where $A_n$ is a sub-factor of type $I_{2n}$ of $FMf$ satisfying $A_n \subseteq A_{n+1}$ ($n = 1, 2, \ldots$). Let $\{f_{1j}^{(n)}\}_{i,j=1}^{2^n}$ be the matrix units of $A_n$. Put $B_n = \rho(A_n)$, then $\rho(FMf) = N = \bigcup B_n$ $eMe$ and $B_n$ is a factor of type $I_{2n}$. Furthermore, put $e_{ij}^{(n)} = \rho(f_{ij}^{(n)})$, then $\{e_{ij}^{(n)}\}$ is the matrix units for $B_n$. It is sufficient for us to show that, for an arbitrary finite set $\{a_1, \ldots, a_k\}$ in $FMf$ and each $\epsilon > 0$, there exists an element $u \in M$ such that $\|\rho(a_j) - u^*a_ju\|_2 < \epsilon$ ($j = 1, 2, \ldots, k$). Given any finite set $\{a_1, \ldots, a_k\}$ in $FMf$ and $\epsilon > 0$, then there exist a positive integer
m and \( \{b_1, \ldots, b_k\} \) in \( A_m \) such that \( \|a_j - b_j\|_2 < \varepsilon/2 \) (\( j = 1, 2, \ldots, k \)). Since \( \text{Tr}(e) = \text{Tr}(f) \),

\[
\sum_{i=1}^{2^m} e_{i1}^{(m)} = e \quad \text{and} \quad \sum_{i=1}^{2^m} f_{i1}^{(m)} = f,
\]

\( \text{Tr}(f_{i1}^{(m)}) = \text{Tr}(e_{i1}^{(m)}) \). And so, there exists a partial isometry \( v \) in \( M \) such that \( vv^* = f_{i1}^{(m)} \) and \( v^*v = e_{i1}^{(m)} \). Put \( u = \sum_{i=1}^{2^m} f_{i1}^{(m)}v_{i1}^{(m)}e_{i1}^{(m)} \), then \( u \) is an element of \( M \) and \( u^*u = e \).

Furthermore, we have the following;

\[
u_{i1}^{(m)}f_{i1}^{(m)}u = \sum_{s,t=1}^{2^m} s_{s1}^{(m)}v_{i1}^{(m)}f_{i1}^{(m)}t_{i1}^{(m)}e_{i1}^{(m)} = \sum_{s,t=1}^{2^m} s_{s1}^{(m)}v^*(\delta_{s1}^{(m)}\delta_{t1}^{(m)}f_{i1}^{(m)}e_{i1}^{(m)})
\]

\[
= e_{i1}^{(m)}v_{i1}^{(m)}f_{i1}^{(m)}e_{i1}^{(m)} = e_{i1}^{(m)}v_{i1}^{(m)}e_{i1}^{(m)} = e_{i1}^{(m)}e_{i1}^{(m)} = e_{i1}^{(m)}.
\]

Thus, \( u_{i1}^{(m)}f_{i1}^{(m)}u = e_{i1}^{(m)} \) for \( i,j = 1, 2, \ldots, 2^m \). And so, \( u^*xu = \rho(x) \) for every \( x \in A_m \). In particular, \( \rho(b_j) = u^*b_ju \) (\( j = 1, 2, \ldots, k \)). Furthermore, we have the following relations;

\[
\|\rho(a_j) - \rho(b_j)\|_2 = \text{Tr}((\rho(a_j - b_j)\rho(a_j - b_j))^{1/2})
\]

\[
= \text{Tr}(f)^{1/2} \text{Tr}((a_j - b_j)^*(a_j - b_j))^{1/2} = \text{Tr}(f)^{1/2}\|a_j - b_j\|_2
\]
\[ \|a_j - b_j\|_2 < \varepsilon/2 \quad \text{and} \]
\[ \| u^*a_j u - u^*b_j u \|_2 = \text{Tr}(u^*(a_j - b_j)*(a_j - b_j)u)^{1/2} \]
\[ = \text{Tr}(uu^*(a_j - b_j)*(a_j - b_j))^{1/2} = \text{Tr}((a_j - b_j)*(a_j - b_j))^{1/2} \]
\[ = \|a_j - b_j\|_2 < \varepsilon/2. \]

Thus, we have

\[ \| \rho(a_j) - u^*a_j u \|_2 \]
\[ \leq \| \rho(a_j) - \rho(b_j) \|_2 + \| \rho(b_j) - u^*b_j u \|_2 + \| u^*b_j u - u^*a_j u \|_2 \]
\[ < \varepsilon/2 + \varepsilon/2 < \varepsilon \quad \text{for} \quad j = 1, 2, \ldots, k. \]

Therefore, we have the complete proof of Theorem 4. \quad \text{q.e.d.}

Remark. In the former work [6] by the author, we have the error in the proof of Theorem 1 in [6] and so we must replace that. Because the results in this report are closely related to the results in [6]. Consider the results in this report and [2] and [3] in the references we have the following considerations for [6]. We replace Theorem 1 in [6] as Theorem 4 in this report and Proposition 2 in [6] as Theorem 3 in this report. Further-
more, if we consider a Haagerup's result [3], the C*-subalgebra $A$ appeared in Theorem 2 in [6] was an MAF-C*-subalgebra but we must replace the algebra $A$ as an AF-C*-subalgebra. The last result (Corollary 4) in [6] is right by [2].

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References


