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Optimal Control for Linear and Semi-linear Partial Functional Differential Equations

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1. Introduction

There exists a great number of literatures which study optimal control problems of abstract control systems in Banach and Hilbert spaces (see books [1,2,3] and the references cited therein). The most studies have been done for the systems without delay, and the papers treating the systems with retardation are not many [4,5,6,7,8].

In this paper we study some standard optimal control problems, namely the fixed time integral convex cost problem and the time optimal control problem for linear and semi-linear retarded systems in reflexive Banach spaces.

The content of this paper is as follows: After system descriptions and formulation of the control problems are given, the retarded adjoint system is introduced and the representation of the adjoint state is given in Section 2. In Section 3 two existence theorems of optimal controls are given, one is for bounded control set and the other is for unbounded control set. In Section 4 the necessary conditions for optimality are described by the adjoint state and integral inequality. Some examples of necessary optimality conditions for technologically important costs are also given in Section 4. In Section 5 the maximum principle for Bolza problem is established with some examples. In Section 6 the bang-bang principle for terminal value problem with time varying control domain and its applications to uniqueness and expression of the optimal control are given under some regularity conditions of adjoint system. Section 7 deals with the time optimal control problem to a target set. An existence theorem, the maximum principle and the bang-bang principle are established for time optimal controls. A convergence theorem of time optimal controls is also given in Section 7. Finally in Section 8 a general integral cost problem is considered for semi-linear control systems and the 'extended' maximum principle is shown. All proofs of the results in this paper are sketched or omitted. Detailed proofs will appear in [10].

2. System Description, Control Problem and Adjoint System

First we give the notations and terminology used in this paper. Let X and Y be real (separable) Banach spaces with norms $|\cdot|$ and $|\cdot|_Y$, respectively. The adjoint spaces of X, Y are denoted by X^*, Y^* and their norms are denoted by $|\cdot|_{X^*}$ and $|\cdot|_{Y^*}$. We write the duality pairing between X and X^* by $\langle\ ,\ \rangle$ and the pairing between Y and Y^* by $\langle\ ,\ \rangle_{Y,Y^*}$. Let $\mathcal{L}(X,Y)$ be the Banach space of bounded linear operators from X into Y. When $X = Y$, $\mathcal{L}(X,Y)$ is denoted by $\mathcal{L}(X)$. Their operator norms are denoted by $\|\cdot\|$. Given an interval $I \subset R$, we denote by $L_p(I; X)$ and $C(I; X)$ the usual Banach spaces of measurable functions which are p-Bochner integrable ($1 \le p < \infty$) or essentially bounded ($p=\infty$) on I and strongly continuous on I, respectively. The norm of $L_p(I; X)$ is denoted by $\|\cdot\|_{L_p}$. The function χ_I means the characteristic function of the interval I.

Let $T > 0$, $h > 0$ be fixed and let $I = [0, T]$, $I_h = [-h, 0]$. We consider the following linear hereditary control system on X:

$$
(\mathrm{CS})\begin{cases} \dfrac{dx(t)}{dt} = A_0 x(t) + \displaystyle\int_{-h}^{0} d\eta(s)x(s+t) + f(t) + B(t)u(t) \quad \text{a.e. } t \in I, & (2.1)\\ x(0) = g^0, \quad x(s) = g^1(s) \quad \text{a.e. } s \in [-h, 0), & (2.2)\\ u \in U_{ad}, \end{cases}
$$

where $f \in L_p(I; X)$, $g = (g^0, g^1) \in X \times L_{p'}(I_h; X)$. $U_{ad} \subset L_p(I; Y)$, $p, p' \in [1,\infty]$,

$B \in L_\infty(I; \mathcal{L}(Y,X))$ and A_0 generates a strongly continuous semigroup $T(t)$, $t \geq 0$ on X. As for the retardation term in (2.1) we suppose that the Stieltjes measure η is given by

$$\eta(s) = -\sum_{r=1}^{m} \chi_{(-\infty,\ -h_r]}(s)A_r - \int_s^0 D(\xi)d\xi \qquad s \in I_h, \tag{2.3}$$

where $0 \leq h_1 < \cdots < h_m \leq h$ are non-negative constants, A_r $(r=1,\cdots, m)$ are bounded linear operators on X and $D \in L_1(I_h; \mathcal{L}(X))$.

The quantities $x(t)$, $u(t)$, $B(t)$ and U_{ad} in (CS) denote a system state (or a trajectory), a control, a controller and a class of admissible controls, respectively.

Let $G(t)$ be the fundamental solution of (CS) which is a unique solution of

$$G(t) = \begin{cases} T(t) + \displaystyle\int_0^t T(t-s)\int_{-h}^0 d\eta(\xi)G(\xi+s)ds, & t \geq 0 \\ O, \quad t < 0, \end{cases} \tag{2.4}$$

where O is the null operator on X. We know that $G(t)$ is strongly continous on R^+. If the condition

$$D \in L_q(I_h; \mathcal{L}(X)),\ 1/p' + 1/q = 1 \tag{2.5}$$

is satisfied, then for each $t \in R^+$ the operator valued function U_t on I_h defined by

$$U_t(s) = \sum_{r=1}^{m} G(t-s-h_r)A_r\chi_{[-h_r,\ 0]}(s) + \int_{-h}^{s} G(t-s+\xi)D(\xi)d\xi, \qquad s \in I_h \tag{2.6}$$

belongs to $L_q(I_h; \mathcal{L}(X))$. Hence the function

$$x(t) = x(t; f,g) + \int_0^t G(t-s)B(s)u(s)ds \tag{2.7}$$

is well-defined and is a member of $C(I; X)$, where

$$x(t; f,g) = \int_0^t G(t-s)f(s)ds + \left(G(t)g^0 + \int_{-h}^0 U_t(s)g^1(s)ds \right), \quad t \in I. \tag{2.8}$$

It is proved in [10] that the function $x(t)$ in (2.7) satisfies the integrated form of (2.1), (2.2) in terms of $T(t)$ if (2.5) is satisfied. In this sense we shall call this $x(t)$ the mild (or weak) solution of (CS). Since we use the class of mild solutions (2.7) to investigate the control problems for (CS), the condition (2.5) is always assumed.

In what follows the admissible set U_{ad} is assumed to be closed and convex in $L_p(I; Y)$. We sometimes denote $x(t)$ in (2.7) by $x_u(t)$ to express the dependence on $u \in U_{ad}$. The function x_u is called the trajectory corresponding to u.

Let $J = J(u,x)$ be the integral convex cost given by

$$J = \phi_0(x(T)) + \int_I (f_0(x(t),t) + k_0(u(t),t))dt, \tag{2.9}$$

where $\phi_0 : X \to R$, $f_0 : X \times I \to R$, $k_0 : Y \times I \to R$. We study the following control problems P_1 and P_2 on the finite interval $I = [0, T]$.

P_1. Find a control $u \in U_{ad}$ which minimizes the cost J subject to the constraint (CS).

P_2. Find optimality conditions for $(\bar{u}, x_{\bar{u}})$ such that

$$\inf_{u \in U_{ad}} J(u,x) = J(\bar{u}, x_{\bar{u}}), \quad \bar{u} \in U_{ad}. \tag{2.10}$$

In P_1 such as $u \in U_{ad}$ is called an optimal control for the cost J. In P_2 the pair $(\bar{u}, x_{\bar{u}})$ is called the optimal solution for J. We will solve P_1 partly by showing the existence of optimal controls in Section 3 and solve P_2 by deriving necessary optimality conditions of both integral and pointwise types in Section 4. More further properties such as maximum principle and bang-bang principle are studied in Section 5 and Section 6. To give a difinite form of those optimality conditions it is required some knowledge on the adjoint system.

Now we introduce the retarded adjoint system in the case where X is reflexive. Let X be reflexive and $q_0^* \in X^*$, $q_1^* \in L_1(I; X^*)$. The retarded adjoint system (AS) on X^* is defined by

$$(AS)\quad \begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) - q_1^*(t) = 0, \quad \text{a.e. } t \in I \\ p(T) = -q_0^*, \quad p(s) = 0 \quad s \in (T, T+h], \end{cases} \tag{2.11}$$

where A_0^*, $\eta^*(s)$ denote the duals of A_0, $\eta(s)$, respectively. Since X is reflexive, it is known [11] that the adjoint operator A_0^* generates a c_0-semigroup $T^*(t)$ on X^* which is the adjoint of $T(t)$, $t \geq 0$. Hence we can construct the fundamental solution $G_*(t)$ as in [10]. That is, $G_*(t)$ is characterized as the (unique) solution of

$$G_*(t) = \begin{cases} T^*(t) + \displaystyle\int_0^t T^*(t-s) \int_{-h}^{0} d\eta^*(\xi) G_*(\xi+s)ds, \quad t \geq 0 \\ 0, \quad t < 0. \end{cases} \tag{2.12}$$

We denote by $G^*(t)$ the adjoint of $G(t)$. Then it is verified that $G^*(t) = G_*(t)$. By changing time direction in (AS), we consider the following system on X^*:

$$(CS)^*\quad \begin{cases} \dfrac{dw(t)}{dt} = A_0^* w(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)w(t+s) + q_1^*(T-t) \quad \text{a.e. } t \in I \\ w(0) = -q_0^*, \quad w(s) = 0 \quad s \in [-h, 0). \end{cases} \tag{2.13}$$

The mild solution $w(t)$ of $(CS)^*$ is represented by

$$w(t) = G^*(t)(-q_0^*) + \int_0^t G^*(t-s)q_1^*(T-s)ds \tag{2.14}$$

It is easily seen that the system $(CS)^*$ is transformed to the system (AS) by a change of variable $t \to T-t$. Hence by (2.14) the function $p(t)$ given by

$$p(t) = w(T-t) = G^*(T-t)(-q_0^*) + \int_t^T G^*(s-t)(-q_1^*(s))ds, \quad t \in I \tag{2.15}$$

may be called the mild (or weak) solution of (AS). We often call that $p(t)$ in (2.15) solves (AS) in the weak sense.

Remark 2.1. Even if X is not reflexive, the adjoint system can be constructed by the adjoint theory in [11].

3. Existence of Optimal Control

In what follows we assume that Y is reflexive and $1 < p < \infty$. We consider two cases to solve the problem P_1, one is the case where U_{ad} is bounded and the other is where U_{ad} is unbounded in $L_p(I; Y)$. For a bounded U_{ad} we suppose the following assumption H_1 on ϕ_0, f_0 and k_0.

H_1: (1) $\phi_0 : X \to R$ is continuous and convex;

(2) $f_0 : X \times I \to R$ is measurable in $t \in I$ for each $x \in X$ and continuous and convex in $x \in X$ for a.e. $t \in I$ and further for each bounded set $K \subset X$ there exists a measurable function $m_K \in L_1(I; R)$ such that

$$\sup_{x \in K} |f_0(x,t)| \le m_K(t) \quad \text{a.e. } t \in I;$$

(3) $k_0 : Y \times I \to R$ satisfies that for any $u \in U_{ad}$, $k_0(u(t),t)$ is integrable on I and the functional $\xi_0 : L_p(I; Y) \to R$ defined by

$$\xi_0(u) = \int_I k_0(u(t),t)dt \tag{3.1}$$

is weakly lower semi-continuous.

THEOREM 3.1. Let U_{ad} be bounded and H_1 be satisfied. Then there exists a control $u_0 \in U_{ad}$ that minimizes the cost J in (2.9).

(Proof) Let $\{u_n\}$ be a minimizing sequence of J such that

$$\inf_{u \in U_{ad}} J = \lim_{n\to\infty} J(u_n, x_n),$$

where x_n is the trajectory corresponding to u_n. Since U_{ad} is bounded and weakly closed, there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ and an $u_0 \in U_{ad}$ such that

$$u_{n_k} \to u_0 \quad \text{weakly in } L_p(I;\, Y). \tag{3.2}$$

Using (3.2), H_1 and Legesgue-Fatou's lemma, u_0 is shown to be an optimal control for J.

Next, we consider the case where U_{ad} is unbounded. In this case we suppose H_1 and the following additional assumption H_2.

H_2: (1) there exists a constant c_0 such that $\phi_0(x) \geq c_0$ on X;

(2) there exists a constant $c_1 > 0$ such that $f_0(x,t) \geq -c_1$ on $X \times I$;

(3) there exists a monotone increasing function $\theta_0 \in C(R^+;\, R)$ such that $\lim_{r\to\infty} \theta_0(r) = \infty$ and

$$\xi_0(u) = \int_I k_0(u(t),t)\,dt \geq \theta_0(\| u \|_{L_p}) \quad \text{for } u \in U_{ad}.$$

<u>THEOREM 3.2.</u> Let H_1 and H_2 be satisfied. Then there exists a control $u_0 \in U_{ad}$ which minimizes the cost J in (2.9).

(Proof) Note that

$$J \geq \theta_0(\| u \|_{L_p}) + c_0 - c_1 T \quad \text{for } u \in U_{ad}.$$

4. Optimality Condition

In this section we study the problem P_2, or we seek necessary optimality conditions of the optimal solution (u,x) for J in (2.9). The existence of optimal solutions is assumed in this section. To give two types of optimality conditions we introduce the following two assumptions H_3 and H_3^w.

H_3: (1) $\phi_0: X \to R$ is continuous and Gateau differentiable, and the Gateau derivative $d\phi_0(x) \in X^*$ for each $x \in X$;

(2) $f_0: X \times I \to R$ is measurable in $t \in I$ for each $x \in X$ and continuous and convex on X for a.e. $t \in I$ and further there exist functions $\partial_1 f_0 : X \times I \to X^*$, $\theta_1 \in L_1(I; R)$, $\theta_2 \in C(R^+; R)$ such that

a) $\partial_1 f_0$ is measurable in $t \in I$ for each $x \in X$ and continuous in $x \in X$ for a.e. $t \in I$ and the value $\partial_1 f_0(x,t)$ is the Gateau derivative of $f_0(x,t)$ in the first argument for (x,t) in $X \times I$, and

b) $|\partial_1 f_0(x,t)|_{X^*} \le \theta_1(t) + \theta_2(|x|)$ for $(x,t) \in X \times I$;

(3) $k_0: Y \times I \to R$ is measurable in $t \in I$ for each $u \in Y$ and continuous and convex on Y for a.e. $t \in I$ and further there exist functions $\partial_1 k_0: Y \times I \to Y^*$, $\theta_3 \in L_q(I; R)$ and $M_4 > 0$ such that

a) $\partial_1 k_0$ is measurable in $t \in I$ for each $u \in Y$ and continuous in $u \in Y$ for a.e. $t \in I$ and the value $\partial_1 k_0(u,t)$ is the Gateau derivative of $k_0(u,t)$ in the first argument for (u,t) in $Y \times I$, and

b) $|\partial_1 k_0(u,t)|_{Y^*} \le \theta_3(t) + M_4 |u|_Y^{p/q}$ for $(u,t) \in Y \times I$.

Next we give the condition $(3)^w$ which is different from $H_3(3)$.

$(s)^w$ $k_0: Y \times I \to R$ is measurable in $t \in I$ for each $u \in Y$ and continuous and convex on Y for a.e. $t \in I$ and further there exist a function $\theta_5 \in L_1(I; R)$ and $M_6 > 0$ such that

$$|k_0(u,t)| \le \theta_5(t) + M_6|u|_Y^p \quad \text{for} \quad (u,t) \in Y \times I.$$

The assumption H_3^w is the set of conditions $H_3(1)$, $H_3(2)$ and $(3)^w$. The assumption H_3 is for the differentiable costs and H_3^w is for non-differentiable costs. The following is the main theorem which gives the necessary conditions of optimality for the problem P_2.

THEOREM 4.1. Let H_3 (resp. H_3^w) be satisfied and let $(u,x) \in U_{ad} \times C(I; X)$ be an optimal solution for J in (2.9). Then the integral inequality

$$\int_I \langle v(t)-u(t),\ \partial_1 k_0(u(t),t)-B^*(t)p(t)\rangle_{Y,Y^*} dt \ge 0 \quad \text{for all} \quad v \in U_{ad} \tag{4.1}$$

$$\text{(resp.} \quad \int_I \langle v(t)-u(t), -B^*(t)p(t)\rangle_{Y,Y^*} dt + \int_I (k_0(v(t),t)-k_0(u(t),t))dt \ge 0 \quad \text{for all} \quad v \in U_{ad} \quad (4.2))$$

holds, where

$$p(t) = -G^*(T-t)d\phi_0(x(T)) - \int_t^T G^*(s-t)\partial_1 f_0(x(s),s)ds. \tag{4.3}$$

If $U_{ad} = L_p(I; X)$, then the condition (4.1) is reduced to that

$$\partial_1 k_0(u(t),t) - B^*(t)p(t) = 0 \quad \text{a.e.} \quad t \in I. \tag{4.4}$$

Furthermore if X is reflexive, $p \in C(I; X^*)$ satisfies

$$\text{(AS)} \begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) - \partial_1 f_0(x(t),t) = 0 \quad \text{a.e.} \quad t \in I, \\ p(T) = -d\phi_0(x(T)), \quad p(s) = 0 \quad s \in (T, T+h] \end{cases}$$

in the weak sense.

(Proof) Let H_3 be satisfied. Then the cost J given in (2.9) is Gateau differentiable. The inequality (4.1) follows from the necessary optimality condition

$$J'(u)(v - u) \geq 0 \quad \text{for all} \quad v \in U_{ad}$$

in [1,p.11] and the representation (2.15). Next, let H_3^w be satisfied. Then we can use the optimality condition

$$(J - \xi_0)'(u)(v - u) + (\xi_0(v) - \xi_0(u)) \geq 0 \quad \text{for all} \quad v \in U_{ad}$$

in [1,p.13] to obtain (4.2), where ξ_0 is given in (3.1). The condition (4.4) is obvious from (4.1) and $U_{ad} = L_p(I; X)$.

Remark 4.1. Consider the special case where Y is a Hilbert space, $p = 2$ and $U_{ad} = \{ u \in L_2(I; Y) : \| u \|_{L_2} \leq \alpha \}$. In this case the optimal control u is characterized by the relation

$$u = -\alpha \frac{\Lambda^{-1}K(u)}{\| \Lambda^{-1}K(u) \|_{L_2(I;\, Y)}},$$

where Λ is the cannonical isomorphism of $L_2(I; Y)$ into $L_2(I; Y^*)$ and $K(u)(t) = \partial_1 k_0(u(t),t) - B^*(t)p(t)$ a.e. $t \in I$.

Now we give pointwise necessary conditions for optimality. Let U be a closed convex set in Y and the admissible set U_{ad} be given by

$$U_{ad} = \{ u \in L_p(I; Y) : u(t) \in U \ \text{a.e.} \ t \in I \}. \tag{4.5}$$

The next corollary follows from the Lebesgue density theorem.

COROLLARY 4.1. Let the assumptions in Theorem 4.1 be satisfied and U_{ad} be given by (4.5). Then the condition (4.1) (resp. (4.2)) is reduced to the pointwise optimality condition that for a.e. $t \in I$,

$$\langle v - u(t), \partial_1 k_0(u(t),t) - B^*(t)p(t) \rangle_{Y,Y^*} \geq 0 \quad \text{for all} \quad v \in U$$

(resp. $\langle v - u(t), - B^*(t)p(t)\rangle_{Y,Y^*} + (k_0(v,t) - k_0(u(t),t)) \geq 0$

for all $v \in U$).

(Proof) The proof is similar to that given in [2,p.290-291]. Remark that $\partial_1 k_0(u(t),t) - B^*(t)p(t)$ and $k_0(u(t),t)$ are measurable and integrable on I by H_3 and H_3^W.

Example 4.1. (Regulator problem) Let X and Y be Hilbert spaces with inner products (,) and $\langle\ ,\ \rangle_Y$, respectively. We suppose $U_{ad} = L_2(I;\ Y)$. The spaces X and X* are identified. The cost J_1 is given by

$$J_1 = (x(T),Nx(T)) + \int_I (x(t),W(t)x(t))dt + \xi_Q(u), \tag{4.6}$$

where

$$\xi_Q(u) = \frac{1}{2}\int_I \langle u(t),Q(t)u(t)\rangle_Y dt. \tag{4.7}$$

In (4.6), (4.7) we assume that $N \in \mathcal{L}(X)$, $W(\cdot) \in L_\infty(I;\ \mathcal{L}(X))$, $Q(\cdot) \in L_\infty(I;\mathcal{L}(Y))$; N, W(s), Q(s) are positive and symmetric for each $s \in I$; there exists a constant $c > 0$ such that

$$\langle u,Q(t)u\rangle_Y \geq c|u|_Y^2 \quad \text{for a.e. } t \in I.$$

Under the above conditions it is verified that $\xi_Q(u)$ is strongly continuous and strictly convex in $L_2(I;\ Y)$ ([1,Chapter 3]). Since J_1 is also strictly convex, there exists a unique optimal control for J_1. Then we have

COROLLARY 4.2. Let the cost J_1 be given by (4.6), (4.7). Then there exists a unique optimal solution $(u,x) \in L_2(I;\ Y) \times C(I;\ X)$ for J_1. The optimal control $u(t)$ is given by

$$u(t) = Q^{-1}(t)B^*(t)p(t) \quad \text{a.e. } t \in I,$$

where the pair $(x,p) \in C(I;\ X) \times C(I\text{: } X)$ satisfies the system of equations

$$\begin{cases} \dfrac{dx(t)}{dt} = A_0x(t) + \displaystyle\int_{-h}^{0} d\eta(s)x(t+s) + B(t)Q^{-1}(t)B^*(t)p(t) + f(t) \quad \text{a.e. } t \in I, \\ x(0) = g^0, \quad x(s) = g^1(s) \quad \text{a.e. } s \in [-h, 0), \\ \dfrac{dp(t)}{dt} + A_0^*p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)P(t-s) - W(t)x(t) = 0 \quad \text{a.e. } t \in I, \\ p(T) = - Nx(T), \quad p(s) = 0 \quad s \in (T, T+h], \end{cases}$$

in the weak sense.

The regulator problem is very important in system design and is investigated in many references. We refer to the books [1,2,3] for infinite dimensional systems without delay and [12,13] for finite dimensional retarded systems. The literature dealing infinite dimensional retarded systems are few [4,5].

5. Maximum Principle

The purpose of this section is to establish the maximum principle for the time varying control domain with the convex integral cost

$$J = \phi_0(x(T)) + \int_I (f_0(x(t),t) + k_0(u(t),t))dt. \tag{5.1}$$

We assume the existence of optimal solutions for J and the assumption H_3^W in this and next sections. Let the admissible set U_{ad} be

$$U_{ad} = \{ u \in L_p(I; Y) : u(t) \in U(t) \quad \text{a.e. } t \in I \}, \tag{5.2}$$

where the (time varying) control domain $U(t) \subset Y$, $t \in I$ satisfies

H_4: (1) $U(t)$ is closed and convex in Y for each $t \in I$;

(2) $\bigcup_{t\in I} U(t)$ is bounded in Y;

(3) for any $t \in I$, $v \in \text{Int } U(t)$, there exists an $\varepsilon_0 > 0$ such that

$$v \in \bigcap_{s\in(t,t+\varepsilon)} U(s) \quad \text{for any } 0 < \varepsilon \le \varepsilon_0.$$

It is clear from $H_4(1),(2)$ that U_{ad} is bounded and convex. Furthermore we have the following lemma.

Lemma 5.1. Let $H_4(1),(2)$ be satisfied. Then U_{ad} given by (5.2) is weakly closed and weakly compact in $L_p(I; Y)$.

(Proof) This lemma follows from Mazur's theorem and Eberlein-Smulian's theorem.

Remark 5.1. If $U(t)$ varies continuously with respect to the Hausdorff metric or $U(t)$ is monotone increasing, then the condition $H_4(3)$ is satisfied.

By Lemma 5.1 and H_3^w, Theorem 4.1 holds for the admissible set (5.2). Moreover if H_1 is satisfied, there is an optimal solution $(u,x) \in U_{ad} \times C(I; X)$ for J in (5.1). We now give the maximum principle for the cost J in (5.1) which is deduced from the optimality condition (4.2).

THEOREM 5.1. Let U_{ad} be given by (5.2) and H_4 be satisfied. Let $(u,x) \in U_{ad} \times C(I; X)$ be an optimal solution for J in (5.1). Then

$$\max_{v \in U(t)} \{ \langle B(t)v,p(t)\rangle - k_0(v,t)\} = \langle B(t)u(t),p(t)\rangle - k_0(u(t),t) \quad \text{a.e. } t \in I, \tag{5.3}$$

where $p(t)$ is given by

$$p(t) = - G^*(T-t)d\phi_0(x(T)) - \int_t^T G^*(s-t)\partial_1 f_0(x(s),s)ds, \quad t \in I. \tag{5.4}$$

If X is reflexive, then $p(t)$ in (5.4) belongs to $C(I; X^*)$ and is the mild solution of (AS) in Theorem 4.1.

(Proof) Let $t \in (0, T)$ and $v \in \operatorname{Int} U(t)$. Then by $H_4(3)$, the function

$$v_\varepsilon(s) = \begin{cases} u(s), & s \in I - (t, t+\varepsilon) \\ v, & s \in (t, t+\varepsilon) \end{cases}$$

belongs to U_{ad} for any $\varepsilon \in (0, \varepsilon_0]$. From (4.2) and Lebesgue's density theorem we have by letting $\varepsilon \to 0$ that for a.e. $t \in I$,

$$- \langle v, B^*(t)p(t)\rangle_{Y,Y^*} + k_0(v,t) \geq - \langle u(t), B^*(t)p(t)\rangle_{Y,Y^*} + k_0(u(t),t). \quad (5.5)$$

Let $t \in I$ be fixed for which $u(t) \in U(t)$ and (5.5) holds. Since the duality pairing $\langle v, B^*(t)p(t)\rangle_{Y,Y^*}$ is continuous in v, we have from (5.5) that (5.3) is true for such $t \in I$. The latter part of this theorem may be obvious.

We shall give some applications of Theorem 5.1. We consider the special cost functionals J_2-J_4 in Examples 5.1-5.3. Such costs are important in practical applications and are studied in [1,9,14,15,16] for systems without delay. We assume that U_{ad} is given by (5.2) and H_4 is satisfied in each examples below.

<u>Example 5.1.</u> (Special linearized Bolza problem) The cost J_2 is given by

$$J_2 = \langle x(T), \psi_0^*\rangle + \int_I \langle x(t), \psi_1^*(t)\rangle dt, \quad (5.6)$$

where $\psi_0^* \in X^*$ and $\psi_1^* \in L_1(I; X^*)$. Then we have

<u>COROLLARY 5.1.</u> Let $(u,x) \in U_{ad} \times C(I; X)$ be an optimal solution for J_2. Then

$$\max_{v \in U(t)} \langle B(t)v, p(t)\rangle = \langle B(t)u(t), p(t)\rangle \quad \text{a.e. } t \in I,$$

where $p(t)$ is given by

$$p(t) = - G^*(T-t)\psi_0^* - \int_t^T G^*(s-t)\psi_1^*(s)ds, \quad t \in I. \quad (5.7)$$

If X is reflexive, $p(t)$ in (5.7) belongs to $C(I; X^*)$ and satisfies

$$\begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) - \psi_1^*(t) = 0 \quad \text{a.e.} \quad t \in I, \\ p(T) = -\psi_0^*, \quad p(s) = 0 \quad s \in (T, T+h] \end{cases}$$

in the weak sense.

Example 5.2. (Terminal value control problem) Let X be a Hilbert space. As usual we identify X and X^*. The cost J_3 is given by

$$J_3 = \frac{1}{2}|x(T) - x_d|^2, \quad x_d \in X. \tag{5.8}$$

COROLLARY 5.2. Let $(u,x) \in U_{ad} \times C(I: X)$ be an optimal solution for J_3 in (5.8). Then

$$\max_{v \in U(t)} (B(t)v, p(t)) = (B(t)u(t), p(t)) \quad \text{a.e. } t \in I,$$

where $p(t)$ is given by

$$p(t) = G^*(T-t)(x_d - x(T)), \quad t \in I. \tag{5.9}$$

The adjoint state $p \in C(I; X^*)$ in (5.9) satisfies

$$\begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) = 0 \quad \text{a.e.} \quad t \in I \\ p(T) = x_d - x(T), \quad p(s) = 0 \quad s \in (T, T+h] \end{cases}$$

in the weak sense ($p(t)$ may be identically zero).

Example 5.3. (Minimum energy problem) Let X and Y be Hilbert spaces. The cost J_4 is given by

$$J_4 = \int_I (\lambda^2|x(t)|^2 + |u(t)|_Y^2)dt, \tag{5.10}$$

where $\lambda > 0$. Then we have

COROLLARY 5.5. Let $(u,x) \in U_{ad} \times C(I; X)$ be an optimal solution for J_4. Then

$$\max_{v \in U(t)} \{ (B(t)v, p(t)) - |v|_Y^2 \} = (B(t)u(t), p(t)) - |u(t)|_Y^2 \quad \text{a.e. } t \in I,$$

where

$$p(t) = -\int_t^T G^*(s-t)(2\lambda^2 x(s))ds \qquad X^* = X, \quad t \in I$$

satisfies

$$\begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) - 2\lambda^2 x(t) = 0 \quad \text{a.e.} \quad t \in I \\ p(s) = 0 \quad s \in [T, T+h] \end{cases}$$

in the weak sense.

6. Bang-Bang Principle

Let the admissible set U_{ad} be given in Section 5. In this section we consider the terminal value cost J given by

$$J = \phi_0(x(T)), \tag{6.1}$$

where ϕ_0 satisfies $H_1(1)$ and $H_3(1)$. We investigate the possibility of the socalled bang-bang control for J in (6.1) under the time varying control domain $U(t)$. In general the bang-bang control does not hold for the retarded systems even in finite dimensional space [17,p.60]. However by restricting the cost J to the terminal value cost (6.1), we can prove that the bang-bang control is possible under some regularity condition for the adjoint system. Let X be reflexive in this section. Consider the adjoint system (AS) in (2.11). We denote by $p(t; q_0^*, q_1^*)$ the mild solution of (AS). Now we give the following condition

C_w: $q_0^* = 0$ in X^* follows from the existence of a set $E \subset I$ such that

$$\text{meas } E > 0 \quad \text{and} \quad p(t; q_0^*, 0) = 0 \quad \text{for all} \quad t \in E. \tag{6.2}$$

We say that the adjoint system (AS) is weakly regular if the condition C_w is satisfied. Examples for which the system (AS) is weakly regular are given in [9,p.41], but such systems do not involve time delay.

Example 6.1. Consider the control system (CS) enjoying the following conditions i), ii) and iii):

i) A_0 generates an analytic semigroup;

ii) the Stieltjes measure η is given by $\eta(s) = -\chi_{(-\infty,-h]}(s)A_1$;

iii) the system (CS) is pointwise complete for all $t > 0$.

The condition iii) means that for any $f \in L_p^{loc}(R^+; X)$,

$$Cl\,\{\, x(t;\, f,g)\ :\ g \in X \times L_{p'}(I_h;\, X)\,\} = X \quad \text{for each} \quad t > 0,$$

where Cl M denotes the closure of M. If i), ii), iii) are satisfied, then the adjoint system of (CS) is weakly regular [10].

The following assumption is needed in proving the bang-band principle.

H_5: $d\phi_0(x_u(T)) \neq 0$ in X^* for each $u \in U_{ad}$, where $x_u(t)$ is the trajectory corresponding to $u \in U_{ad}$.

THEOREM 6.1. Let the cost J be given by (6.1). Assume that the adjoint system (AS) is weakly regular and $B^*(t)$ is one to one for each $t \in I$. If H_5 is satisfied, then the optimal control $u(t)$ for J in (6.1) is a bang-bang control, i.e., $u(t)$ satisfies

$$u(t) \in \partial U(t) \quad \text{a.e.} \quad t \in I. \tag{6.3}$$

(Proof) This theorem is a consequence from the maximum principle (Theorem 5.1) and weak regularity.

Example 6.2. Let the assumptions in Theorem 6.1 be satisfied and let X be a Hilbert space. We consider two costs $J_3 = \frac{1}{2}|x(T) - x_d|^2$ and $J_5 = (x(T), \psi_0)$, $\psi_0 \in X$. If there exists no trajectory x_u, $u \in U_{ad}$ such that $x_u(T) = x_d$ ($\psi_0 \neq 0$ in X), then the optimal control $u(t)$ for J_3 (J_5) is a bang-bang control, i.e., $u(t)$ satisfies (6.3).

Let U be a convex set in Y. The convex set U is said to be strictly convex if $u, v, (u + v)/2 \in \partial U$ imply $u = v$. The following corollaries follow immediately from Theorem 6.1.

COROLLARY 6.1. Let the assumptions in Theorem 6.1 be satisfied and let $U(t)$ be strictly convex for all $t \in I$. Then the optimal control $u(t)$ for J in (6.1) is unique.

COROLLARY 6.2. Let the assumption in Theorem 6.1 be satisfied. Let Y be a Hilbert space and

$$U(t) = \{ u \in Y : |u - y(t)|_Y \leq r(t)\}, \quad t \in I, \tag{6.4}$$

where $y(\cdot) \in C(I; Y)$ and $r(\cdot) \in C(I; R^+ - \{0\})$. Then the optimal control $u(t)$ for J in (6.1) is unique and is given by

$$u(t) = y(t) + r(t) \cdot \frac{\Lambda_Y^{-1} B^*(t) p(t)}{|\Lambda_Y^{-1} B^*(t) p(t)|_Y} \qquad \text{a.e. } t \in I,$$

where Λ_Y is the cannonical isomorphism of Y onto Y^* and

$$p(t) = G^*(T-t) d\phi_0(x(T)), \quad t \in I.$$

(Proof) Notice that the nonvoid closed ball in a Hilbert space is strictly convex and $U(t)$ in (6.4) is Hausdorff continuous in $t \in I$.

7. Time Optimal Control

In this section we study the time optimal control problem. Throughout this section it is assumed that X is reflexive and U_{ad} is bounded in $L_p(I;Y)$.

Let W be a target set in X. Define

$$U_0 = \{ u \in U_{ad} : x_u(t) \in W \text{ for some } t \in I \}$$

and suppose that $U_0 \neq \phi$. For each $u \in U_0$ we can define the transition time that is the first time $\tilde{t}(u)$ such that $x_u(\tilde{t}) \in W$. The time optimal control problem P_3 is formulated as

P_3. Find a control $\bar{u} \in U_0$ such that $\tilde{t}(\bar{u}) \leq \tilde{t}(u)$ for all $u \in U_0$ subject to the constraint (CS).

In P_3 such an $\bar{u} \in U_{ad}$ is called a time optimal control and $\tilde{t}(\bar{u})$ is called an optimal time.

THEOREM 7.1. Assume that W is weakly compact in X and $U_0 \neq \phi$. Then there exists a time optimal control for P_3.

Now we consider the possibility of maximum principle and bang-bang principle for time optimal controls. The most simple case in which the maximum principle holds is given by the following

THEOREM 7.2. Assume that W is convex, closed, bounded and has non-empty interior. Let u be a time optimal control for P_3 and let t_0 be its optimal time. Then there exists a non-zero $q^* \in X^*$ such that

$$\max_{v \in U_{ad}} \int_0^{t_0} \langle v(s), B^*(s)G^*(t_0-s)q^* \rangle_{Y,Y^*} ds = \int_0^{t_0} \langle u(s), B^*(s)G^*(t_0-s)q^* \rangle_{Y,Y^*} ds.$$

Furthermore if U_{ad} is given by (5.2) and the control domain $U(t)$ satisfies H_4, then

$$\max_{v \in U(t)} \langle v, B^*(t)G^*(t_0-t)q^*\rangle_{Y,Y^*} = \langle u(t), B^*(t)G^*(t_0-t)q^*\rangle_{Y,Y^*}$$

$$\text{a.e. } t \in [0, t_0].$$

(Proof) This theorem is proved by using the separating hyperplane theorem [18].

COROLLARY 7.1. Let W satisfy the assumption in Theorem 7.2 and let the assumption in Theorem 6.1 with $T = t_0$ be satisfied, where t_0 is the optimal time for the problem P_3. Then the time optimal control $u(t)$ for P_3 is a bang-bang control, i.e., $u(t)$ satisfies

$$u(t) \in \partial U(t) \quad \text{a.e.} \quad t \in [0, t_0].$$

(Proof) The proof is similar to that given in Theorem 6.1. Note that $q^* \neq 0$.

Lastly we consider the case $W = \{g_1\}$, a single point. In this case the time optimal control problem can be considered as a limit of those problems for target sets with non-empty interior. Let $\{W_n\}$ be a sequence of convex and weakly compact sets in X such that

$$g_1 \in \bigcap_{n=1}^{\infty} W_n , \quad \text{Int } W_n \neq \phi , \ n = 1,2,\cdots, \quad W_1 \supset W_2 \supset \cdots \supset W_n \supset \cdots$$
$$\text{dist }(g_1, W_n) = \sup_{x \in W_n} |x - g_1| \to 0 \quad \text{as} \quad n \to \infty. \tag{7.1}$$

Put $U_0^n = \{ u \in U_{ad} : x_u(t) \in W_n \quad \text{for some} \quad t \in I \}$.

THEOREM 7.3. Let $\{W_n\}$ be a sequence of convex and weakly compact sets in X satisfying the condition (7.1). Assume $U_0^n \neq \phi$ for all $n = 1,2,\cdots$ and let $\{u_n\}$, be a sequence such that u_n is the time optimal control with the optimal time t_n to the target set W_n, $n = 1,2,\cdots$. Then there exists a time optimal control $u_0(t)$ to a point target set $\{g_1\}$ which is given by the weak limit of some subsequence of $\{u_n\}$ in $L_p([0, t_0]; Y)$, where $t_0 = \lim_{n\to\infty} t_n$ is the optimal time to the target $\{g_1\}$.

8. Semi-linear System

In this section we consider the following semi-linear control system:

$$\begin{cases} \dfrac{dx(t)}{dt} = A_0 x(t) + \displaystyle\int_{-h}^{0} d\eta(s)x(s+t) + f(x(t),u(t),t) \quad \text{a.e.} \quad t \in I, & (8.1) \\ x(0) = g^0, \quad x(s) = g^1(s) \quad \text{a.e.} \quad s \in [-h, 0), & (8.2) \\ \quad u \in U_{ad} \subset L_\infty(I; Y), & (8.3) \end{cases}$$

where $f: X \times Y \times I \to X$ is a nonlinear control term. By using suitable modifications we can develop optimal control theory as in previous sections for the semi-linear system (8.1)-(8.3). As a part of the theory we shall give the maximum principle for a general integral cost.

A continuous solution $x(t) = x_u(t)$ of the integral equation

$$x(t) = \int_0^t G(t-s)f(x(s),u(s),s)ds + \left(G(t)g^0 + \int_{-h}^{0} U_t(s)g^1(s)ds \right) \quad t \in I$$

is called the mild solution of (8.1)-(8.3). We define the set U_{ad}^0 and the cost $J = J(u,x)$ by

$$U_{ad}^0 = \{ u \in U_{ad} \colon \text{the mild solution } x_u(t) \text{ exists on } I \}$$

and

$$J = \int_I w(x(t),u(t),t)dt, \tag{8.4}$$

respectively. Here in (8.6) $w: X \times Y \times I \to R$ is a cost integrand. We shall call a pair $(\bar{u}, x_{\bar{u}}) \in U_{ad}^0 \times C(I; X)$ the optimal solution for J in (8.4) if $\bar{u}$ satisfies

$$\inf_{u \in U_{ad}^0} J(u,x) = J(\bar{u}, x_{\bar{u}}).$$

To state the maximum principle precisely, we require the following assumption H_6 on f and w.

H_6: (1) $f: X \times Y \times I \to X$ and $w: X \times Y \times I \to R$ are continuous in $(x,u) \in X \times Y$ and measurable in $t \in I$;

(2) for each $u(\cdot) \in U_{ad}$ there exists a function $\theta_7: R^+ \times I \to R^+$ such that for all $x \in X$

$$|f(x,u(t),t)|\ ,\ |w(x,u(t),t)| \le \theta_7(|x|,t) \quad \text{a.e. } t \in I,$$

and $\theta_7(r,\cdot) \in L_1(I;\ R)$, $\theta_7(\cdot,t)$ are monotonically increasing for all $(r,t) \in R^+ \times I$;

(3) f and w are continuously Fréchet differentiable in the first argument and the corresponding derivatives $\partial_1 f(x,u,t) \in L(X)$ and $\partial_1 w(x,u,t) \in X^*$ are continuous in $(x,u) \in X \times Y$ and measurable in $t \in I$ and further for each $u(\cdot) \in U_{ad}$,

$$\| \partial_1 f(x,u(t),t) \| \le \theta_8(|x|,t) \quad \text{a.e. } t \in I$$

$$| \partial_1 w(x,u(t),t) |_{X^*} \le \theta_8(|x|,t) \quad \text{a.e. } t \in I,$$

where $\theta_8: R^+ \times I \to R^+$ is as in (2).

Let $(\bar{u},\bar{x}) \in U^0_{ad} \times C(I;\ X)$ be an optimal solution for J in (8.4). By virtue of $H_6(3)$ we can construct a family of bounded operators $U(t,s) \in L(X)$, $0 \le s \le t \le T$ by the solution of the operator integral equation

$$U(t,s)x = G(t-s)x + \int_s^t G(t-\xi)\partial_1 f(\bar{x}(\xi),\bar{u}(\xi),\xi)U(\xi,s)x\,d\xi \quad \text{for any } x \in X. \tag{8.5}$$

The following theorem gives a general form of the maximum principle [19].

<u>THEOREM 8.1.</u> Let U_{ad} be given by (5.2) and assumptions H_4 and H_6 be satisfied. Let $(\bar{u},\bar{x}) \in U^0_{ad} \times C(I;\ X)$ be an optimal solution for J in (8.4). Then

$$\max_{u \in U(t)} H(t,u) = H(t,\bar{u}(t)) \quad \text{a.e. } t \in I,$$

where

$$p(t) = -\int_t^T U^*(s,t)\partial_1 w(\bar{x}(s),\bar{u}(s),s)\,ds \quad t \in I. \tag{8.6}$$

Furthermore if X is reflexive, $p \in C(I; X^*)$ satisfies

$$\begin{cases} \dfrac{dp(t)}{dt} + A_0^* p(t) + \displaystyle\int_{-h}^{0} d\eta^*(s)p(t-s) + \partial_1 f^*(\bar{x}(t),\bar{u}(t),t)p(t) \\ \qquad\qquad\qquad\qquad = \partial_1 w(\bar{x}(t),\bar{u}(t),t) \quad \text{a.e. } t \in I \\ p(s) = 0 \qquad s \in [T, T+h] \end{cases} \tag{8.7}$$

in the weak sense. Here in (8.6) and (8.7) $U^*(s,t)$ and $\partial_1 f^*(\bar{x}(t),\bar{u}(t),t)$ denote the adjoint operators of $U(s,t)$ given in (8.5) and $\partial_1 f(\bar{x}(t),\bar{u}(t),t)$ given in $H_6(3)$, respectively.

(Proof) This theorem can be proved by calculating the first variation of J in (8.4) and applying the Lebesgue's density theorem.

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