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Robust Stability of Linear Quadratic Regulators under System Uncertainties

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1. Introduction

For the past decade, considerable interest has beed directed to the robustness of multivariable systems designed based on the LQ technique in the presence of system uncertainties [1]-[7]. Safonov and Athans [1] have considered the robustness of LQ regulators againt large dynamical, time-varying, and nonlinear perturbations in the feedback gains; they show that if the control weighting matrix is diagonal, any LQ regulator has at least ± 60 degrees phase margin, infinite gain margin, and 50 percent gain reduction tolerance in each feedback channel. In [2], Lehtomaki, Sandell, and Athans have derived frequencey domain robustness results for LQ as well as LQG designs in terms of the minimum singular value of the return difference transfer matrix.

Wong and Athans [3] have explicitly parametrized constant perturbations in system matrices A, B such that LQ regulators will not be destabilized. In [4], Patel, Toda and Sridhar have developed the robustness results of LQ regulators in terms of bounds on the perturbations in the system matrices A, B

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such that the feedback system will remain stable. Combining the results of [1] and [4], Ikeda [5] has developed robustness results for LQ regulators, showing that the stability margin will be reduced under the parameter perturbations. Moreover, Safonov [6] has derived stability conditions based on the abstract results on the input/output stability of such general systems that may include nonlinear time-varying dynamical perturbations in the A matrix. And recently, Soroka and Shaked [7] have shown by an example that LQ regulators may not be robust where small changes in system parameters may lead to fast unstable closed loop modes. Although the stability results are obtained for general nonlinear, time-varying, and dynamical perturbations in [2],[4]-[6], they are implicit relations between bounds and the weighting matrices, since the solution of the algebraic Riccati equation (ARE) is included. Hence, to obtain the stability conditions, we have to solve the ARE's for several times, thereby limiting the usefulness of the stability results. In fact, in [1] the results are stated without resort to the solution of the ARE.

This paper considers the problem of robust stability of LQ regulators in the presence of system uncertainties under the assumption that the perturbations satisfy the matching conditions [8]. Some stability results for linear, nonlinear, time-varying, and dynamical perturbations are stated explicitly in terms of perturbations and the weighting matrices in the performance index. Thus a designer can easily select appropriate weighting matrices for the LQ regulator design such that the closed loop system remains robustly stable. The problem is formulated in section 2. Robust stability results for constant linear perturbations are derived in section 3.1. Section 3.2 deals with the case of nonlinear, time-varying perturbations, and section 3.3 is devoted to the dynamical perturbations. The concluding remarks are given in section 4.

2. Problem Statement

Consider a linear time-invarinat system

$$(\Sigma) \qquad \dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_o \tag{1}$$

where $x(t)$ is the n-dimensionl state vector, $u(t)$ is the m-dimensional control vector, and A and B are constant matrices of dimensions $n \times n$ and $n \times m$, respectively.

The optimal LQ regulator problem is to find the optimal control that minimizes the performance index

$$J = \int_0^\infty [x^T(t)Qx(t) + u^T(t)Ru(t)]dt \tag{2}$$

subject to the constraint of (1), where Q is an $n \times n$ nonnegative definite matrix, R is an $m \times m$ positive definite matrix, and $(\cdot)^T$ denotes the transpose of a vector or a matrix. Then the optimal control $u^*(t)$ and the optimal state trajectory $x^*(t)$ are given by

$$u^*(t) = -Kx^*(t) = -R^{-1}B^TPx^*(t) \tag{3}$$

$$\dot{x}^*(t) = (A - BK)x^*(t), \qquad x^*(0) = x_o \tag{4}$$

where K is the $m \times n$ optimal feedback gain, and P is the solution of the ARE

$$A^TP + PA - PBR^{-1}B^TP + Q = 0 \tag{5}$$

It is well known [9],[10] that the ARE of (5) has the unique nonnegative definite solution, and the associated closed loop system (4) is asymptotically stable if and only if (A, B) is stabilizable and $(\sqrt{Q}, A)$ is detectable, where $\sqrt{Q}$ is a square root of Q.

In this paper, we consider the following three types of perturbations or system uncertainties.

Case A: Constant perturbations

We consider a perturbed linear system

$$(\tilde{\Sigma}_L) \qquad \dot{x} = (A + BF)x + BGu, \quad u = -Kx, \quad x(0) = x_o \qquad (6)$$

where F is an m x n constant matrix, and G is an m x m nonsingular matrix. It should be noted that the parameter perturbations satisfy the matching conditions[8], namely, $\Delta A = BF$, $\Delta B = B(G - I)$, where $F = 0$ and $G = I$ are nominal values. It is also interesting to note that the perturbed system $(\tilde{\Sigma}_L)$ is stabilizable (or controllable) if the nominal system (Σ) of (1) is stabilizable (or controllable). We observe that although the parameter perturbations are restricted to the range space of B, this type of perturbations can cover a fairly large class of perturbations, including the parameter variations in mechanical systems.

Case B: Nonlinear and time-varying perturbations

Consider a perturbed nonlinear time-invariant system

$$(\tilde{\Sigma}_{NL}) \qquad \dot{x} = Ax + Bf(x) + Bg(u), \quad u = - Kx, \quad x(0) = x_o \qquad (7)$$

where $f: R^n \to R^n$ and $g: R^m \to R^m$ are memoryless nonlinearities satisfying the Lipschitz conditions with $f(0) = 0$ and $g(0) = 0$, so that the perturbed system $(\tilde{\Sigma}_{NL})$ has a unique solution defined for all $t > 0$. Note that $f(x) \equiv 0$ and $g(u) = Iu$ are the nominal functions.

We also consider a nonlinear time-varying system

$$(\tilde{\Sigma}_{NLT}) \qquad \dot{x} = Ax + Bf(t,x) + Bg(t,u), \quad u = -Kx, \quad x(0) = x_o \qquad (8)$$

where $f: R \times R^n \to R^n$ and $g: R \times R^m \to R^m$ are time-varying memoryless nonlinear perturbations that are continuous in t, x and satisfy the uniform Lipschitz conditions, namely, there exists a constant $c > 0$ such that

$$\|f(t,x) - f(t,y)\| \leq c \|x - y\| \qquad (9)$$

$$\|g(t,u) - g(t,v)\| \leq c\,\|u - v\| \tag{10}$$

for all $t > 0$, and $x, y \in R^n$, $u, v \in R^m$. It is also assumed that $f(t,0) = 0$ and $g(t,0) = 0$. Under the above conditions, the perturbed system $(\tilde{\Sigma}_{NLT})$ has a uniuqe solution defined for all $t > 0$, and $x_e = 0$ is an equilibrium solution.

Case C: Linear dynamical perturbations

We consider a linear perturbed system described by

$$(\tilde{\Sigma}_{LD}) \qquad \dot{x} = Ax + (B\mathcal{F}x)(t) + (B\mathcal{G}u)(t), \quad u = -Kx, \quad x(0) = x_o \tag{11}$$

where $\mathcal{F}: L^n_{2e}[0, \infty) \to L^m_{2e}[0, \infty)$ and $\mathcal{G}: L^m_{2e}[0, \infty) \to L^m_{2e}[0, \infty)$ are finite-gain linear time-invariant operators with stable rational transfer function matrices $F(s)$ and $G(s)$, respectively [1],[6]. Thus the Laplace transform of the weighting pattern of (11) is given by

$$W(s) = [sI - A - BF(s) + BG(s)K]^{-1} \tag{12}$$

For each of Cases A, B, C above, we wish to derive conditions such that the feedback controller $u(t) = -Kx(t)$ guarantees the asymptotic stability in the large of the perturbed systems of (6), (7), (8) and (11) in terms of perturbations and the weighting matrices Q and R.

3. Main Results

3.1 Constant perturbations

Theorem 1: Suppose that (A, B) is stabilizable and $(\sqrt{Q}, A)$ is detectable. If there exist some positive numbers α and β such that

$$G^TR + RG - (1+\beta)R \geqq 0 \tag{13}$$

and

$$\beta(1-\alpha)Q \geqq F^TRF \tag{14}$$

then the closed loop system $(\tilde{\Sigma}_L)$ of (6) is asymptotically stable.

Proof: Suppose that (6) is not asymptotically stable. Then $A + BF - BGR^{-1}B^TP$ has an eigenvalue λ with nonnegative real part, namely,

$$(A + BF - BGR^{-1}B^TP)w = \lambda w, \qquad \mathrm{Re}\,\lambda \geqq 0, \quad w \neq 0 \tag{15}$$

By manipulating the ARE of (5), we have

$$\begin{aligned}(A + BF - BGR^{-1}B^TP)^TP &+ P(A + BF - BGR^{-1}B^TP)\\ &+ PBR^{-1}[G^TR + RG - (1+\beta)R]R^{-1}B^TP\\ &+ (F - \beta B^{-1}B^TP)^TR(F - \beta R^{-1}B^TP)/\beta\\ &+ [\beta(1-\alpha)Q - F^TRF]/\beta + \alpha Q = 0\end{aligned} \tag{16}$$

Premultiplying w^H and postmultiplying w to (16), and using (15) yield

$$\begin{aligned}2(\mathrm{Re}\,\lambda)w^HPw &+ w^HPBR^{-1}[G^TR + RG - (1+\beta)R]R^{-1}B^TPw\\ &+ w^H(F - \beta R^{-1}B^TP)^TR(F - \beta R^{-1}B^TP)w/\beta\\ &+ w^H[\beta(1-\alpha)Q - F^TRF]w/\beta + \alpha w^HQw = 0\end{aligned} \tag{17}$$

where $(\cdot)^H$ denotes the conjugate transpose or the adjoint of a vector or matrix. From hypothesis, each term of (17) is nonnegative, so that

$$\begin{aligned}&w^H[\beta(1-\alpha)Q - F^TRF]w = 0\\ &(F - \beta R^{-1}B^TP)w = 0, \quad w^HQw = 0\end{aligned} \tag{18}$$

Thus it follows that $\sqrt{Q}w = 0$, $Fw = 0$, $R^{-1}B^TPw = 0$, so that we have from (15)

$$Aw = \lambda w, \quad \sqrt{Q}w = 0, \qquad \mathrm{Re}\,\lambda \geqq 0, \quad w \neq 0 \tag{19}$$

This contradicts the detectability of $(\sqrt{Q}, A)$. □

Remark 1: We observe from (17) that (13) and

$$(F - \beta R^{-1}B^TP)^TR(F - \beta R^{-1}B^TP)/\beta + (1 - \alpha)Q - F^TRF/\beta$$

$$= (1 - \alpha)Q - PBF - F^TB^TP + \beta PBR^{-1}B^TP > 0 \tag{20}$$

may be less restrictive conditions for the asymptotic stability of $(\tilde{\Sigma}_L)$. But since (20) includes the solution P of the ARE, it is necessary to solve the ARE of (5) iteratively to obtain a possibly better stability condition. On the other hand, the conditions (13) and (14) express explicitly the stability region in terms of the perturbations and weighting matrices, enabling a designer to select appropriate Q and R quite easily, although the stability region may be conservative. □

Taking $F = 0$ in Theorem 1, we have the well known result [1],[3].

<u>Corollary 1</u>: Suppose that (A, B) is stabilizable and $(\sqrt{Q}, A)$ is detectable. Let $F = 0$ in Theorem 1. If G satisfies

$$G^TR + RG - R > 0 \tag{21}$$

then the closed loop system $(\tilde{\Sigma}_L)$ with $F = 0$ is asymptotically stable. Moreover, if $Q > 0$, the condition (21) can be weakened as $G^TR + RG - R \geq 0$.

<u>Proof</u>: For $F = 0$, (16) is reduced to

$$(A - BGR^{-1}B^TP)^TP + P(A - BGR^{-1}B^TP)$$

$$+ PBR^{-1}[G^TR + RG - R]R^{-1}B^TP + Q = 0 \tag{22}$$

The rest of the proof is similar to that of Theorem 1. □

<u>Corollary 2</u>: Suppose that (A, B) is stabilizable and $(\sqrt{Q}, A)$ is detectable. Let $G = I$ in Theorem 1. If F satisfies

$$(1 - \alpha)Q \geq F^TRF, \qquad \alpha > 0 \tag{23}$$

then the closed loop system $(\tilde{\Sigma}_L)$ with $G = I$ is asymptotically stable.

<u>Proof</u>: A proof is omitted. □

3.2 Nonlinear and time-varying perturbations

In this section, we present stability results for perturbed nonlinear systems $(\tilde{\Sigma}_{NL})$ and $(\tilde{\Sigma}_{NLT})$.

Theorem 2: Suppose that (A, B) is stabilizable and $(\sqrt{Q}, A)$ is observable. If there exist positive constants α, β such that

$$2u^TRg(u) - (1+\beta)u^TRu \geqq 0 \tag{24}$$

for all $u \in R^m$, and

$$f^T(x)Rf(x) \leqq \beta(1-\alpha)x^TQx \tag{25}$$

for all $x \in R^n$, then the closed loop system $(\tilde{\Sigma}_N)$ of (7) is asymptotically stable in the large.

Proof: Note that the unique nonnegative definite solution P of the ARE of (5) becomes positive definite, since $(\sqrt{Q}, A)$ is observable. Thus $V(x) = x^TPx > 0$ for $x \neq 0$. Using the ARE of (5), the time derivative of $V(x)$ along the motion of (7) is given by

$$\begin{aligned}\dot{V}(x) &= [Ax + Bf(x) + Bg(u)]^TPx + x^TP[Ax + Bf(x) + Bg(u)] \\ &= -[2u^TRg(u) - (1+\beta)u^TRu] - [f(x)+\beta u]^TR[f(x)+\beta u]/\beta \\ &\quad - [\beta(1-\alpha)x^TQx - f^T(x)Rf(x)]/\beta - \alpha x^TQx\end{aligned} \tag{26}$$

where $u = -R^{-1}B^TPx$. It follows from (24) and (25) that $\dot{V}(x) \leqq 0$ for all $x \in R^n$. Thus the perturbed system $(\tilde{\Sigma}_N)$ is stable in the sense of Lyapunov.

To prove the asymptotic stability, we need the following lemma.

Lemma (LaSalle and Lefschetz [11])

Consider an autonomous system

$$\dot{x} = F(x), \quad F(0) = 0, \quad x(0) = x_o \tag{FA}$$

If the solution of (FA) is bounded for all $t > 0$, then its positive limit set Γ^+ is nonempty, compact and invariant. □

Note that since our solution $x(t)$ of the perturbed system $(\tilde{\Sigma}_N)$ is bounded, we can apply the above lemma. Let Γ^+ be the positive limit set of (7). Since $V(x(t)) \geqq 0$ is non-increasing, there exists a constant ℓ such that

$$\lim_{t\to\infty} V(x(t)) = \ell \geqq 0 \tag{27}$$

Because Γ^+ is nonempty, we can take an element $\xi \in \Gamma^+$. Let $\phi(t)$ be the solution of $(\tilde{\Sigma}_N)$ with the initial value ξ. Since Γ^+ is invariant, $\phi(t)$ belongs to Γ^+ for all $t > 0$. But by the definition of the positive limit set,

$$\Gamma^+ \subset C_\ell = \{x \mid V(x) = \ell\} \tag{28}$$

Thus $V(\phi(t)) = \ell$, so that $dV(\phi(t))/dt = 0$. Hence, it follows from (26) that

$$\begin{aligned} &\beta(1-\alpha)\phi^T(t)Q\phi(t) - f^T(\phi(t))Rf(\phi(t)) = 0 \\ &f(\phi(t)) + \beta u(t) = 0, \qquad \phi^T(t)Q\phi(t) = 0 \end{aligned} \tag{29}$$

where $u(t) = -R^{-1}B^TP\phi(t)$. Hence we have $\sqrt{Q}\phi(t) = 0$, $f(\phi(t)) = 0$, $u(t) = 0$. Therefore, from (7)

$$\dot{\phi}(t) = A\phi(t), \quad \phi(0) = \xi, \quad \sqrt{Q}\phi(t) = 0 \tag{30}$$

Since $(\sqrt{Q}, A)$ is observable, we get $\xi = 0$. Hence $\xi \in C_\ell$, so that ξ must be zero. Thus from (27), $\lim_{t\to\infty} x(t) = 0$ for any $x_o \in R^n$. This implies that the perturbed system $(\tilde{\Sigma}_{NL})$ is asymptotically stable in the large. □

Remark 2: It should be noted that the detectability of $(\sqrt{Q}, A)$ is not sufficient for ensuring the asymptotic stability of $(\tilde{\Sigma}_{NL})$. Also, note that if Q is positive definite, the proof of Theorem 2 is immediate, because it follows from (26) that $\dot{V}(x) < 0$ for all $x \neq 0$. □

Corollary 3: Suppose that (A, B) is stabilizable and $(\sqrt{Q}, A)$ is observable. Let $f(x) \equiv 0$ in Theorem 2. If

$$2u^T Rg(u) - u^T Ru > 0 \tag{31}$$

for all $u \in R^m$, then the closed loop system $(\tilde{\Sigma}_N)$ with $f(x) \equiv 0$ is asymptotically stable in the large. Furthermore, if $Q > 0$, then (31) is weakened as $2u^T Rg(u) - u^T Ru \geqq 0$.

Proof: A proof is omitted. □

Corollary 4: Let $g(u) = Iu$ in Theorem 2. If there exists a positive constant α such that

$$f^T(x)Rf(x) \leqq (1-\alpha)x^T Qx \tag{32}$$

for all $x \in R^n$, then the closed loop system $(\tilde{\Sigma}_{NLT})$ with $g(u) = Iu$ is asymptotically stable in the large.

Proof: A proof is omitted. □

The second observation in Remark 2 suggests that the above results can be extended to the case of nonlinear time-varying perturbations.

Theorem 3: Suppose that (A, B) is stabilizable and Q is positive definite. If there exist positive constants α, β, k such that

$$\frac{1+\beta}{2} u^T Ru \leqq u^T Rg(t,u) \leqq k\, u^T u \tag{33}$$

for all $u \in R^m$ and $t > 0$, and

$$f^T(t,x)Rf(t,x) \leqq \beta(1-\alpha)x^T Qx \tag{34}$$

for all $x \in R^n$ and $t > 0$, then the perturbed system $(\tilde{\Sigma}_{NLT})$ of (8) is asymptotically stable in the large.

Proof: Let $V(x) = x^T Px$. Since P is positive definite, $V(x) > 0$ for all $x \neq 0$. Referring to (26), we see from (33) and (34) that the time derivative $\dot{V}(x)$ of $V(x)$ along the motion of (8) is evaluated as

$$\dot{V}(x) \leqq -\alpha x^T Q x < 0, \quad x \neq 0 \tag{35}$$

since Q is positive definite. Thus we observe that $V(x) = x^T P x$ satisfies all the conditions necessary for ensuring the asymptotic stability in the large (Kalman and Bertram [12, Theorem 1]).

Corollary 5: Let $f(t,x) \equiv 0$ in Theorem 3. If

$$\frac{1}{2} u^T R u \leqq u^T R g(t,u) \leqq k\, u^T u \tag{36}$$

holds for some $k > 0$ and for all $x \in R^n$, then the perturbed system $(\tilde{\Sigma}_{NLT})$ is asymptotically stable in the large.

Proof: For $f(t,x) \equiv 0$, (34) is satisfied with $\beta = 0$. □

Corollary 6: Let $g(t,u) = Iu$ in Theorem 3. If

$$f^T(t,x) R f(t,x) \leqq (1-\alpha) x^T Q x \tag{37}$$

holds for some $\alpha > 0$ and for all $x \in R^n$, then the perturbed system $(\tilde{\Sigma}_{NLT})$ is asymptotically stable in the large.

Proof: For $g(t,u) = Iu$, (33) is satisfied with $\beta = 1$. □

3.3 Linear dynamical perturbations

In this section, we consider the robust stability of the perturbed system $(\tilde{\Sigma}_{LD})$ of (11) based on the approach due to Safonov and Athans [1].

Theorem 4: Suppose that (A, B) is stabilizable and Q is positive definite. If there exist positive constants α, β such that

$$G^T(-j\omega)R + RG(j\omega) - (1+\beta)R \geqq 0 \tag{38}$$

and

$$\beta(1-\alpha)Q \geqq F^T(-j\omega)RF(j\omega) \tag{39}$$

for all ω, then the perturbed system $(\tilde{\Sigma}_{LD})$ is asymptotically stable in the large.

Proof: We follow Safonov and Athans [1]. Let x_τ be the trancation defined by

$$x_\tau = \begin{cases} x(t), & 0 \leqq t \leqq \tau \\ 0, & \text{otherwise} \end{cases} \tag{40}$$

and similarly for u_τ. Let $<\ ,\ >$ be the inner product in $L_2[0, \infty)$, namely

$$<x, y> = \int_0^\infty x^T(t)y(t)dt \tag{41}$$

for $x, y \in L_2[0, \infty)$. Let $V(x) = x^T Px$. Then, by using (11) and (5)

$$\begin{aligned} x_o^T Px_o - x^T(\tau)Px(\tau) &= -\int_0^\tau \dot{V}(x)dt \\ &= -2<Px\ ,\ (Ax_\tau + B\mathcal{F}x_\tau + B\mathcal{G}u_\tau)> \\ &= -<x_\tau, (PA + A^TP)x_\tau> + 2<Ru_\tau, \mathcal{F}x_\tau + \mathcal{G}u_\tau> \\ &= <x_\tau, Qx_\tau> - <x_\tau, PBR^{-1}B^TPx_\tau> \\ &\quad + 2<Ru_\tau, \mathcal{F}x_\tau> + <Ru_\tau, 2\mathcal{G}u_\tau> \\ &= \alpha<x_\tau, Qx_\tau> + <Ru_\tau, [2\mathcal{G} - (1+\beta)I]u_\tau> \\ &\quad + \beta<u_\tau + \beta^{-1}\mathcal{F}x_\tau, R(u_\tau + \beta^{-1}\mathcal{F}x_\tau)> \\ &\quad + <x_\tau, [(1-\alpha) - \beta^{-1}\mathcal{F}^*R\mathcal{F}]x_\tau> \end{aligned} \tag{42}$$

where $\mathcal{F}^*$ is the adjoint operator of $\mathcal{F}$. But it follows from Parseval theorem and (38), (39) that since x_τ, u_τ are square integrable,

$$\begin{aligned} <Ru_\tau, [2\mathcal{G} - (1+\beta)I]u_\tau> &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{U}^H(j\omega)[G^H(j\omega)R + RG(j\omega) - (1+\beta)R]\tilde{U}(j\omega)d\omega \\ &\geqq 0 \end{aligned} \tag{43}$$

$$\begin{aligned} <x_\tau, [(1-\alpha)Q - \beta^{-1}\mathcal{F}^*R\mathcal{F}]x_\tau> &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{X}^H(j\omega)[(1-\alpha)Q - \beta^{-1}F^H(j\omega)RF(j\omega)]\tilde{X}(j\omega)d\omega \\ &\geqq 0 \end{aligned} \tag{44}$$

where $\tilde{U}(j\omega)$ and $\tilde{X}(j\omega)$ are the Fourier transforms of u_τ and x_τ, respectively. Thus we have

$$x_o^T P x_o \geq \alpha \langle x_\tau, Qx_\tau \rangle \tag{45}$$

Taking $\tau \to \infty$ in (45), we see that $x(t)$ is square integrable, since Q is positive definite. Since $\mathcal{F}$, $\mathcal{G}$ are of finite gains, $\dot{x}(t)$ is also square integrable. Thus we have $\lim_{t\to\infty} x(t) = 0$ for any $x_o \in R^n$. It follows that the weighting pattern $w(t)$, the inverse Laplace transform of (12), is asymptotically stable. Therefore, if there are no hidden unstable modes which will be due to pole-zero cancellations in $W(s)$, then $(\tilde{\Sigma}_{LD})$ is asymptotically stable.

The dynamics of $(\tilde{\Sigma}_{LD})$ are expressed as

$$\begin{bmatrix} sI - A & -B \\ L_N(s) & L_D(s) \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = 0 \tag{46}$$

where $L(s) = G(s)K - F(s)$, and $L_N(s)$ and $L_D(s)$ are coprime polynomial matrices satisfying $L(s) = L_D^{-1}(s)L_N(s)$ with $L_D(s)$ row reduced [13]. Note that all the roots of $\det[L_D(s)]$, the poles of $L(s)$, have negative real parts, since $G(s)$ and $F(s)$ are stable. It suffices to require that the roots of the characteristic polynomial $p(s)$ for (46) have all negative real parts for $(\tilde{\Sigma}_{LD})$ to be asymptotically stable in the large. As in [1], it follows from (46) and (12) that

$$\begin{aligned} p(s) &= \det \begin{bmatrix} sI - A & -B \\ L_N(s) & L_D(s) \end{bmatrix} \\ &= \det[L_D(s)]\det[sI - A + BL(s)] \\ &= \det[L_D(s)]/\det[W(s)] \end{aligned} \tag{47}$$

so that

$$\det[W(s)] = \frac{\det[L_D(s)]}{p(s)} \tag{48}$$

Thus we see that possible pole-zero cancellations in (48) is associated with stable zeros of $p(s)$, since all the roots of $\det[L_D(s)]$ have negative real parts, so that there are no unstable hidden modes in $W(s)$. Also, all the roots of $\det[W(s)]$ have negative real parts, since $w(t)$ is asymptotically stable. Therefore $p(s)$ is a stable polynomial. □

<u>Remark 3</u>: Suppose that

$$G^T(-j\omega)R + RG(j\omega) - (1+\beta+\gamma)R \geqq 0 \tag{49}$$

holds for all ω and $\gamma > 0$ instead of (38), then we have

$$x_o^T P x_o \geqq \alpha \langle x_\tau, Qx_\tau \rangle + \gamma \langle u_\tau, Ru_\tau \rangle \tag{50}$$

which is similar to the estimate obtained in [1]. □

We consider the special case where

$$R = \mathrm{diag}(r_1, \ldots, r_m), \quad r_i > 0 \tag{51}$$

and the perturbation $\mathcal{G}$ satisfies

$$\mathcal{G}u = \begin{bmatrix} \mathcal{G}_1 u_1 \\ \vdots \\ \mathcal{G}_m u_m \end{bmatrix}, \text{ or } G(s) = \mathrm{diag}[G_1(s), \ldots, G_m(s)] \tag{52}$$

where $G_i(s)$ are stable rational transfer function.

<u>Corollary 7</u>: If the perturbed system $(\tilde{\Sigma}_{LD})$ satisfies (51) and (52), and if $\mathrm{Re}[G_i(j\omega)] \geqq (1+\beta)/2$ holds for all ω together with (39), then $(\tilde{\Sigma}_{LD})$ is asymptotically stable in the large. □

Put $G_i(j\omega) = e^{j\theta_i(\omega)}$ in Corollary 7. Then it follows that

$$\cos\theta_i(\omega) \geqq \frac{1+\beta}{2} \quad \text{for all } \omega,\ i = 1, \ldots, m \tag{53}$$

Therefore the phase margin in each channel becomes less than ± 60 degrees, because, in general, $\beta > 0$ due to the perturbation $\mathcal{F}$. But we see from (39) that ± 60 degrees phase margin can be recovered by taking Q sufficiently large while keeping β arbitrarily small. This is a generalization of the result of [1].

Example: We consider a single-input system, namely, $m = 1$. Let

$$R = r \text{ (scalar)}$$

$$F(j\omega) = (f_1(j\omega),\ldots, f_n(j\omega)) \quad [1 \times n \text{ vector}]$$

Then the conditions (38) and (39) are reduced to

$$\mathrm{Re}[G(j\omega)] \geqq \frac{1+\beta}{2} \tag{54}$$

$$\frac{\beta(1-\alpha)}{r} Q \geqq \begin{bmatrix} |f_1|^2 & \bar{f}_1 f_2 & \cdots & \bar{f}_1 f_n \\ \bar{f}_2 f_1 & |f_2|^2 & \cdots & \bar{f}_2 f_n \\ & \cdots & & \\ \bar{f}_n f_1 & \bar{f}_n f_2 & \cdots & |f_n|^2 \end{bmatrix} \tag{55}$$

for all ω and $\alpha, \beta > 0$. Since $F^T(-j\omega)F(j\omega) \leqq (\Sigma_{i=1}^n |f_i(j\omega)|^2)\cdot I$, if Q satisfies

$$Q = \mathrm{diag}(q_1,\ldots,q_n), \qquad q_i > \frac{r}{\beta}\rho^2, \; i = 1,\ldots, n \tag{56}$$

where $\rho^2 = \sup_\omega \Sigma_{i=1}^n |f_i(j\omega)|^2$, then (55) is satisfied. □

Several examples for the control of mechanical systems with constant, nonlinear perturbations are found in Sasaki [14].

4. Concluding Remarks

This paper has considered the robust stability of LQ regulators in the presence of system uncertainties. Under the assumption that the perturbations satisfy the matching conditions, the robustness results are derived explicitly in terms of perturbations and weighting matrices for constant,nonlinear, time-varying, and linear dynamical perturbations. Theorem 3 and 4 generalize the results due to Safonov and Athans [1].

Robust stablity of Kalman filter and nonlinear filters will be treated in a future paper.

References

[1] M.G. Safonov and M. Athans, "Gain and phase margins for multiloop LQG regulators," IEEE Trans. Automat. Contr. vol. AC-22, pp. 173-179, April 1977.

[2] N.A. Lehtomaki, N.R. Sandell, and M. Athans, "Robustness results in linear-quadratic Gaussian based multivariable control designs," IEEE Trans. Automat. Contr., vol. AC-26, pp. 75-93, Feb. 1981.

[3] P.K. Wong and M. Athans, "Closed-loop structural stability for linear-quadratic optimal systems," IEEE Trans. Automat. Contr., vol. AC-22, pp. 94-99, Feb. 1977.

[4] R.V. Patel, M. Toda, and B. Sridhar, "Robustness of linear quadratic state feedback designs in the presence of system uncertainty," IEEE Trans. Automat. Contr., vol. AC-22, pp. 945-949, Dec. 1977.

[5] M. Ikeda, "On robust properties of optimal regulators," 8th Summer Seminar on Systems Analysis, Kobe University, July 1982 (in Japanese).

[6] M.G. Safonov, Stability and Robustness of Multivariable Feedback Systems. Cambridge, Massachusetts: The MIT Press, 1980.

[7] E. Soroka and U. Shaked, "On the robustness of LQ regulators," IEEE Trans. Automat. Contr., vol. AC-29, pp. 664-665, July 1984.

[8] G. Leitmann, "Guaranteed asymptotic stability for some linear systems with bounded uncertainties," Trans. ASME, J. Dynamic Systems, Measurement, and Contr., vol. 101, pp. 212-216, Sept. 1979.

[9] H. Kwakernaak and R. Sivan, Linear Optimal Control Systems. New York: Wiley-Interscience, 1972.

[10] V. Kucera, "A contribution to matrix quadratic equations," IEEE Trans. Automat. Contr., vol. AC-17, pp. 344-347, June 1972.

[11] J. LaSalle and S. Lefschetz, Stability by Lyapunov's Direct Method. New New York: Academic, 1961.

[12] R.E. Kalman and J.E. Bertram, "Control system analysis and design via the 'second method' of Lyapunov I - continuous-time systems," Trans. ASME, J. Basic Eng., vol. 82D, pp. 371-393, June 1960.

[13] T. Kailath, Linear Systems. Englewood Cliffs, NJ: Prentice-Hall, 1980.

[14] S. Sasaki, "On the robust stability analysis of LQ regulators under parameter variations," MS. thesis, Department of Applied Mathematics and Physics, Faculty of Engineering, Kyoto University, Feb. 1985.