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AN ELEMENTARY PROOF OF H^{∞}-MINIMIZATION THEOREM

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1. INTRODUCTION

The H^{∞}-minimization theory, which was initiated by Zames [1] and developed by Zames and Francis [2][3], receives increasing attention as a new methodology for control system design. The core of the present H^{∞}-minimization theory is a generalization of the classical interpolation theory which dates back to Pick [4] and Nevanlinna [5]. Since the application of the classical interpolation theory was innitiated by Youla and Saito [6], it plays important roles in many fields of system theory such as network synthesis [7][8], Hankel-norm model reduction [9], modeling filter of stochastic process [10][11] and robust stabilization of feedback systems [12][13][14].

Recently, Francis and Zames [3] used an advanced result in interpolation theory due to Sarason [15] to obtain an efficient algorithm of H^{∞}-minimization. The Sarason's theorem gave an essentially new formulation of the classical interpolation problem and solved its generalized version. Though the underlying idea of Sarason's theorem is clear, its proof (particulary, of its multivariable version) is not easy to follow for a non-mathematician, because it heavily relies on some advanced tools in functional analysis [16].

The purpose of this paper is to give an elementary proof of the key theorem in H^{∞}-minimization theory with no recourse to Sarason's theorem. Functional analysis is not used except some basic notions like operator norm and inner product. Since this theorem, as a generalization of the classical interpolation theorem, plays a key role, not only in the H^{∞}-minimization theory but also in many branches of system theory mentioned above, to make its proof understable for engineers is believed to be warranted. Also, the proof itself gives a new light on the fundamental structure of H^{∞}-minimization theory.

After the statement of the problem in Section 2, we give a characterization of H^{∞}-norm using the adjoint operator of the multiplication operator in H^2 (Section 3). This is the most essential point of departure of our approach from the Sarason's theorem, which uses the characterization of H^{∞}-norm via a multiplication operator in H^2. The highlight of our arguments is the direct proof of the stability of the optimal interpolation function (Lemma 7 in Section 4). This generalizes the theorem by Genin and Kung [9] with much simpler proof.

2. STATEMENT OF THE PROBLEM

For a complex function h(z), we define its H^{∞}-norm as

$$\|h\|_{\infty} = \text{ess.} \sup_{\omega} |h(e^{j\omega})|. \tag{1}$$

The set of all analytic functions in the open unit disc satisfying $\|h\|_{\infty} < \infty$ is called the Hardy space (of exponent ∞) and denoted by H^{∞}. For complex functions h(z) and g(z), we define an inner product as

$$\langle h, g\rangle = \frac{1}{2\pi j}\int_U \tilde{h}(z^{-1})g(z)z^{-1}dz, \tag{2}$$

where U denotes the unit circle and $\tilde{h}(z) = \overline{h(\bar{z})}$. This is usually referred to as the L^2-inner product. The set of all complex functions h(z) analytic on the open unit disc satisfying $\|h\|_2 = \langle h, h\rangle^{1/2} < \infty$ becomes a Hilbert space under the definition of the inner product (2). This is the Hardy space (of exponent 2) and is denoted by H^2. Obviously, $H^{\infty} \subset H^2$. It is well-known that any $h \in H^2$ has a Taylor series representation

$$h(z) = h_0 + h_1 z + h_2 z^2 + \cdots, \quad \sum_{i=1}^{\infty} |h_i|^2 < \infty. \tag{3}$$

For the sake of notational simplicity, we drop the suffix 2 in the expression of L^2-norm. Therefore, $\|h\|$ always denotes the L^2-norm $\|h\|_2$ of h.

A function $u \in H^\infty$ is called inner if

$$|u(e^{j\omega})| = 1, \quad -\pi \le \omega \le \pi. \tag{4}$$

If u(z) is a real rational inner function which has no zero on U, then it is always of the form

$$u(z) = \left(\frac{z - \lambda_1}{1 - \bar{\lambda}_1 z}\right)^{\nu_1} \left(\frac{z - \lambda_2}{1 - \bar{\lambda}_2 z}\right)^{\nu_2} \cdots \left(\frac{z - \lambda_m}{1 - \bar{\lambda}_m z}\right)^{\nu_m}. \tag{5}$$

$$|\lambda_i| < 1, \quad 1 \le i \le m.$$

Note that, for each z,

$$u(z)u(z^{-1}) = 1. \tag{6}$$

A function $f \in H^\infty$ is called outer if it has no zero on the open unit disc.

Now we state the H^∞-minimization problem:

<u>H^∞-minimization problem</u>

Assume that t(z), $u(z) \in H^\infty$ and u(z) is inner without zeros on U. Find $v(z) \in H^\infty$ such that $\|t - uv\|_\infty$ is minimum.

This problem is originally formulated as the problem of minimizing the sensitivity of closed-loop systems with respect to its weighted H^∞-norm [3]. The reduction of the sensitivity minimization problem to the above problem is done by the parametrization of all stabilizing controllers and the inner-outer factorization of H^∞-functions. For detail, see [3].

3. CHARACTERIZATION OF H^∞-NORM AS AN OPERATOR NORM

In comparison with $H^2(L^2)$-norm, the calculation of H^∞-norm is not easy even for simple functions. Straightforward maximization of $|h(e^{j\omega})|$ with respect to ω is usually not feasible. An effective way of calculating $\|h\|_\infty$ is given by regarding it as the norm of an operator.

Let Π be the projection operator from L^2 to H^2. Associated with a given $h(z) \in H^\infty$, we define an operator $\Gamma_h : H^2 \to H^2$ as

$$\Gamma_h x(z) = \Pi h(z^{-1}) x(z). \tag{7}$$

If both h(z) and x(z) are rational functions in H^∞, $\Gamma_h x(z)$ represents the portion of the partial fraction expansion of $h(z^{-1})x(z)$ corresponding to the poles of x(z). In particular, we have

$$\Gamma_h \frac{1}{1 - \lambda z} = h(\lambda)\frac{1}{1 - \lambda z}, \quad \text{for } |\lambda| < 1. \tag{8}$$

We state some properties of Γ_h for later use.

Lemma 1

(i) *For any* $f \in H^2$,

$$\|\Gamma_h f\| \leq \|h(z^{-1})f\|,$$

the equality holds if and only if $h(z^{-1})f(z) \in H^2$.

(ii) *If* $h, g \in H^\infty$, *then*

$$\Gamma_{hg} = \Gamma_h\Gamma_g = \Gamma_g\Gamma_h$$

(iii) If $g \in H^\infty$ *is outer, then* $\Gamma_g f = 0$, $f \in H^2$, *implies* $f = 0$

(Proof) (i) Since $x - \Pi x$ is orthogonal to H^2 for any $x \in L^2$, we have

$$\langle h(z^{-1})f - \Gamma_h f, \quad \Gamma_h f\rangle = 0.$$

Therefore, $\|h(z^{-1})f - \Gamma_h f\|^2 = \langle h(z^{-1})f - \Gamma_h f, h(z^{-1})f\rangle = \|h(z^{-1})f\|^2 - \langle\Gamma_h f,$

$h(z^{-1})f\rangle = \|h(z^{-1})f\|^2 - \|\Gamma_h f\|^2$, from which the assertion follows immediately.

(ii) For any $f \in H^2$, $\Gamma_{hg} f = \Pi h(z^{-1})g(z^{-1})f(z) = \Pi h(z^{-1})\Pi g(z^{-1})f(z) + \Pi h(z^{-1})(g(z^{-1})f(z) - \Pi g(z^{-1})f(z)) = \Pi h(z^{-1})\Pi g(z^{-1})f(z)$. This proves the assertion.

(iii) $\Gamma_g f = 0$ implies that all the poles of $f(z)$ must be the zeros of $g(z^{-1})$. This is impossible because $g(z^{-1})$ does not vanish for $|z| > 1$. ■

The norm $\|\Gamma_h\|$ of the operator Γ_h is defined as usual by

$$\|\Gamma_h\| = \sup\{\frac{\|\Gamma_h \xi\|}{\|\xi\|} \ ; \ \xi \in H^2\}. \tag{9}$$

The following result gives a characterization of $\|h\|_\infty$ as an operator norm.

Lemma 2 *For* $h \in H^\infty$, $\|h\|_\infty = \|\Gamma_h\|_\infty$.

(Proof) Due to (8), $\|\Gamma_h(1 - \lambda z)^{-1}\| = |h(\lambda)| \cdot \|(1 - \lambda z^{-1})\|$ for each $|\lambda| < 1$. This implies $\|\Gamma_h\| \geq \sup\{|h(\lambda)| \ ; \ |\lambda| < 1\} = \|h\|_\infty$. From Lemma 1 (i), $\|\Gamma_h f\| \leq \|h(z^{-1})f\| \leq \|h\|_\infty \|f\|$, which implies $\|h\|_\infty \geq \|\Gamma_h\|$. This establishes the assertion. ■

Lemma 2 provides an effective way of computing $\|h\|_\infty$. Assume that $h(z)$ is represented by a Taylor series $h(z) = h_0 + h_1 z + h_2 z^2 + \cdots$. We define an infinite matrix of Toeplitz type

$$\tilde{\Gamma}_h = \begin{bmatrix} h_0 & h_1 & h_2 & \cdots \\ 0 & h_0 & h_1 & \cdots \\ 0 & 0 & h_0 & \cdots \\ & \cdots & & \end{bmatrix}$$

If $x(z) \in H^2$ is also represented by a Taylor series $x(z) = x_0 + x_1 z + x_2 z^2 + \cdots$, then, if we define $x = (x_0, x_1, \cdots)^T$, we see that $(\tilde{\Gamma}_h x)_{i+1} = h_0 x_i +$

$h_1 x_{i+1} + \cdots$ is equal to the coefficient of z^i in the Laurènt series of $h(z^{-1}) x(z)$. Therefore, the matrix $\tilde{\Gamma}_h$ is a representation of Γ_h in the space of coefficient sequences. This gives an alternative definition of $\|\Gamma_h\|$ as

$$\|\Gamma_h\| = \sup\{\frac{\|\tilde{\Gamma}_h x\|}{\|x\|} \; ; \; \|x\|^2 = \sum_{i=0}^{\infty} |x_i|^2 < \infty\}.$$

If we take the n×n principal submatrix of $\tilde{\Gamma}_h$ and calculate its maximum singular value $\sigma_n(h)$, then it can be shown that $\sigma_n(h) \to \|\Gamma_h\| = \|h\|_\infty (n \to \infty)$. The convergence is expected to be nice because $\sigma_n(h)$ is obviously non-decreasing with respect to n. A function theoretic proof of Lemma 2 along this line was given by Grenander and Szegö [17].

For $x, y \in H^2$, we have $\langle y, \Gamma_h x\rangle = \langle y, h(z^{-1})x\rangle - \langle y, h(z^{-1})x - \Pi h(z^{-1})x\rangle = \langle y, h(z^{-1})x\rangle = \langle h(z)y, x\rangle$. This implies that the adjoint operator Γ_h^* of Γ_h is the multiplication operator

$$\Gamma_h^* x = h(z)x. \tag{10}$$

Usual characterization of H^∞ norm is $\|\Gamma_h^*\|$, which is dual to Lemma 2. Our characterization of $\|h\|_\infty$ has some advantages over the usual one, which will be made clear in the subsequent developements.

In the definition (9) of $\|\Gamma_h\|$, the supremum is usually not attained by any $\xi \in H^2$; in other words, the supremum is usually not replaced by the maximum. In the case where the supremum is actually attained by some $\xi \in H^2$, $\|\Gamma_h\|^2$ becomes equal to the maximum eigenvalue of $\Gamma_h^* \Gamma_h$ and ξ is the corresponding eigenfunction. More generally, consider the eigenvalue problem

$$\Gamma_h^* \Gamma_h \xi = \mu^2 \xi, \quad \xi \in H^2. \tag{11}$$

If this eigenvalue problem is solvable, it follows from (10) that

$$h(z) = \frac{\xi(z)}{\Gamma_h \xi(z)} \mu^2. \tag{12}$$

The relation (12) suggests that the eigenvalue problem (11) is solvable only for a limited class of h(z). This is clarified by the following lemma.

Lemma 3 *The eigenvalue problem (11) is solvable, if and only if h(z) is of the form*

$$h(z) = \mu b(z) \tag{13}$$

where b(z) is an inner function. The eigenvalue μ^2 *is always equal to* $\| h \|_\infty^2 = \| \Gamma_h \|^2$.

(Proof) Assume that h(z) is given by (13). Since b(z) is inner, $b(z) b(z^{-1}) = 1$ for each z. Therefore, $h(z^{-1}) b(z) = \mu$, from which $h(z) \Pi h(z^{-1}) b(z) = \mu^2 b(z)$. This implies that b(z) is an eigenfunction corresponding to the eigenvalue μ^2.

To prove tha converse, it is sufficient to show that

$$b(z) = \frac{\xi(z)}{\Gamma_h \xi(z)} \mu \tag{14}$$

is an inner function. From (12), it follows that

$$b(z) b(z^{-1}) = 1 + \frac{h(z^{-1})\xi(z) - \Pi h(z^{-1})\xi(z)}{\Pi h(z^{-1})\xi(z)}. \tag{15}$$

The second term in the right-hand side of (15) should be invariant if z and z^{-1} are exchanged. But it is impossible because its denominator is in H^2 and its numerator is orthogonal to H^2. Therefore, the second term should vanish. This establishes the assertion. The last statement of the lemma follows immediately from $|h(e^{j\omega})| = \mu$, $\forall \omega$. ■

4. SOLUTION TO THE H^∞-MINIMIZATION THEOREM

The H^∞-minimization problem formulated in Section 2 is now treated based on the results obtained in the preceding section.

We assume that u(z) is given by (5). Since u(z) is real for real z, each non-real λ_i in (5) must be accompanied by its complex conjugate $\bar{\lambda}_i$. The integer $n = \nu_1 + \nu_2 + \cdots + \nu_m$ denotes the total order of u(z). Let

$$h(z) = t(z) - u(z)v(z), \quad v(z) \in H^\infty. \tag{16}$$

In view of Lemma 2, our objective is to find $v(z) \in H^\infty$ for which $\|\Gamma_h\|$ is minimized.

In order to exploit a convenient form for representing the constraint on h(z) dictated from the form (16), we denote by F the kernel of the operator Γ_u in H^2, i.e.,

$$F = \{f \in H^2 : \Gamma_u f = \Pi u(z^{-1})f = 0\}. \tag{17}$$

The structure of F is quite simple. Since $u(z) \in H^\infty$, $\Pi u(z^{-1})f$ vanishes if and only if the denominator of f(z) is cancelled out by the numerator of $u(z^{-1})$, which equals $(1 - \bar{\lambda}_1 z)^{\nu_1}(1 - \bar{\lambda}_2 z)^{\nu_2} \cdots (1 - \bar{\lambda}_m z)^{\nu_m}$. Therefore, F is spanned by the n functions

$$f_i(z) = \frac{1}{(1 - \bar{\lambda}_j z)^k}, \quad 1 \le k \le \nu_j \ ; \ 1 \le j \le m. \tag{18}$$

In other words, F is the n-dimensional subspace of H^2 spanned by the basis $\{f_1(z), f_2(z), \cdots, f_n(z)\}$.

<u>Lemma 4</u> *h(z) $\in H^\infty$ is of the form (16), if and only if*

$$\Gamma_h f_i = \Gamma_t f_i, \quad 1 \le i \le n. \tag{19}$$

(Proof) If h(z) is of the form (16), then, due to Lemma 1 (ii), $\Gamma_h f_i = (\Gamma_t - \Gamma_{uv}) f_i = \Gamma_t f_i - \Gamma_v \Gamma_u f_i = \Gamma_t f_i$. Conversely, assume that $(\Gamma_h - \Gamma_t) f_i = \Gamma_{h-t} f_i = 0$. This implies that $h(z^{-1}) - t(z^{-1})$ cancelles out all the denominator of f_i. Hence, $h(z^{-1}) - t(z^{-1})$ has a factor $(1 - \bar{\lambda}_1 z)^{\nu_1} \cdots (1 - \bar{\lambda}_m z)^{\nu_m}$. Thus, taking into account that non-real λ_i appears with its complex conjugate, we conclude that $h(z) - t(z)$ is divisable by $(z - \lambda_1)^{\nu_1}(z - \lambda_2)^{\nu_2} \cdots (z - \lambda_m)^{\nu_m}$. This implies that h(z) is of the form (16). ■

For later use, we need the following result.

<u>Lemma 5</u> *For* $g(z) \in H^2$, $\langle f_i, g\rangle = 0$ *if and only if* $\Gamma_g \check{f}_i = 0$.

(Proof) Since $\langle f_i(z), g(z)\rangle = \langle 1, f_i(z^{-1}) g(z)\rangle$, $\langle f_i, g\rangle = 0$, if and only if $\check{f}_i(z^{-1}) g(z) z^{-1} \in H^2$. This implies that $g(z^{-1}) \check{f}_i(z)$ has no H^2 part, which establishes the assertion. ■

Since $t(z) \in H^\infty$, the partial fraction expansion of $t(z^{-1})(1 - \lambda_j z)^{-k}$ has the H^2 projection as a linear combination of $(1 - \lambda_j z)^{-1}$, $(1 - \lambda_j z)^{-2}$, $\cdots$, $(1 - \lambda_j z)^{-k}$. Therefore, F is an invariant subspace of Γ_t, i.e.,

$$\Gamma_t F \subset F. \tag{20}$$

Denote by $\hat{\Gamma}_t$ the restriction of Γ_t on F. In view of Lemma 4, the H^∞-minimization problem is restated as follows:

Find $h \in H^\infty$ such that $\|\Gamma_h\|$ is minimum satisfying

$$\Gamma_h | F = \hat{\Gamma}_t. \tag{21}$$

The constraint (21) obviously implies

$$\inf \|\Gamma_h\| = \inf\{\|t - uv\|_\infty \ ; \ v \in H^\infty\} \geq \|\hat{\Gamma}_t\|. \tag{22}$$

The rest of this section is devoted to showing that the equality in (22) is actually achieved, i.e., there exists Γ_h which "extends" $\hat{\Gamma}_t$ without increasing the norm.

Since $\hat{\Gamma}_t : F \to F$ is finite dimensional operator, its norm $\|\hat{\Gamma}_t\|$ is easily calculated. In order to derive its explicit characterization, define an n-tuple $F(z) + [f_1(z), \cdots, f_n(z)]$. Due to (20), we can find an n×n constant matrix S such that

$$\Gamma_t F(z) = F(z)S. \tag{23}$$

If all the zeros of u(z) are simple, i.e., $\nu_1 = \nu_2 = \cdots = \nu_m = 1$, then

$$S = \mathrm{diag}[t(\bar{\lambda}_1), t(\bar{\lambda}_2), \cdots, t(\bar{\lambda}_m)]. \tag{24}$$

This is easily seen from $\Pi t(z^{-1})(1 - \bar{\lambda}_i z)^{-1} = t(\bar{\lambda}_i)(1 - \bar{\lambda}_i z)$.

From the definition of $\hat{\Gamma}_t$, $\|\hat{\Gamma}_t\| = \mu$ where

$$\mu^2 = \max\{ \frac{\|\Gamma_t \xi\|^2}{\|\xi\|^2} \ ; \ \xi \in F \} = \max\{ \frac{\|\Gamma_t F(z)x\|^2}{\|F(z)x\|^2} \ ; \ x \in \mathcal{C}^n \}. \tag{25}$$

Here, $\mathcal{C}^n$ denotes the set of complex n-tuples.

Define a n×n Hermitian matrix Φ as

$$(\Phi)_{ij} = \langle f_i, f_j \rangle.$$

Then, $\|F(z)x\|^2 = x^* \Phi x$. Also, due to (23), $\|\Gamma_t F(z)x\|^2 = \|F(z)Sx\|^2 = x^* S^* \Phi Sx$. Therefore, $\|\hat{\Gamma}_t\|^2$ is equal to the maximum number μ^2 for which

$$\det(\mu^2 \Phi - S^* \Phi S) = 0. \tag{26}$$

The maximum in (24) is attained by

$$x(z) = F(z)x, \tag{27}$$

where x is the eigenvector corresponding to μ^2, i.e.,

$$(\mu^2\Phi - S^*\Phi S)x = 0. \tag{28}$$

If the equality in (22) holds, $\Gamma_h^*\Gamma_h$ have the eigenvector corresponding to $\|\hat{\Gamma}_t\| = \mu$ in F, which is equal to x(z) given by (27). Thus, we conclude that the equality in (22) holds, only if

$$\Gamma_h^*\Gamma_h x = \mu^2 x. \tag{29}$$

Due to (12) and $\Gamma_h x = \Gamma_t x$, we have

$$h(z) = \frac{x(z)}{\Gamma_t x(z)}\mu^2. \tag{30}$$

In order to show that h(z) given by (30) is actually optimal, it is necessary to prove that (i) $h(z) \in H^\infty$, (ii) h(z) is of the form (16). This will be shown in the following lemmas. We can assume, without loss of generality, that x(z) and $\Gamma_t x(z)$ have no common zero. Indeed, if they have a common zero, the reduced pair $\hat{x}(z)$ and $\Gamma_t\hat{x}(z)$ after cancelling it constitutes another solution to the eigenvalue problem (29). Hence, we can consider $\hat{h}(z) = (\hat{x}(z)/\Gamma_t\hat{x}(z))\mu^2$ instead of h(z).

<u>Lemma 6</u> *For x(z) given by (27), $\Gamma_t x(z)$ is outer.*

(Proof) It is sufficient to show that $\Gamma_t x(z)$ does not vanish for $|z| < 1$. Assume that $\Gamma_t x(z)$ vanish at $z = \lambda$, $|\lambda| < 1$. If we define an inner function $b(z) = (z - \bar{\lambda})/(1 - \lambda z)$ and $\hat{x}(z) = \Gamma_b x(z)$, we have, from Lemma 1 (ii), $\Gamma_t\hat{x}(z) = \Gamma_t\Gamma_b x(z) = \Gamma_b\Gamma_t x(z)$. From the assumption, $b(z^{-1})\Gamma_t x(z) \in H^2$. This implies $\Gamma_t\hat{x}(z) = b(z^{-1})\Gamma_t x(z)$. From Lemma 1 (i) and $x(\lambda) \neq 0$, it follows that $\|\hat{x}(z)\| < \|b(z^{-1})x(z)\| = \|x(z)\|$. Therefore, we have

$$\frac{\|\Gamma_t \hat{x}\|}{\|\hat{x}\|} = \frac{\|b(z^{-1})\Gamma_t x\|}{\|\hat{x}\|} = \frac{\|\Gamma_t x\|}{\|\hat{x}\|} > \frac{\|\Gamma_t x\|}{\|x\|} .$$

Since $\hat{x}(z) \in F$, this contradicts the maximality of μ in (25). The proof has been completed. ■

<u>*Lemma 7*</u> *h(z) given by (30) belongs to* H^∞.

(Proof) In view of Lemma 6, it is sufficient to prove that $\Gamma_t x(z)$ has no zero on the unit circle $|z| = 1$. This can be easily shown by a slight modification of the proof of Lemma 6 with the use of some limiting arguments. The details are omitted. ■

Lemma 7 is a generalization of the stability theorem on classical interpolation problem which was proved by Genin and Kung [9]. Actually, we can prove the converse of Lemma 7, that is, the function h(z) given by (30) belongs to H^∞, only if μ^2 is the maximum eigenvalue of (26). In fact, we can go further as in [9]. Let $\mu_1^2 \geq \mu_2^2 \geq \cdots \geq \mu_n^2$ be the solutions of the equation (26). Then, the corresponding $h_i(z)$, which is defined by (30) with x being replaced by the eigenvector x_i corresponding to μ_i^2, has exactly i-1 zeros on the open unit disc. The proof of this fact will be reported in the forthcoming paper. We just remark that an analogous connection between the maximality and the minimum-phase property of the solution exists in the context of the spectrum factorization through Riccati equation [18].

It remains to show that h(z) given by (30) is of the form (16). According to Lemma 4 it is sufficient to show the following assertion.

<u>*Lemma 8*</u> *For each* f_i,

$$\Gamma_{h-t} f_i = 0. \tag{31}$$

(Proof) Due to (28) and the definition of Φ and S, we have, for each $i = 1, \cdots, n$,

$$<f_i, \mu^2 x> - <\Gamma_t f_i, \Gamma_t x> = 0,$$

where x(z) is given by (27). Since $<\Gamma_t f_i, \Gamma_t x> = <f_i, \Gamma_t^* \Gamma_t x> = <f_i, t(z)\Gamma_t x>$, we have $<f_i, \mu^2 x - t\Gamma_t x> = 0$ for each i. Due to Lemma 5, this implies that $\Gamma_\psi f_i = 0$, for each i, where $\psi(z) = \mu^2 x(z) - t(z)\Gamma_t x(z)$. Here we used the fact that each non-real λ_i appears as a conjugate pair. From the definition of h(z) in (30), $\psi(z) = (h(z) - t(z))\Gamma_t x(z)$. Since $\Gamma_t x(z)$ is outer, Lemma 1 (iii) implies that $\Gamma_{h-t} f_i = 0$ for each i. This completes the proof. ■

Now we state the main theorem which solves the H^∞-minimization problem.

<u>Theorem</u> *Let μ be the largest positive number which satisfies (26) and x be the corresponding eigenvector satisfying (28). Then,*

$$\min\{\| t(z) - u(z)v(z) \|_\infty \; ; \; u \in H^\infty\} = \mu^2, \tag{32}$$

and the optimal $h(z) = t(z) - u(z)v(z)$ for which the minimum (32) is attained is given by (30). The optimal $h(z)$ is of the form

$$h(z) = \mu b(z),$$

where $b(z)$ is an inner function.

(Proof) The main part of the proof has already been done in Lemmas 6 to 8. The last statement follows immediately from Lemma 3. ■

In the case where all the zeros of u(z) are simple, i.e., $\nu_1 = \nu_2 = \cdots = \nu_m = 1$ in (5), the eigenequation (26) can be explicitly written. Since in this case $\langle f_i, f_j\rangle = (1 - \lambda_i\bar{\lambda}_j)^{-1}$ and S is given by (24), we have

$$(\mu^2\Phi - S^*\Phi S)_{ij} = \frac{\mu^2 - t(\lambda_i)t(\bar{\lambda}_j)}{1 - \lambda_i\bar{\lambda}_j} .$$

For $\mu = 1$, this is the well-know Pick matrix.

5. CONCLUSION

An elementary proof of the H^∞-minimization theorem is given based on the adjoint characterization of the H^∞-norm. This characterization enables to represent the constraint on the interpolation function in a clear way and leads naturally to the eigenvalue problem. The stability argument gives a new light on the fundamental structure of H^∞-optimization problem.

Using the adjoint formulation given in this paper, we can solve more complicated H^∞-minimization problem, such as the two-sensitivity problem discussed by Kwakernaak [19], in a straightforward way. This will be reported in the forthcoming paper.

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