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Verification of the Discharging procedure in the Four Color Theorem

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ABSTRACT

Appel and Haken published the proof of the Four Color Theorem in 1977. This proof is divided into two parts, i.e., discharging and reducibility. By using a computer, we tried rigorous verification of the discharging procedure. And it is confirmed that there are some errors in the proof.

1. Introduction.

The Four Color Conjecture had been one in the most famous unsolved conjectures of mathematics. In 1976, the proof of the Four Color Theorem was announced by Appel and Haken [2]. First, we will trace the history of its studies.

The statement of the Four Color Conjecture is as follows. Every planar map M is four-colorable (that is, the regions of M can be colored with four colors so that any two regions receive different colors if they have a common border line). By use of triangulations, which are plane graphs all of whose regions are triangles, the Four Color Conjecture is often formulated as follows. Every triangulation without loop is (vertex-)four-colorable (that is, the vertices of G can be colored with four colors so that any adjacent two vertices receive different colors). From now on, we will phrase the results in the second terminology of triangulations and vertex-coloring.

The first published attempt to prove the Four Color Conjecture was made by A. B. Kempe [9] in 1879. He proved that the problem can be restricted to the consideration of normal planar maps which correspond to triangulations. For triangulations, he derived from Euler's formula, the equation

$$4n_2+3n_3+2n_4+n_5=\sum_{k=7}^{k_{max}}(k-6)n_k+12 \tag{1.1}$$

where n_i is the number of vertices with precisely i neighbors and k_{max} is the maximum degree of the triangulation. This equation immediately implies that every triangulation contains a vertex with fewer than six neighbors.

In order to prove the Four Color Theorem by induction on the number n of vertices in the triangulation, Kempe assumed that every triangulation with $n \leq r$ is four-colorable and considered a triangulation G_{r+1} with $r+1$ vertices. He considered four cases;

(1) G_{r+1} contains a vertex v_2 of degree two.

(2) G_{r+1} contains a vertex v_3 of degree three.

(3) G_{r+1} contains a vertex v_4 of degree four.

(4) G_{r+1} contains a vertex v_5 of degree five.

At least one of these cases occurs by (1.1). In each case, he produced a triangulation G_r with r vertices by removing one vertex from G_{r+1}. By the induction hypothesis, G_r admits a four-coloring, say c_r, and Kempe attempted to derive a four-coloring c_{r+1} of G_{r+1} as an extension of c_r. This task is very easy in cases (1) and (2). To treat cases (3) and (4), Kempe invented a tool called a "Kempe chain". Let C be a component which is obtained by removing all vertices which colored by some two colors from G_r. Then two colors of vertices of C can be exchanged and we can obtain a new coloring c'_r. Such a component C is called a Kempe chain. By such exchanges, a four coloring c_{r+1} of G_{r+1} is obtained.

Kempe's argument is correct in case (3). But it is incorrect in case (4), as was pointed out by Heawood [6] in 1890. Kempe's argument proves, however, that five colors suffice for coloring triangulations and that a minimal counter-example to the Four Color Conjecture (minimal with respect to the number of vertices in the triangulation) can contain no vertex of degree 2, 3, or 4. This restricts the Four Color Problem to consideration of triangulations in which each vertex has at least five neighbors. Each such triangulation must contain at least twelve vertices of degree five, since we have $n_2=n_3=n_4=0$ in (1.1) and thus

$$n_5=\sum_{k=7}^{k_{max}}(k-6)n_k+12. \tag{1.2}$$

From now , we consider such triangulations only.

Birkhoff [4] improved Kempe's reduction technique. He described the methods by which it can be proved that some configuration cannot occur in a minimal counter-example of the Four Color Theorem. Such a configuration is said to be "reducible". He showed that certain configurations are reducible by his methods. These methods were improved by many investigators and many reducible configurations have been discovered.

In 1904, Wernicke [11] proved that any triangulation must contain at least one vertex of degree five which is adjacent to another vertex of degree five or six. In 1922, Franklin [5] showed that every triangulation contains two adjacent vertices of degree five or some vertex of degree five which is adjacent to two vertices of degree six. The further improvement was made by Lebesque in 1940 by displaying a large collection of configurations at least one of which must occur in any triangulation. We refer to such a set of configurations as an "unavoidable set".

Heesch [7,8] exhibited several sets of reducible configurations which are unavoidable in those triangulations which satisfy certain restrictive conditions. He used a method called a discharging procedure to prove the unavoidability. As an example of the discharging procedure, we consider triangulations without vertices of degree six or seven. Initially, to each vertex v_i of degree i in a triangulation, we assign a "charge" $q_0(v_i)=60\times(6-i)$. Then by (1.2), the sum of the charges is positive 720. Now we discharge all v_5's, i.e., we obtain a new charge distribution q which assigns every vertex v of G a charge $q(v)$ such that $\sum q(v)=\sum q_0(v)$ as follows. The positive charges of the vertices of degree five are distributed in equal fractions to their neighbors v_k which have $deg(v_k)\geq 8$. Then it is shown that positive charges can occur only in sixteen special cases provided that the triangulation does not contain one of twenty reducible configurations. Then as the second step, the positive charges of these vertices are distributed to their negative neighbors, and it can be proved that no positive charge occurs, provided that the triangulation does not contain any member of the list of twenty reducible configurations. This implies that there exists no triangulation without vertices of degree six or seven which does not contain at least one of the twenty reducible configurations (since the sum of all the charges must be positive).

In 1974, Appel and Haken [1] proved existence of a finite unavoidable set of configurations called geographically good by constructing a discharging procedure. In 1976, Appel and Haken [2] announced that they found a finite

unavoidable set of reducible configurations and proved the Four Color Theorem.

Appel and Haken used computers to prove reducibility of the configurations. But they carried out the discharging procedure by hand. From 1976 through now, they found many errors in the computation of the discharging procedure. All errors which were found have been corrected, but they think that other possible errors remain, since detailed checking of the discharging procedure is a task that require enumeration of a large number of cases by hand.

We tried to verify the discharging procedure of Appel and Haken by using a computer. The reasons why we use a computer are:

(1) Verification by hand takes two months according to Appel and Haken.

(2) Simple works are repeated for verification.

(3) Simple works for a long time cause human trifling errors.

(4) In order to modify the discharging procedure by a trial-and-error method, it is effective to use a computer.

Now it is possible to verify discharging procedures for a few hours. We believe that our method is useful for verification and modification of the discharging procedure.

2. Notation and Definitions.

We consider finite plane graphs. The terminology is essentially the same as that of Behzad, Chartrand, and Lesniak-Foster [3].

Let G be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G, respectively. A graph is said to be ***planar***, if it can be drawn in the plane so that its edges intersect only at their end-vertices. Such a drawing of a planar graph G is called a ***planar embedding*** of G. A planar embedding of G can itself be regarded as a graph. We refer to a planar embedding of a planar graph as a *plane graph*. A plane graph G divides the plane into connected components, which are called *regions* of G. The unbounded region is called an *exterior region*. If a vertex v is incident to a edge which is a part of the boundary of a region r, v is called a ***boundary vertex*** of r. A region which has exactly three boundary vertices is called a ***triangle***. A ***near-triangulation*** is a plane graph all of whose regions are triangles except for the exterior region. A *triangulation* is a near-triangulation whose exterior region is a triangle.

A ***degree specification*** of a near-triangulation G is a function from $V(G)$ into the set of all the subset of $\{ x \mid x$ is an integer, $x \geq 5 \}$. A degree specification of G is denoted by ds_G. The value of degree specifications used in this paper is restricted (See Figure 2.1). Let v be a vertex of G. We say that v is ***specified*** by $ds_G(v)$. If $|ds_G(v)|=1$ for a vertex v of G, v is called a ***fully specified vertex***. Otherwise, v is called a ***partially specified vertex***. The member of $ds_G(v)$ of a fully specified vertex v is called the ***specified degree*** of v. If a vertex v is specified by $DS=ds_G(v)$, v is called a DS-vertex. If an edge e is incident to two vertices which are specified by DS and DS' respectively, e is called a ***DS-DS' edge***. If a vertex v is specified by DS such that $DS \subset DS_m$, v is called a ***minor vertex***. If $DS \subset DS_U$, then v is called a ***major vertex***.

If a near-triangulation C with degree specification satisfies the following four properties, C is called a ***configuration***.

(1) $deg_C(v) \leq \max ds_C(v)$ for every vertex v of C.

(2) If $deg_C(v)=\max ds_C(v)$ for a vertices v of C, v is not a boundary vertex of the exterior region of C.

(3) If v is not a boundary vertex of the exterior region of C, v is a fully specified vertex and the specified degree of v is equal to $deg_C(v)$.

(4) If a vertex v is a cut vertex in C, then v is fully specified and the sum of $deg_C(v)$ and the number of components in $G-v$ is equal to the specified degree of v.

The triangulation can be considered as a configuration whose vertices are all fully specified.

Let C and D be configurations. A ***immersion*** is a mapping $f: C \to D$ of C into D defined by the following five properties:

(1) The images of vertices and edges of C are vertices and edges of D, respectively.

(2) If an edge e and a vertex v of C are incident, then the edge $f(e)$ and the vertex $f(v)$ are incident in D.

(3) It is satisfied that $ds_C(v) \supset ds_D(f(v))$ for every vertex v of C.

(4) If e_1 and e_2 are distinct edges of C which are incident to a common vertex v, then $f(e_1)$ and $f(e_2)$ are also distinct edges of D.

specified set	denotation	specified degree	symbol in computer input	symbol in computer output	symbol in drawings
{5}	DS_5	5	5	5	●
{6}	DS_6	6	6	6	
{7}	DS_7	7	7	7	○
{8}	DS_8	8	8	8	□
{9}	DS_9	9	9	9	△
{10}	DS_{10}	10	0	0	⬠
{5,6}	DS_m		M	M	m 
{5,6,...}	DS_F		F	*	
{6,7,...}	DS_S		S	.	• 
{7,8,...}	DS_U		U	U	
{8,9,...}	DS_E		E	E	

Figure 2.1

(5) Let $e_1, e_2, \ldots, e_n$ be the edges incident to a vertex v of C in consecutive order, reading clockwise or counter-clockwise. Then $f(e_1), f(e_2), \ldots, f(e_n)$ are consecutively ordered around $f(v)$ in D.

(6) Let F be a induced subgraph of D by $f(V(C))$. If a vertex v is a cut vertex of C, the number of component in $C-v$ is equal to that in $F-f(v)$.

A configuration C is said to be ***contained in*** a configuration D if there is a immersion $f: C \to D$. A immersion of a configuration C into a triangulation G is similarly defined by considering G as a configuration.

3. Review of Appel and Haken's Work.

3.1. Outline.

The Four Color Theorem is formulated as follows.

Four Color Theorem. *Every triangulation G with no loop is colored by four colors so that two adjacent vertices of G are colored by distinct colors.* •

If the following theorem is proved, the Four Color Theorem is proved.

Theorem 1. *There exists a set U of configurations which satisfies the following two properties:*

(1) *The configurations of U are contained in no minimal triangulation which is not colored by four colors.*

(2) *Every loop-free triangulation contains at least one member of U.* •

We call the set U of configurations which satisfy the condition (2) of Theorem 1 an *unavoidable set*.

Let G be a triangulation considered in the above theorem. We may confine ourselves to the case that every vertex v of G satisfy $deg(v) \geq 5$ by Kempe's argument [9]. The details about basic results of the Four Color Theorem are described in [10].

Appel and Haken proved Theorem 1. They gave a set U which consists of 1482 configurations (for example, in Figure 3.1.1). They proved Theorem 1 (1) by computers. We have not verified the proof of this part. Theorem 1 (2) is proved by a method called a discharging procedure. First we explain the discharging rule and proceed to the proof of Theorem 1 (2).

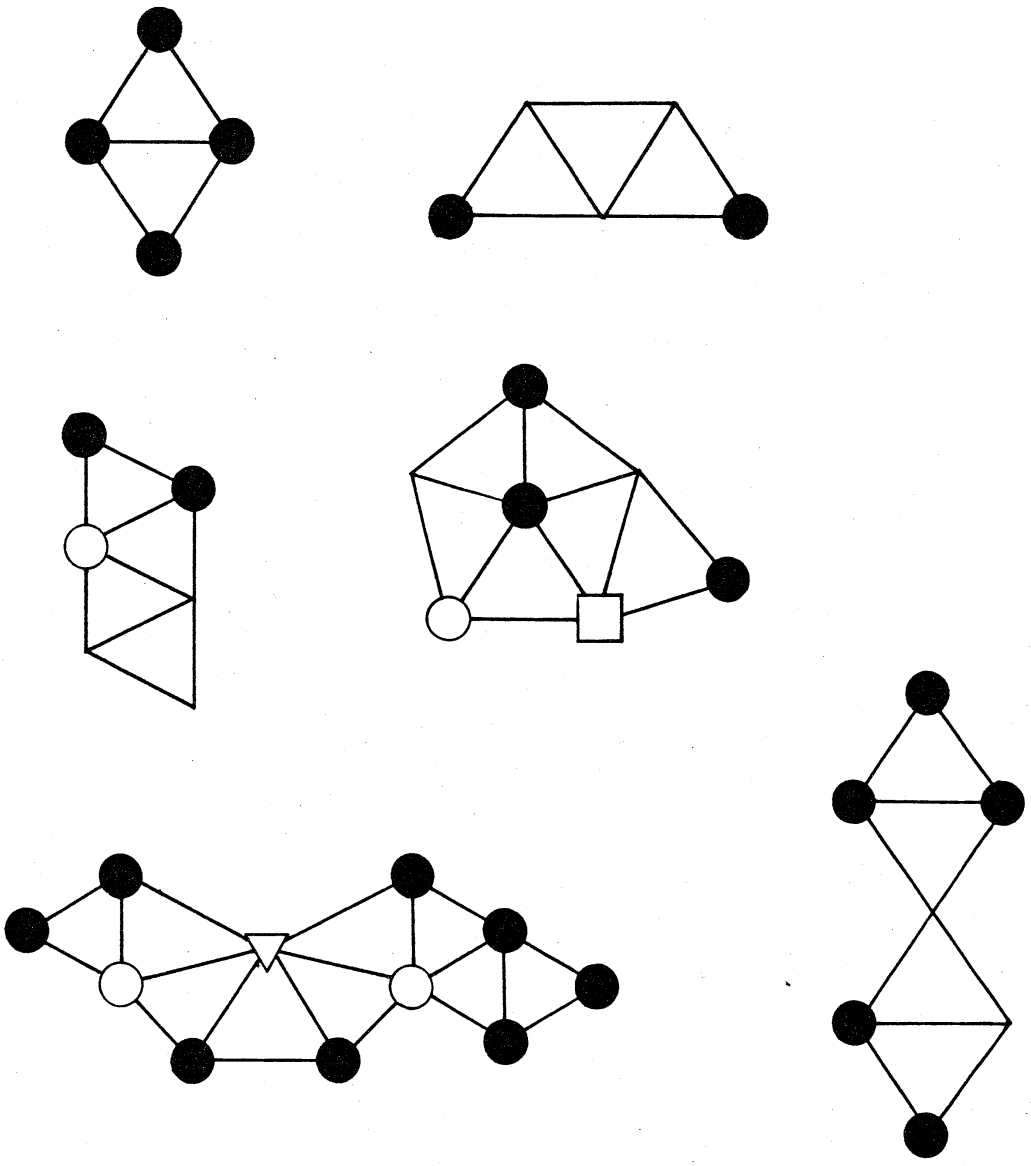

Figure 3.1.1 Configurations in U

3.2. Discharging Rule.

Let G be a triangulation. We consider an integer-valued function q_0 on $V(G)$. For every vertex v of G, q_0 is defined as $q_0(v)=60\times(6-deg(v))$. Then by (1.2), we have

$$\sum_{v\in V(G)} q_0(v)=720 \tag{3.1}$$

Note that $q_0(v)$ is negative for every major vertex ($deg(v)\geq 7$), zero for every vertex of degree six, and positive for every vertex of degree five.

Let $\{(v_1,w_1,c_1),(v_2,w_2,c_2), \ldots ,(v_n,w_n,c_n)\}$ be the set of triples such that v_i and w_i $(1\leq i\leq n)$ are vertices of G, and c_i $(1\leq i\leq n)$ is a nonnegative integer. Let q be an integer-valued function on $V(G)$ defined by $q(v)=q_0(v)-\sum_{v_i=v} c_i+\sum_{w_i=v} c_i$.

Then by (3.1) we have

$$\sum_{v\in V(G)} q(v)=720 \tag{3.2}$$

We call q a *terminal charge function* of G. The triple (v_i,w_i,c_i) is called a *discharging* from v_i to w_i. The integer c_i is called a ***discharging value*** from v_i to w_i. Let $t=(v,w,c)$ be a discharging. If v and w are not adjacent, we call the discharging t a *T-discharging*. Otherwise, three cases occur. If $c=30$, then the discharging t is called an *R-discharging*. If $c<30$, then the discharging t is called an *S-discharging*. If $c>30$, then the discharging t is called an *L-discharging*.

Next we explain rules for deciding dischargings for every triangulation which Appel and Haken used in [2]. Rules to define dischargings for every triangulation are called ***discharging rules***. Let G be a triangulation. We consider a method to make the dischargings (v,w,c) of G. Every discharging of G are defined from a 5-vertex to a major vertex.

First we consider about T-dischargings. Seven situations are described in [2] to define the T-dischargings. (See Figure 3.2.1.) They are called *T-situations*. T-situations with solid arrows are called *T2-situations*. T-situations with open arrows are called *T1-situations*. If a T-situation is contained in G, we define a T-discharging between two vertices indicated by the arrow. The discharging value c is defined as 20 if the arrow is solid, and as 10 if the arrow is open, with the following exception (see Figure 3.2.2). In the case that two T-dischargings leave the same 5-vertex across the same 6-6 edge (but arrive at different major

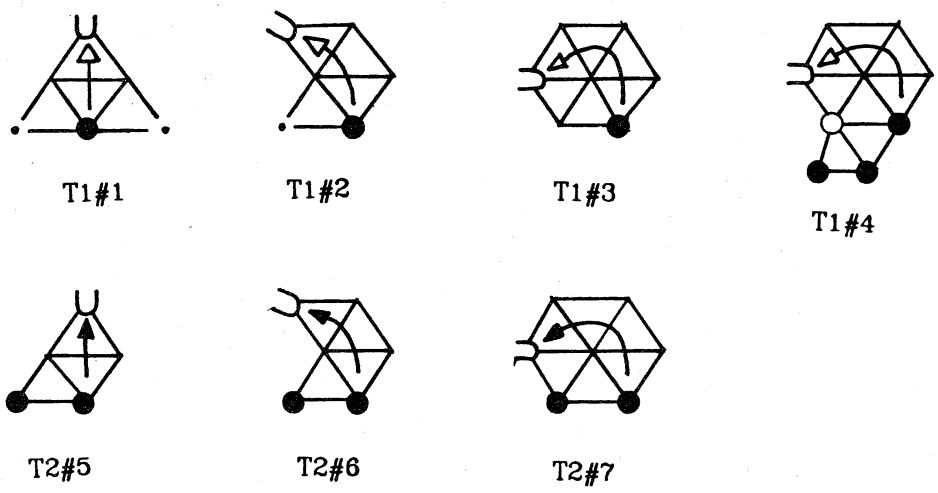

Figure 3.2.1 T-situations

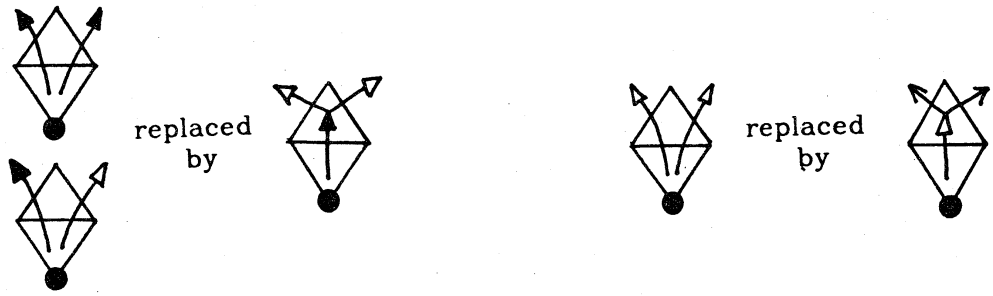

Figure 3.2.2 Exceptional rules for T-dischargings

vertices), the discharging value is defined as follows. If at least one of the two arrows is solid then the discharging value is 10. If both of the arrows are open , the discharging value is 5. If a T-discharging t has discharging value 5 or 10, t is called *T1-discharging*. If a T-discharging t has discharging value 20, t is called *T2-discharging*.

Next we consider about R-dischargings, S-dischargings, and L-dischargings. As a general rule, R-dischargings are defined from 5-vertices to major vertices. And as exception, S-dischargings and L-dischargings are defined. 484 situations are described in [2]. (Some examples are given in Figure 3.2.3 and Figure 3.2.4.) Each of situations has a ***distinguished edge*** which is drawn vertical and marked by a number (0, 5, 10, 15, 20, 25, 35, 40, 50, or 60). Let n be a marked number. We distinguish three classes of S-situations, and three classes of L-situations as follows;

S0-situation means $n=0$ or 5,

S1-situation means $n=10$ or 15,

S2-situation means $n=20$ or 25,

L4-situation means $n=35$ or 40,

L5-situation means $n=50$,

L6-situation means $n=60$.

Let e be an edge of G, and v,w be two vertices incident to e such that $deg(v)=5$ and $deg(w)\geq 7$. If an S-situation (L-situation) C is contained in G in such a way that the distinguished edge of C is identified to e, we say that C is ***applied at*** e. In this case, we also say that C is ***applied at*** v (or w). We denote the discharging value from v to w by $d(e)$.

Regarding the edges e, we now have three possibilities.

(1) No S-situations or L-situations are applied at e. In this case we call e an *R-edge* and we define $d(e)$ to be 30.

(2) One or more S-situations, but no L-situations, are applied at e. Then we call e an *S-edge* and we define $d(e)$ to be equal to the smallest of the marked numbers of the applied S-situations.

(3) One or more L-situations and zero, one, or more S-situations are applied at e. Then we call e an *L-edge* and we define $d(e)$ to be equal to the largest of the marked number of the applied L-situations.

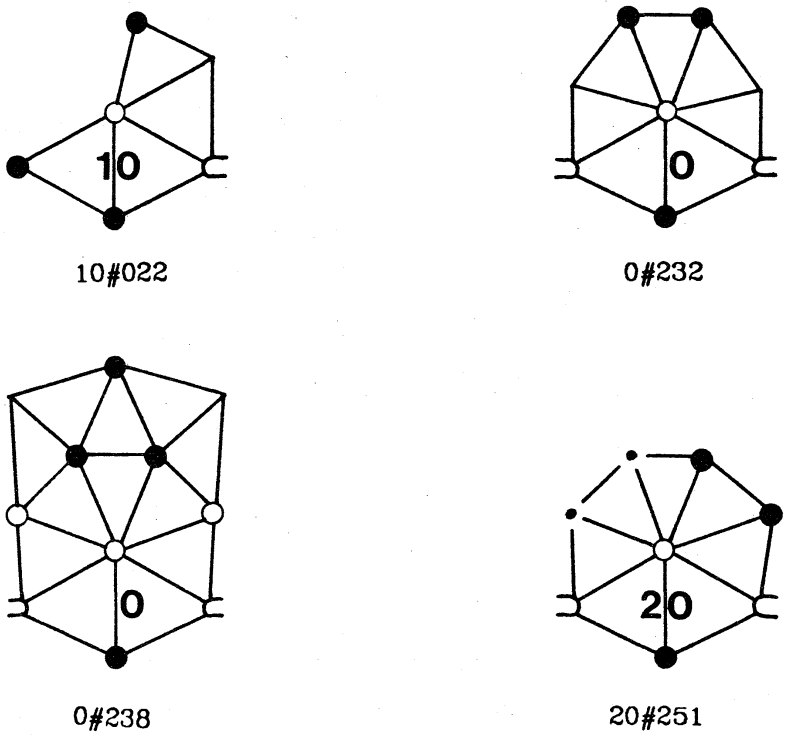

Figure 3.2.3 Examples of S-situations

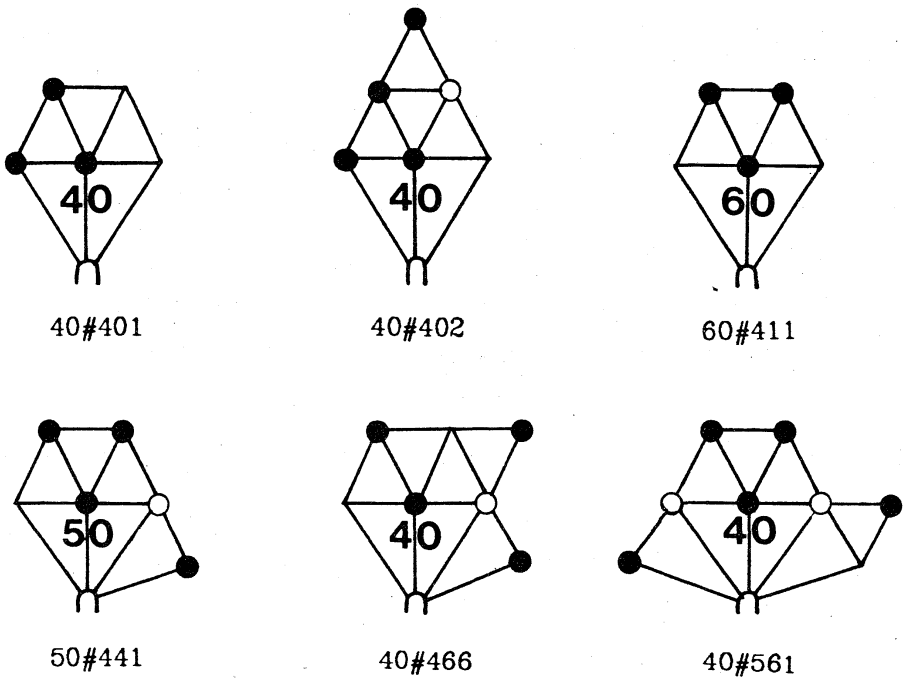

Figure 3.2.4 Examples of L-situations

Now every edge e of G which joins a 5-vertex to a major vertex has a uniquely defined discharging value $d(e)$ and we can define a discharging from v to w.

We define dischargings of G only by dischargings obtained by the above rules. We distinguish three classes of S-dischargings and three classes of L-dischargings by the discharging value c as follows.

S0-discharging means $n=0$ or 5,

S1-discharging means $n=10$ or 15,

S2-discharging means $n=20$ or 25,

L4-discharging means $n=35$ or 40,

L5-discharging means $n=50$,

L6-discharging means $n=60$.

Note that a discharging rule DR depends essentially on the set of T-situations T, on the set of S-situations S, and on the set of L-situations L. Thus, to be precise, we should denote the discharging rules by $DR(T,S,L)$, indicating that we would obtain different discharging rules by using different sets of T-situations, S-situations, or L-situations. A terminal charge function q of G obtained by the dischargings which are made by applying $DR(T,S,L)$ to G, is called a terminal charge function of G with respect to $DR(T,S,L)$.

3.3. Discharging Theorem.

Appel and Haken gave a set T of T-situations, a set S of S-situations, and a set L of L-situations. The number of the sets are 7, 269, and 215, respectively. And they gave a set U of 1482 configurations. Then they prove the following theorem by using the discharging rule (T,S,L).

Discharging Theorem for (T,S,L). *Let G be any triangulation. Let q be a terminal charge function of G with respect to (T,S,L). If the triangulation G contains no configuration belonging to U, then $q(v)\leq 0$ for every vertex v of G.* ▪

But the conclusion of Discharging Theorem contradicts (3.1). Therefore Theorem 1 (2) is proved.

We describe the outline of the proof for Discharging Theorem. It follows immediately from the definition of (T,S,L) that $q(v_6)=0$ for every 6-vertex v_6 of G. The proof consists of two parts;

(A) $q(v_5) \leq 0$ for every 5-vertex v_5 of G and

(B) $q(v_k) \leq 0$ for every k-vertex v_k of G ($k \geq 7$).

Part (A) is called V_5-Lemma and part (B) is called V_k-Lemma.

First we explain the proof of (A) in brief. We consider the discharging rule (T,S,ϕ) (which uses T-situations and S-situations but no L-situations) and we denote the terminal charge function with respect to (T,S,ϕ) by q_{TS}. By straightforward enumeration of all the possible cases of $q_{TS}(v_5)>0$, it is proved that G contains one of 33 configurations if G contains a 5-vertex v_5 which satisfies $q_{TS}(v_5)>0$. In the enumeration, all cases which imply the presense of a configuration of U are deleted. And then it is proved that there is no 5-vertex v such that $q(v)>0$ in the 33 configurations by applying some L-dischargings to the configurations.

Next, we explain the proof of (B) in brief. First, some lemmas are proved. The proofs of the lemmas are easily obtained by straightforward enumeration of several cases. From the consequences of these lemmas, it is shown that $q(v) \leq 0$ for every vertex v of G such that $deg(v) \geq 11$. It remains to prove (B) for a vertex v which has degree seven, eight, nine, or ten. We consider the discharging procedure (T,ϕ,L) (i.e. we ignore the S-situations) and the terminal charge function of G with respect to (T,ϕ,L) is denoted by q_{TL}. It is shown that G contains one of the 152 configurations if G has a vertex v which satisfies $q_{TL}(v)>0$. In the enumeration, all cases which imply the presense of a configuration of U are deleted. This proof is also obtained by straightforward case enumeration. Then we assure that every vertex v of the configurations has $q(v) \leq 0$ by applying some S-situations to the configurations.

In July 1984, Appel and Haken informed to the author about the status of the proof of the Four Color Theorem. According to this, a number of errors were found in the case enumeration. They have been corrected in most cases by correcting trivial miswritings, but in one case by modifying the discharging rule. They say that the proof has no specific errors that they know but they expect to find a few more errors of the same type as above.

4. Results.

Three errors are found in the proof of Discharging Theorem in [2]. The following three facts are claimed in the proof:

(1) When the degree of the distinguished major vertex of an L-situation 35#564 is specified by seven, I4-5 is contained in it. (See class check list for (5l) in the microfiche supplement of [2].)

(2) When an L-situation 40#453 is attached at Z in CTL#134, an S-situation 20#117 is contained in it. (See class check list for CTL#134 in the microfiche supplement of [2].)

(3) When an S-situation 10#252n is attached at A and an S-situation 20#173r is attached at E in Figure 4.1, I4-26 is contained in it. (See class check list for (2a) in the microfiche supplement of [2]).

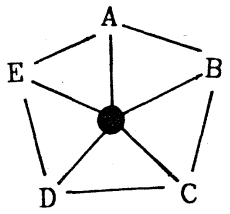

Figure 4.1

However, we found that these facts are false.

5. Conclusions.

We wrote computer programs in order to help verifying Discharging Theorem. The programs were written in PASCAL 8000, and HITAC 8700 is used.

We explain the way of verification in brief. First we must prove Lemma(5-6-6), Lemma(6-6-6), Lemma(5^5-7-6-6), Lemma(6-6), Lemma(T), Lemma(5,L,5), and Lemma(5,L) (see [2]) by hand. Lemma(5-6-6), Lemma(6-6-6), Lemma(5^5-7-6-6), Lemma(T), Lemma(5,L,5), and Lemma(5,L) are used in programs. Then we verify Lemma(I), Lemma S^+, V_5-Lemma, V_7-Lemma, V_8-Lemma, V_9-Lemma, V_{10}-Lemma, and V_{11}-Lemma. Every case of Lemma S^+, V_{10}-Lemma, and V_{11}-Lemma is proved by a computer. Almost every case of Lemma(I), V_5-Lemma, V_7-Lemma, V_8-Lemma, and V_9-Lemma is verified by a computer. The remaining cases are calculated by hand. Sometimes we use Lemma(6-6) in these cases. Verification takes about thirty hours for inputting all the drawings in [2] by hand, about an hour for computer calculation, and a few hour for calculation by hand. Incidentally, if a person carries out all the tasks, it will take two months according to Appel and Haken.

We consider merits of such a method by a computer. There are three merits as follows:

(1) From now, we can verify the proof with respect to a modified discharging rule in a few hours. So we can repeat modification of discharging rules and verification alternately in a few hours.

(2) The calculation is more precise. In particular, when we use a trial-and-error method on discharging rules, verification by a computer is more precise.

(3) The Lemmas used in the programs are so simple, that verification of the programs is easier.

A remaining problem is that we are apt to make a mistake in data input. In order to settle the problem, we must verify reducibility by using a computer.

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