

グラフの正則成分因子

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1. Introduction

We consider a finite graph G which may have multiple edges but has no loops. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of G , respectively. We write $d_G(x)$ for the degree of a vertex x in G . Let a , b and r be integers such that $0 \leq a \leq b$ and $r > 0$. A spanning subgraph F of G is called an $[a, b]$ -factor of G if $a \leq d_F(x) \leq b$ for all $x \in V(G)$, and we call an $[r, r]$ -factor an r -factor. An r -regular graph is a graph in which each vertex has degree r .

Tutte [8] ([3], p.77) proved that for any odd integer r and any integer k ($0 \leq k \leq r$), every r -regular graph has a $[k-1, k]$ -factor. It was proved in [5], [9] that every regular graph has a $[1, 2]$ -factor each of whose components is regular. Enomoto and Saito [4] gave the following conjecture: Every r -regular graph has a $[k-1, k]$ -factor each of whose components is regular for any k , $0 < k < r$. Note that this conjecture is true when r is even by Petersen's 2-factorable theorem (see Lemma 1). So the essential part of this conjecture is the case that r is odd. We obtain the following two theorems.

Theorem 1. Let r and k be positive integers. If $k \leq 2(2r+1)/3$, then every $(2r+1)$ -regular graph has a $[k-1, k]$ -factor each

of whose components is regular.

Theorem 2. Let k and r be positive integers. If $2r+3 - \sqrt{2r+1} < k \leq 2r$, then there exists a simple $(2r+1)$ -regular graph that has no $[k-1, k]$ -factor each component of which is regular.

It seems that there exists a $(2r+1)$ -regular graph that has no $[k-1, k]$ -factor with regular components if $2(2r+1)/3 < k \leq 2r$. Some results related to our results can be found in a survey article [1].

2. Proofs of Theorems

Let G be a graph, and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a (g, f) -factor if $g(x) \leq d_F(x) \leq f(x)$ for all $x \in V(G)$. A (g, f) -factor satisfying $g(x) = f(x)$ for all $x \in V(G)$ is briefly called an f -factor. For a vertex subset X of G , we write $G-X$ for the graph obtained from G by deleting the vertices in X together with their incident edges. Similarly, for an edge subset Y of G , $G-Y$ denotes the graph obtained from G by deleting all the edges in Y . For two disjoint subsets S and T of $V(G)$, we denote by $e_G(S, T)$ the number of edges of G joining S and T .

Lemma 1. (Petersen [7], [2] p.166) Every $2r$ -regular graph has a $2k$ -factor for every integer k , $0 < k < r$.

Lemma 2 [6] Let G be an n -edge-connected graph ($n \geq 1$), θ be a real number such that $0 \leq \theta \leq 1$, and f be an integer-valued function defined on $V(G)$. Suppose (1) and (2) hold. Moreover, if one of (3a) and (3b) holds, then G has an f -factor.

$$(1) \quad \sum_{x \in V(G)} f(x) \equiv 0 \pmod{2}.$$

$$(2) \quad \varepsilon = \sum_{x \in V(G)} |f(x) - \theta d_G(x)| < 2.$$

(3a) $\{f(x) \mid x \in V(G)\}$ consists of even numbers, and $m(1-\theta) \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod{2}$.

(3b) $\{d_G(x), f(x) \mid x \in V(G)\}$ consists of odd numbers, and $m\theta \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod{2}$.

Lemma 3. Let G be an n -edge-connected graph ($n \geq 1$), θ be a real number such that $0 < \theta < 1$, and g and f be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. Suppose (1) and (2) hold. Moreover, if one of (3a) and (3b) holds, then G has a (g, f) -factor.

(1) G has at least one vertex v such that $g(v) < f(v)$.

$$(2) \quad \varepsilon = \sum_{x \in V(G)} (\max\{0, g(x) - \theta d_G(x)\} + \max\{0, \theta d_G(x) - f(x)\}) < 1.$$

(3a) $\{f(x) \mid f(x) = g(x), x \in V(G)\}$ consists of even numbers, and $m(1-\theta) \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod{2}$.

(3b) $\{f(x), d_G(x) \mid f(x) = g(x), x \in V(G)\}$ consists of odd numbers, and $m\theta \geq 1$, where $m \in \{n, n+1\}$ and $m \equiv 1 \pmod{2}$.

Lemma 4. Let G be a 2-edge-connected $(2r+1)$ -regular graph, and h be a positive integer. If $2h \leq 2(2r+1)/3$, then G has a $2h$ -factor. If $(2r+1)/3 \leq 2h+1 \leq (2r+1)$, then G has a $(2h+1)$ -factor. In particular, for every integer k , $0 < k < 2r+1$, G has a $[k-1, k]$ -factor each component of which is regular.

Proof Define a function f on $V(G)$ by $f(x) = 2h$ for all $x \in V(G)$, and set $\theta = 2h/(2r+1)$. We show that G , f and θ satisfy conditions (1), (2) and (3a) of Lemma 2. Since G is of even order, (1) holds, and (2) is trivial as $\varepsilon = 0$. Furthermore, (3a) follows from $m = 3$ and $2h \leq 2(2r+1)/3$. Hence G has an f -factor,

that is, G has a $2h$ -factor. Similarly, we can prove that G has a $(2h+1)$ -factor if $2h+1 \geq (2r+1)/3$ by using (3b) instead of (3a). Since one of $\{k-1, k\}$ is odd and the other is even, the last statement is an easy consequence of the two results proved above.

Lemma 5. Let G be a 2-edge-connected $[2r, 2r+1]$ -graph having exactly one vertex w of degree $2r$. Then

- (1) if $0 < 2k \leq 2(2r+1)/3$, then G has a $2k$ -factor; and
- (2) if $(2r+1)/3 \leq 2k+1 \leq 2r+1$, then G has a $[2k, 2k+1]$ -factor F such that $d_F(w) = 2k$ and $d_F(x) = 2k+1$ for all $x \in V(G) \setminus \{w\}$.

Proof We prove only (2). It is clear that we may assume $2r > 2k$. Define two functions g and f on $V(G)$ by

$$g(x) = \begin{cases} 2k & \text{if } x=w \\ 2k+1 & \text{otherwise,} \end{cases} \quad \text{and } f(x) = 2k+1 \text{ for all } x \in V(G).$$

Put $\theta = (2k+1)/(2r+1)$. We show that G , g , f and θ satisfy conditions (1), (2) and (3b) of Lemma 3. Since $g(w) > f(w)$, (1) holds. It is immediate that $g(w) < \theta d_G(w) < f(w)$. Thus (2) holds. Since $\{d_G(x), f(x) \mid f(x) = g(x), x \in V(G)\} = \{2r+1, 2k+1\}$ and $m=3$, (3b) follows. Therefore, G has a (g, f) -factor F , which is a $[2k, 2k+1]$ -factor. Since G is of odd order, we have $d_F(w) = 2k$. Consequently, F is a desired factor.

The next lemma plays an important role in the proof of Theorem 1, and its proof is not so short.

Lemma 6. Let G be a connected $(2r+1)$ -regular graph with at least two bridges, and k be a positive integer. If $(2r+1)/3 \leq k \leq 2(2r+1)/3$, then G has a $[k-1, k]$ -factor each component of which is regular.

Proof of Theorem 1 We prove the theorem by induction on $2r+1$. Let G be a $(2r+1)$ -regular graph and k be an integer such that $2 \leq k \leq 2(2r+1)/3$. Note that every regular graph has a $[0,1]$ -factor with regular components since it has a 0-factor. By Lemma 4, we may assume G is not 2-edge-connected. Suppose G has one bridge vw . Then each component C of $G-vw$ is a 2-edge-connected $[2r, 2r+1]$ -graph possessing one vertex of degree $2r$. Thus C has a k -factor or a $(k-1)$ -factor by Lemma 5. Therefore G has a k -factor or a $(k-1)$ -factor, and the theorem holds. Consequently, we may assume G has at least two bridges.

By Lemma 6, a 3-regular graph with at least two bridges has a $[1,2]$ -factor with regular components. Hence every 3-regular graph has a $[1,2]$ -factor with regular components, and so the theorem is true if $2r+1=3$. Similarly, we can show that every 5-regular graph has a $[2,3]$ -factor F_1 with regular components. Since 3-regular components of F_1 has a $[1,2]$ -factor with regular components, F_1 has a $[1,2]$ -factor with regular components, which is of course a desired $[1,2]$ -factor of G . Hence the theorem follows for $2r+1=5$. In general, if a $(2r+1)$ -regular graph G has an $[h-1, h]$ -factor F_2 with regular components and if each component of F_2 has a $[k-1, k]$ -factor with regular components, then G has a $[k-1, k]$ -factor with regular components. By this argument, we can verify that if $2r+1 \leq 13$, then the theorem holds. Suppose $2r+1 \geq 15$. If $(2r+4)/3 \leq k \leq 2(2r+1)/3$, then a $(2r+1)$ -regular graph G has a $[k-1, k]$ -factor with regular components by Lemma 6. Hence we may assume $k < (2r+4)/3$. Let h be the greatest integer not exceeding $2(2r+1)/3$. Then G has an

$[h-1, h]$ -factor F with regular components. Since $2(h-1)/3 \geq 2(4r-1)/9$ and $2r+1 \geq 15$, we have $2(h-1)/3 \geq (2r+4)/3 > k$. Hence each component of F has a $[k-1, k]$ -factor with regular components by Lemma 1 or by the inductive hypothesis. Therefore G has a $[k-1, k]$ -factor with regular components, and we conclude that the proof of Theorem 1 is complete.

Proof of Theorem 2. Let k and r be positive integers such that $2r+2 - \sqrt{2r+1} < 2k \leq 2r$. Let k' be an odd integer that is one of $\{2k-1, 2k+1\}$ and not equal to $2r+1$. Let K_{2r+3} denote the complete graph with vertex set $\{a_1, \dots, a_{2r+3}\}$. We obtain the graph R from K_{2r+3} by deleting edges $a_1a_2, a_1a_3, \dots, a_1a_{2r-2k+5}, a_{2r-2k+6}a_{2r-2k+7}, \dots, a_{2r+2}a_{2r+3}$. It is clear that $d_R(a_1) = 2k-2$ and $d_R(a_i) = 2r+1$ for all $i, i \neq 1$. Let $R(1), \dots, R(2r)$ be copies of R , and let b_i be the vertex of R_i whose degree is $2k-2$ for all i . We construct a graph H with vertex set $V(R(1)) \cup \dots \cup V(R(2r)) \cup \{c_1, \dots, c_{2r-2k+2}, v\}$ as follows. Join every b_i to all c_j ($1 \leq j \leq 2r-2k+2$) and v by new edges, and add new edges $c_1c_2, c_3c_4, \dots, c_{2r-2k+1}c_{2r-2k+2}$ (see Figure). Then $d_H(v) = 2r$ and $d_H(x) = 2r+1$ for all $x \in V(H) \setminus \{v\}$.

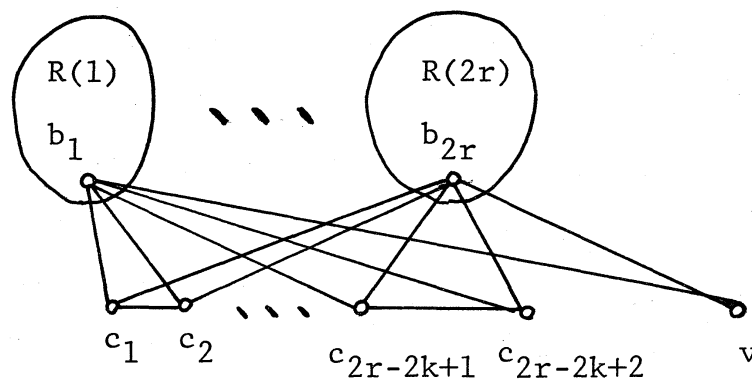


Figure Graph H .

Let H_1, \dots, H_{2r+1} be copies of H , and let v_i be the vertex of H_i whose degree is $2r$ for every i . We now construct a $(2r+1)$ -regular graph G as follows, which has the required property. Set $V(G) = V(H_1) \cup \dots \cup V(H_{2r+1}) \cup \{w\}$, and join each v_i to w by a new edge. We omit the proof of the non-existence of $[k-1, k]$ -factor with regular components in G .

Acknowledgements The author would like to thank Professor H. Enomoto for his helpful suggestions.

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本稿で述べた定理の完全な証明は下記の論文におよ。

M. Kano, Factors of regular graphs, to appear.