On the Unit Groups of Burnside Rings (Topology and Transformation Groups)

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On the Unit Groups of Burnside Rings.

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1. Introduction.

Let $G$ be a finite group and let $\mathsf{Set}_f^G$ be the category of finite (right) $G$-sets and $G$-maps. The Grothendieck ring of this category (with respect to coproduct $+$ and product $\times$) is called the Burnside ring of $G$ and is denoted by $A(G)$.

A super class function is a map of the set of subgroups of $G$ to $\mathbb{Z}$ which is constant on each conjugate class of subgroups. Let $\tilde{A}(G) = \mathbb{Z}^{\text{Cl}(G)}$ be the ring of super class functions. For any subgroup $S$ of $G$, the map $[X] \mapsto |X^S|$ extends to a ring homomorphism $\varphi_S : A(G) \longrightarrow \mathbb{Z}$, and so we have a ring homomorphism

$$\varphi = \prod_{(S)} \varphi_S : A(G) \longrightarrow \tilde{A}(G) = \mathbb{Z}^{\text{Cl}(G)}.$$

This map is injective. Thus we can identify any element $x$ of $A(G)$ with the super class function $\varphi(x)$, and so we simply write $x(S) := \varphi(x)(S) = \varphi_S(x)$ for a subgroup $S$. See tom Dieck [Di79] Chapter 1.
Now, by geometric methods, tom Dieck proved that for an \( \mathbb{R}G \)-module \( V \), the function

\[
u(V) : S \longrightarrow \text{sgn \ dim} V^S
\]

belongs to the Burnside ring \( A(G) \), where \( \text{sgn} \ n := (-1)^n \) ([Di79] Proposition 5.5.9). The first purpose of this paper is to prove this fact by purely algebraic methods. In fact we shall prove the following theorem in Section 2.

**Theorem A.** Let \( G \) be a finite group and let \( V \) be a \( \mathbb{C}G \)-module with real valued character. Then the function

\[
u(V) : S \longmapsto \text{sgn \ dim}_\mathbb{C} V^S
\]

is a member of the Burnside ring \( A(G) \).

Since clearly \( \nu(V)^2 = 1 \) and \( \nu(V \oplus W) = \nu(V) \nu(W) \), we have a group homomorphism into the unit group:

\[
u = \nu_G : \overline{R}(G) \longrightarrow A(G)^* ,
\]

where \( \overline{R}(G) \) is the ring of real valued virtual characters of \( G \). We call this map \( \nu_G \) a **tom Dieck homomorphism**.

There are various maps between Burnside rings (and unit groups), and the assignment \( A^* : H \longmapsto A(H)^* \) together with restrictions and multiplicative inductions forms a so called \( G \)-functor (= a Mackey functor from the
category of finite $G$-sets) and further that the tom Dieck homomorphism gives a morphism between $G$-functors. Since $A^*$ is a $G$-functor, we have that $A(G)^*$ is a module over $A(G)$ (and also over $A(G)_{(2)}$, the localization at 2). In fact, the action $A(G)^* \times A(G) \rightarrow A(G)^*$ is induced by the exponential map $(Y, X) \rightarrow Y^X$ (the set of all maps of $X$ to $Y$). From the theory of Burnside rings and $G$-functors, we can show some transfer theorems about $A(G)^*$. See Section 3. The proof will appear in another paper.

**Notation and terminology.** We always denote by $G$ a finite group. The set of $G$-conjugate classes $(H)$ of subgroups $H$ of $G$ is denoted by $\text{Cl}(G)$. For subsets $A$, $B$ of $G$, we mean by $A =_G B$ (resp. $A \leq_G B$) that $A$ and $B$ are conjugate in $G$ (resp. $A$ is $G$-conjugate to a subgroup of $B$). We put $A^g := g^{-1}Ag$. When a group $G$ acts on a set $X$, we denote by $X^G$ the set of elements fixed by $G$. The ordinary character ring of $G$ is denoted by $R(G)$. For a ring $R$, the unit group of $R$ is denoted by $R^*$. The inner product of characters $\chi$ and $\varphi$ is denoted by $\langle \chi, \varphi \rangle$. Other notation and terminology are standard. See [Go68], [Di79].
2. Proof of Theorem A.

In this section, we prove Theorem A. As in the introduction, let $G$ be a finite group and let $Cl(G)$ denote the set of conjugate classes of subgroups of $G$. We mean by $(S)$ the class of a subgroup $S$. We set $WS := N_G(S)/S$ for a subgroup $S$ of $G$.

Lemma 2.1. There is an exact sequence of abelian groups:

$$0 \rightarrow \Lambda(G) \xrightarrow{\varphi} \mathbb{Z}^{Cl(G)} \xrightarrow{\chi^r} \coprod_{(S)} \mathbb{Z}/|WS| \mathbb{Z} \rightarrow 0,$$

where $\varphi$ is the injective ring homomorphism given in the introduction, and for a super class function $x$, the $S$-component of $\chi^r(x)$ is defined by

$$(x)_S := \sum_{gS \in WS} x(<g>S) \mod |WS|.$$

This lemma is well-known and its proof is found in, for example, tom Dieck [Di79] 1.3. We can now prove Theorem A.

Proof of Theorem A. Let $\chi$ be the real valued character afforded by the $\mathbb{C}G$-module $V$, and let $u(V)$ be the present super class function:
\[ u(V) : (S) \mapsto \text{sgn dim } V^S. \]

By Lemma 2.1, we must show that for each subgroup \( S \) of \( G \),

\[ \sum_{gS \in WS} u(V)(\langle g \rangle S) \equiv 0 \pmod{|WS|}. \tag{1} \]

Let \( \chi' \) be the character afforded by the \( gWS \)-module \( V^H \), so that by an easy representation theory, we have that

\[ \chi'(gS) = \frac{1}{|S|} \sum_{h \in S} \chi(gh), \quad gS \in WS, \]

and so \( \chi' \) is also a real valued character. Thus in order to prove (1), we may assume that \( S = 1 \). Set \( u_\chi(g) := u(V)(\langle g \rangle) \), then it has the value

\[ u_\chi(g) = \text{sgn } \langle \chi', \langle g \rangle \rangle, \]

where \( \langle , \rangle \) stands for the inner product of characters.

Now when \( S = 1 \), (1) becomes

\[ \sum_{g \in G} u_\chi(g) \equiv 0 \pmod{|G|}. \tag{2} \]

In order to prove (2), it will suffice to show that \( u \) is a virtual character of \( G \). In fact, we can show that

\[ u_\chi = (-1)^{X(1)} \det \chi, \tag{3} \]

where \( \det \chi \) is the linear character of \( G \) defined by the
composition

\[
\text{det } \chi : G \longrightarrow \text{GL}(V) \longrightarrow \mathbb{C}^*.
\]

See Yoshida [Yo78]. In order to prove (3), we may assume that \( G \) is cyclic. Since \( u_{\chi + \varphi} = u_{\chi} u_{\varphi} \) and \( \text{det } (\chi + \varphi) = \text{det } \chi \cdot \text{det } \varphi \), we may further assume that either \( \chi \) is a real valued linear character or \( \chi = \lambda + \overline{\lambda} \) for some nonreal linear character \( \lambda \). In the first case, we have that \( \langle \chi \mid_{<g>}, 1_{<g>} \rangle = 1 \) if \( g \) is in the kernel of and \( = 0 \) otherwise, and so \( u = -1 \) or \( +1 \), respectively. Since \( \text{det } \chi = \chi \), (3) holds in this case. Next assume that \( \chi = \lambda + \overline{\lambda} \), so that \( \langle \chi \mid_{<g>}, 1_{<g>} \rangle = 0 \) or \( 2 \) and \( \text{det } \chi = 1_G \). Thus (3) holds also in this case. The theorem is proved.

3. Some transfer theorems for the unit groups.

Let \( p \) be a prime. We put

\[
\mathbb{Z}(p) := \{ a/b \in \mathbb{Q} \mid a \in \mathbb{Z}, \ b \in \mathbb{Z} - p\mathbb{Z} \},
\]

\[
\Lambda(G)(p) := \mathbb{Z}(p) \bigotimes \mathbb{Z} \Lambda(G).
\]

For a finite group \( H \), the subgroup generated by all \( p' \)-elements of \( H \) is denoted by \( O^p(H) \). When \( O^p(Q) = Q \) (that is, \( Q \) has no normal subgroup of index \( p \)), the group
\( Q \) is called to be \( p \)-perfect. Let \( \text{Cl}_p(G) \leq \text{Cl}(G) \) denote the classes of \( p \)-perfect subgroups.

There is a one-to-one correspondence between primitive idempotents of \( \text{A}(G)_{(p)} \) and \( \text{Cl}_p(G) \) (cf. [Di79] 1.4). An explicit formula of primitive idempotents was obtained by Gluck [Gl81] and Yoshida [Yo83]. Let \( \mu \) be the Mobius function of the subgroup lattice of \( G \) and \( \delta_G \) the function defined by

\[
\delta_G(H, K) := \begin{cases} 
1 & \text{if } H = G \setminus K \\
0 & \text{otherwise.}
\end{cases}
\]

Each primitive idempotent of \( \text{A}(G)_{(p)} \) is then written in the form

\[
e_{G, Q}^p = \sum_{(D) \in \text{Cl}(G)} \lambda_{G, Q}^{(D)} [D \setminus G],
\]

where \( (Q) \in \text{Cl}_p(G) \) and

\[
\lambda_{G, Q}^{(D)} := \frac{1}{|N_G(D)|} \sum_{K \leq G} \mu(D, K) \delta_G(O^P(K), Q).
\]

As a super class function, \( e_{G, Q}^p \) has the value

\[
e_{G, Q}^p(S) = \begin{cases} 
1 & \text{if } O^P(S) = G \setminus Q \\
0 & \text{otherwise.}
\end{cases}
\]

For the finite group \( G \), let \( \text{St}_G \) denote the
category of finite (right) $G$-sets and $G$-maps. For two
$G$-sets $X$ and $Y$, let $Y^X$ be the $G$-set consisting of all
mappings of $X$ to $Y$ with $G$-action defined by $\alpha^g(x) :=
\alpha(xg^{-1}) \cdot g$. For an element $a = [A] - [B]$ of $A(G)$, we
furthermore define the exponential map

$$(-)^a : A(G)^* \to A(G)^* ; u \mapsto u^{A+B}.$$ 

We often write $u^a$ for $u^a$. By this action, $A(G)^*$ is
an $A(G)_{(2)}$-module whose annihilator contains $2A(G)_{(2)}$.

**Theorem B.** (i) There is a decomposition

$$A(G)^* = \bigoplus_{(Q)} A(G)^* \uparrow e^2_{G,Q},$$

where $(Q)$ runs over $Cl_2(G)$, classes of 2-perfect
subgroups.

(ii) Let $Q$ be a 2-perfect subgroup of $G$ and let $P$
be a subgroup of $N := N_G(Q)$ such that $P/Q$ is a Sylow
2-subgroup of $N/Q$. Then there are group isomorphisms:

$$A(G)^* \uparrow e^2_{G,Q} \cong A(N)^* \uparrow e^2_{N,Q} \cong (A(P)^*)^N \uparrow e^2_{P,Q},$$

where the last group is the subgroup of $A(P)^* \uparrow e^P_{P,Q}$
consisting of all elements $x$ such that

$$\text{res}^P_{P \cap P} \text{con}^n_{P \cap P} (x) = \text{res}^P_{P \cap P} (x)$$
for any element $n$ of $N$.

**Theorem C.** Let $N$ be a finite group with 2-perfect normal subgroup $Q$. Put $W := N/Q$. Let $\bar{P} = P/Q$ be a Sylow 2-subgroup of $W$. Then the following groups are isomorphic:

(a) $A(N)^* \uparrow e^2_{N,Q}$,
(b) $\{ \bar{u} \in A(W)^* \uparrow e^2_{W,1} \mid \bar{u}(S/Q) = 1 \text{ if } o^{2'}(S) \nmid Q \}$,
(c) $\{ \bar{v} \in A(\bar{P})^{*W} \mid \bar{v}(S/Q) = 1 \text{ if } o^{2'}(S) \nmid Q \}$,

where $o^{2'}(S)$ is the subgroup of $S$ generated by all 2-elements and $A(\bar{P})^{*W}$ is the set of elements $v$ of $A(\bar{P})^*$ such that

$$\text{res}_{\bar{P}^w \cap \bar{P}} \text{ con}^g(x) = \text{res}_{\bar{P}^w \cap \bar{P}} (x) \text{ for all } w \text{ in } W.$$ 

**Theorem D.** Assume that the finite group $G$ has an abelian Sylow 2-subgroup. Let $Q$ be a 2-perfect subgroup of $G$. Put $N := N_G(Q)$, $W := N/Q$, and let $P/Q$ be an (abelian) Sylow 2-subgroup of $W$. Put $L := N_G(P) (\subseteq N)$, $\bar{L} := L/Q'$, $\bar{Q} := Q/Q'$, and $\bar{P} := P/Q'$, where $Q'$ is the intersection of subgroups of $Q$ of odd prime index. Then the following hold:

(i) $A(G)^* \uparrow e^2_{GQ} \cong A(\bar{L})^* \uparrow e^2_{LQ}$. 

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(ii) If $Q$ is perfect, then
\[ A(\bar{L})^* \uparrow_{LQ}^{2} \cong A(P/Q)^*_{L/Q} \cong C_{(P/QP^2)}(L/Q), \]
where $C$ stands for the centralizer group.

(iii) Assume that $Q$ is not perfect. If $P$ is generated by elements $t$ with $C_Q(t) \neq 1$, then
\[ A(\bar{L})^* \uparrow_{LQ}^{2} = 1, \]
and otherwise it is of order 2.

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Appendix

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