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# The witt groups of orthogonal representation

by

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§0. Introduction. Let D be a Dedekind domain with the quotient field K. Let $\Lambda$ be a D-order in the semisimple K-algebra A. We assume that $\Lambda$ (and so does A) has an involution (-) with $\overline{a+b} = \bar{a}+\bar{b}$, $\overline{ab}=\bar{b}\bar{a}$, and $\bar{\alpha} = \alpha$ for all $a,b\in\Lambda$, $\alpha\in D$. Most interesting case is when $\Lambda$ is the integral group ring ZG of a finite groiup G.

J.P.Alexander, P.E.Conner, and G.C.Hamrick[1] have got the following exact sequence by defining the witt groups $W_*(D,\Lambda)$, $W_*(K,A)$, and $W_*(K/D,\Lambda)$ of orthogonal representations:

(0.1) $$0 \to W_*(D,\Lambda) \to W_*(K,A) \to W_*(K/D,\Lambda)$$

The above quadratic forms are D, K, K/D-valued, respectively. On the other hand, there are the witt groups $W_*^{\lambda}(\Lambda)$ of $\Lambda$-valued $\lambda$-quadratic form of $\Lambda$-projective modules for $\lambda=\pm1$. For these witt groups, W.Pardon[3] obtained the following periodic exact sequence by giving the higher witt groups $W_1^{\lambda}(\Lambda)$, $W_1^{\lambda}(A)$, and $W_1^{\lambda}(A/\Lambda)$:

(0.2) $$\to W_1^{\lambda}(\Lambda) \to W_1^{\lambda}(A) \to W_1^{\lambda}(A/\Lambda) \to W_0^{\lambda}(\Lambda) \to W_0^{\lambda}(A) \to W_0^{\lambda}(A/\Lambda)$$

$$\to W_1^{-\lambda}(\Lambda) \to W_1^{-\lambda}(A) \to W_1^{-\lambda}(A/\Lambda) \to$$

The motivation for the paper to be described here comes from the difference between hyperbolic and metabolic. The above groups are all defined by using metabolic, but the latter is equal to hyperbolic since those modules are $\Lambda$-projective. We will at first reformulate the definition of the witt group by using hyperbolic, then we will define the higher witt group of orthogonal representation. At last, we will get the following exact sequence:

Theorem A. Under the above hypothesis, we have the following exact sequence:

(0.3) $$W_0^{-\lambda}(K/D,\Lambda) \xrightarrow{\delta} W_1^{\lambda}(D,\Lambda) \xrightarrow{\kappa} W_1^{\lambda}(K,A) \xrightarrow{\nu} W_1^{\lambda}(K/D,\Lambda)$$

$$\xrightarrow{\delta} W_0^{\lambda}(D,\Lambda) \xrightarrow{\kappa} W_0^{\lambda}(K,A) \xrightarrow{\nu} W_0^{\lambda}(K/D,\Lambda)$$

In their paper[1], they conjectured that if $G$ is an odd group, then the map $W_*(Q,QG) \to W_*(Q/Z,ZG)$ is an epimorphism.

Theorem B. If their conjecture is true, then for the group ring $\Lambda=ZG$ of the finite group $G$ of odd order, the exact sequence (0.3) in Theorem A is periodic.

§1. Notations and definitions. We will adopt the notations and definitions from [1] and [3]. For simplifying the notations, we use the letter $R$ to denote $D$, $K$, $K/D$ and the letter $B$ for $\Lambda$, $A$, or $\Lambda$, when $R$ is $D$, $K$, $K/D$, respectively. We will also assume that $B$ is equipped with an anti-involution $(-)$ that reduces to the identity on $R\subset B$.

For left B-module $V$, we call it B-lattice when one of the following holds:

a) $R=D$ and $V$ is a $\Lambda$-lattice,

b) $R=K$ and $V$ is a finitely generated A-module, and

c) $R=K/D$ and $V$ is a finitely generated torsion $\Lambda$-module.

We first recall from [1] the definition of orthogonal representation.

Definition 1.1. A $\lambda$-orthogonal representation ($\lambda$-orth. rep.) of $B$ is a triple $(B,V,b)$ wherein

1) $V$ is a B-lattice,

2) $(V,b)$ is an R-valued $\lambda$-symmetric (i.e. $b(v,w)=\lambda b(w,v)$ ) inner product space structure on $V$, and

3) for $s\in R$, $v,w\in V$, $b(sv,w) = b(v,\bar{s}w)$.

Definition 1.2. For a $\lambda$-orth. rep. $(B,V,b)$, let $b_\phi$ be the homomorphism : $V \to V^*=\mathrm{Hom}_R(V,R)$ with $b_\phi(v)(w)=b(v,w)$.

Definition 1.3. A $\lambda$-orth. rep. $(B,V,b)$ is metabolic if and only if there is a B-submodule $N \subset V$ for which $N = \{v\in V \mid b(v,n)=0 \text{ for all } n\in N\}$. The $\lambda$-orth. rep. $(B,V,b)$ is hyperbolic if and only if $V$ has two totally isotropic B-submodules $W$ and $W^*$ such that $V = W\oplus W^*$ and $b|W\oplus W^*$

is the natural form, that is, $b(v,w)=w(v)$ for $v \in W$, $w \in W^{*}=\mathrm{Hom}_R(W,R)$.

Definition 1.4. $F_0{}^{\lambda}(R,B)$ denotes the set of isometry classes of nonsingular $\lambda$-orth. rep. $(B,V,b)$.

$F_0{}^{\lambda}(R,B)$ is an abelian semigroup under orthogonal sum.

Definition 1.5. For $R=D$, we introduce the following equivalence: Let $(\Lambda,V_1,b_1)$ and $(\Lambda,V_2,b_2)$ be $\lambda$-orth. reps., then $(\Lambda,V_1,b_1)$ $(\Lambda,V_2,b_2)$ if and only if there is a hyperbolic $\lambda$-orth. rep. $(\Lambda,W,c)$ such that $(\Lambda,V_1,b_1)\oplus(\Lambda,V_2,-b_2)\oplus(\Lambda,W,c)=(\Lambda,V_1\oplus V_2\oplus W,\ b_1\oplus(-b_2)\oplus\ c\ )$ is hyperbolic.

Definition 1.6. Let $W_0{}^{\lambda}(D,\Lambda)$ be the quotient of the Grothendieck group on $G_0{}^{\lambda}(A)$ by the subgroup generated by the above equivalence relation. $W_0{}^{\lambda}(K,A)$ and $W_0{}^{\lambda}(K/D,\Lambda)$ are the Grothendieck group on $F_0{}^{\lambda}(A)$ modulo the subgroup generated by metabolics.

Definition 1.7. A $\lambda$-orth. formation of $B$ is a triple $(L,H,\Delta)$ wherein:

1) $L,H$ are B-lattices,

2) if $R=D$ or $K$, then $K\otimes H$ is A-free and if $R=K/D$ then $H$ has a short free resolution. (We will denote the set of these

B-lattices by D(R).)

3) $\Delta=(\xi,\zeta):L \to H\oplus H^*$ is an injection and $\Delta(L)$ is a metabolizer in $H\oplus H^*$, and

4) $(B,L,[\ ,\ ])$ is a $(-\lambda)$-orth. rep., where $[v,w]=\zeta(w)(\xi(v))$ for $v,w\in L$.

Definition 1.8. Let $F_1^{\lambda}(R,B)$ denote the set of isomorphism classes of $\lambda$-orth. reps. (isomorphisms induced by isomorphisms of L and H prserving $\Delta$).

$F_1^{\lambda}(R,B)$ is an abelian semigroup in the obvious way.

We next intrduce the following four operations on $F_1^{\lambda}(R,B)$. Let $\theta=(L,H,\Delta)$ be a $\lambda$-orth. formation with $\Delta=(\xi,\zeta)$.

(A). Let (E): $0\to J\to H_1 \xrightarrow{j} H\to 0$ be the short exact sequence of elements of D(R). We define a $\lambda$-orth. formation $\sigma(\theta)=(L_1,H_1,\Delta_1)$ as follows. Let $L_1$ be the pullback in the diagram:

$$\begin{array}{ccccccccc} 0 & \to & J & \longrightarrow & L_1 & \xrightarrow{j_1} & L & \to & 0 \\ & & \downarrow = & & \downarrow \xi_1 & & \downarrow \xi & & \\ 0 & \to & J & \longrightarrow & H_1 & \xrightarrow{j} & H & \to & 0 \end{array}$$

Define $\zeta_1=j\hat{}\,\zeta j_1$, where $j\hat{}: H^* \to H_1^*$ is given by $j\hat{}(h^*)(h_1)=h(j(h_1))$. Then one verifies that if we set $\Delta_1=(\xi_1,\zeta_1)$, then $\sigma(\theta)=(L_1,H_1,\Delta_1)$ is a $\lambda$-orth. formation and it is well-defined up to isomorphism by $\theta$ and the isomorphism class of (E).

(B). Let $(E'): 0 \to T \to H_1^* \xrightarrow{t} H^* \to 0$ be the exact sequence of elements of $D(R)$. Then by the similar way as above, we can make a $\lambda$-orth. formation $\sigma(\theta) = (L_1, H_1, \Delta_1)$, where $L_1$ is the pullback

$$\begin{array}{ccccccccc} 0 & \to & T & \longrightarrow & L_1 & \xrightarrow{t_1} & L & \to & 0 \\ & & \downarrow = & & \downarrow \zeta_1 & & \downarrow \zeta & & \\ 0 & \to & T & \longrightarrow & H_1^* & \xrightarrow{t} & H^* & \to & 0 \end{array}$$

and $\xi_1 = r^{\wedge}\xi r_1$ and $\Delta_1 = (\xi_1, \zeta_1)$.

(C). If $(B, H, \phi)$ is a $(-\lambda)$-orth. rep., then set $\chi(H, \phi)\theta = (L, H, \Delta')$, where $\Delta' = (\xi, \zeta + d_\phi \xi)$. Then $(L, H, \Delta')$ becomes a $\lambda$-orth. formation.

(D). If $(H^{\wedge}, \phi)$ is a $(-\lambda)$-orth. res., then the triple $\chi(H^{\wedge}, \phi)\theta = (L, H, \Delta')$ given by $\Delta' = (\xi + d_\phi \zeta, \zeta)$ is a $\lambda$-orth. formation.

Definition 1.8. $W_1^{\lambda}(R, B)$ is the semigroup $F_1^{\lambda}(R, B)$ modulo the eqivalence relation generated by the above four operations.

Remark. We will later show that $W_1^{\lambda}(R, B)$ is a group.

§2. Definitions of maps. In this section, we will define the homomorphisms.

(a). $\kappa_0^\lambda$: $W_0^\lambda(D,\Lambda) \longrightarrow W_0^\lambda(K,A)$ and
$\kappa_1^\lambda$: $W_1^\lambda(D,\Lambda) \longrightarrow W_1^\lambda(K,A)$.

These maps are given by tensoring with K.

Lemma 2.1. These maps are well-defined.

(b). $\nu_0^\lambda$: $W_0^\lambda(K,A) \longrightarrow W_0^\lambda(K/D,\Lambda)$

This homomorphism is classical, see [1].

(c). $\delta_0^\lambda$: $W_0^\lambda(K/D,\Lambda) \longrightarrow W^{-\lambda}_1(D,\Lambda)$

Let a nonsingular torsion $\lambda$-orth. rep. $\theta=(\Lambda,V,b)$ be given. Choose a short exact sequence $0 \to Q \xrightarrow{\xi} P \xrightarrow{j} V \to 0$ with a $\Lambda$-free P. Since P is $\Lambda$-free, there is a $\lambda$-orth. rep. $(\Lambda,P,d)$ such that $d(p_1,p_2)\equiv b(j(p_1),j(p_2))$ (mod D). Since $d(Q,P)\subset D$, there is a homomorphism $\zeta:Q\to P^*$. Then $(Q,P,\Delta=(\xi,\zeta))$ becomes a $(-\lambda)$-orth. formation. Set $\delta_0^\lambda(\theta)=[(Q,P,\Delta)] \in W^{-\lambda}_1(D,\Lambda)$.

Lemma 2.2. $\delta_0^\lambda$ is well defined.

(d). $\nu_1^\lambda$: $W_1^\lambda(K,A) \longrightarrow W_1^\lambda(K/D,\Lambda)$

Let $\theta=(L,H,\Delta=(\theta,\zeta))$ be a $\lambda$-orth. formation of A. Since A is semi-simple, we may assume $L\oplus L^*=H\oplus H^*$. Let I be a $\Lambda$-free

full $\Lambda$-lattice in $L$ and $I^*=\{w^*\in L^*: (w^*, I)\subset D\}$. Let $W$ and $W_1$ be $\Lambda$-free full $\Lambda$-lattices in $H\cap(I\oplus I^*)$ and in $H^*\cap(I\oplus I^*)$, respectively. Then we have
$W\oplus W_1\subset I\oplus I^*\subset W_1{}^*\oplus W^*$. Setting the inclusion
$\bar{\Delta}$: $I\oplus I^*/W\oplus W_1 \longrightarrow W_1^*/W\oplus W^*/W_1$, we get an element $(I\oplus I^*/W\oplus W_1, W_1^*/W, \bar{\Delta})$ of $F_1{}^\lambda(K/D,\Lambda)$.

Lemma 2.3. $\nu_1^\lambda$ is well-defined.

(e). $\delta_1^\lambda$: $W_1^\lambda(K/D,\Lambda) \longrightarrow W_0^\lambda(D,\Lambda)$

Let $\theta=(V,W,\Delta=(\xi,\zeta))\in F_1{}^\lambda(K/D,\Lambda)$. Let $0\to Q\to P\to W\to 0$ be a short resolution of $W$. Set $H=K\otimes P$ and we assume $P\subset H$ and $Q^*\subset H^*$. Since $P,Q$ are $\Lambda$-free, there is a short resolution of $W^*$: $0\to P^*\to Q^*\to W^*\to 0$. (see [3]). Considering $V$ as a submodule in $W\oplus W^*\cong P/Q\oplus Q^*/P^*$, let $L$ be the inverse image of $V$ in $P\oplus Q^*$ and $b$ be the bilinear form on $L$ induced from $P\oplus Q^*$. Then we have that $(\Lambda,V,b)$ is a nonsingular $\lambda$-orth. rep. and set $\delta\ (\theta)=[(\Lambda,V,b)]$.

Lemma 2.4. $\delta_1^\lambda$ is well-defined.

Reference.

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