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Transitive points under the modular group
and continued fractions

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0. Introduction.

A Fuchsian group is a discrete subgroup of linear fractional transformations each of which preserves a unit disk \( D = \{ z | |z|<1 \} \) (or upper half plane \( H = \{ z = x+iy | y>0 \} \)). Denote the boundary of \( D \) and of \( H \) by \( S \) and \( \hat{\mathbb{R}} \) respectively. Since a Fuchsian group acting on \( H \) is conjugate to some Fuchsian group acting on \( D \) by some linear fractional transformation, we consider a Fuchsian group acting on \( D \) or \( H \) case by case.

We think that \( D \) and \( H \) are both equipped with Poincaré metric. The ergodic properties of Fuchsian groups have been investigated by many authors (e.g. [2],[6]). In this paper, we consider the following property. Let \( \Gamma \) be a Fuchsian group acting on \( D \).

We call a point \( \zeta \in S \) is a transitive point under \( \Gamma \) if, for all ordered pair \( (\zeta_1, \zeta_2) \) of two distinct points of \( S \) and all \( z \in D \) and for all \( \varepsilon > 0 \), there exists an element \( \gamma \in \Gamma \) such that \( |\zeta_1 - \gamma(z)| + |\zeta_2 - \gamma(z)| < \varepsilon \). In fact, the transitivity is independent of the choice of \( z \) (see [4]). The transitivity associated to a Fuchsian group acting on \( H \) is defined similarly. If \( \zeta \) is not a transitive point, we call it an intransitive point under \( \Gamma \).

For example, parabolic fixed points of \( \Gamma \) are intransitive points
under $\Gamma$ (see [4]). In [5], it showed that if $\Gamma$ is a finitely generated Fuchsian group of the first kind, almost all points of $S$ are transitive points under $\Gamma$. But what points are transitive under $\Gamma$? We consider this problem in the case of the modular group $G$. In this case, Artin [1] investigated the transitivity of geodesic lines as Quasiergodizität. The modular group is a Fuchsian group acting on $\mathbb{H}$ and each of whose elements is of the form

$$g(z) = \frac{az+b}{cz+d} \quad a, b, c, \text{ and } d \text{ are integers } \& \text{ ad-bc = 1.}$$

By $[n_0, n_1, n_2, \cdots]$ we denote the continued fraction $x = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \cdots}}$, where $n_0$ is non-negative integer and $n_i$, $i \geq 0$, is a positive integer. If $x < 0$, we define $x = -[n_0, n_1, n_2, \cdots]$ for $-x = n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \cdots}}$. All rational numbers are parabolic fixed points of $G$, so we consider only irrational numbers. Using continued fractions, we give a characterization of transitive points under the modular group, as follows.

Theorem 1. A point $x = \pm[n_0, n_1, n_2, \cdots]$ is a transitive point under the modular group $G$ if and only if, for an arbitrary finite sequence $a_0, a_1, \cdots, a_m$, where $a_i$ is a positive integer, there exists $k$ such that $n_k = a_0, n_{k+1} = a_1, \cdots, n_{k+m} = a_m$.

In §1, we prove some lemmas and a theorem on transitive points under an arbitrary Fuchsian group. In §2, we shortly explain the cutting sequence which is defined in [7] by Series. In §3, we give the proof of Theorem 1. In §4, using the cutting
sequence, we give another proof of a certain theorem concerning to continued fractions.

1. Theorem on transitivity.

Let \( \xi, \xi_1 \) and \( \xi_2 \) be in \( S \) and let \( z \) be in \( D \). By \( L(\xi_1, \xi_2) \) and \( R(z, \xi) \) we denote the oriented geodesic line whose initial point \( \xi_1 \) and whose terminal point \( \xi_2 \) and the oriented geodesic ray whose initial point \( z \) and whose terminal point \( \xi \) respectively. We say that \( R(z, \xi) \) (or \( L(\xi_1, \xi_2) \)) converges to \( L(\theta_1, \theta_2) \) if, for all \( \epsilon > 0 \), there exists an element \( \gamma \in \Gamma \) such that \( |\gamma(z) - \theta_1| + |\gamma(\xi) - \theta_2| < \epsilon \) (or \( |\gamma(\xi_1) - \theta_1| + |\gamma(\xi_2) - \theta_2| < \epsilon \)). Using this notation, we say the definition of transitivity in \( S \) as follows. A point \( \xi \) is called a transitive point under \( \Gamma \) if, for an arbitrary geodesic line \( L(\theta_1, \theta_2) \) and an arbitrary geodesic ray \( R(z, \xi) \), \( R(z, \xi) \) converges to \( L(\theta_1, \theta_2) \). We also say that \( L(\xi_1, \xi_2) \) is a transitive geodesic line under \( \Gamma \) if, for an arbitrary geodesic line \( L(\theta_1, \theta_2) \), the geodesic line \( L(\xi_1, \xi_2) \) converges to \( L(\theta_1, \theta_2) \). If \( L(\xi_1, \xi_2) \) is not a transitive geodesic line under \( \Gamma \), we call it an intransitive geodesic line under \( \Gamma \). For example, let \( \xi_1 \) and \( \xi_2 \) be fixed points of a hyperbolic element of \( \Gamma \). Then \( L(\xi_1, \xi_2) \) is an intransitive geodesic line.

In this section, we assume that \( \Gamma \) is an arbitrary Fuchsian group, but not an elementary group. Hence \( \Gamma \) has hyperbolic elements. Let \( \xi_1 \) and \( \xi_2 \) be fixed points of a hyperbolic element of \( \Gamma \). The geodesic ray \( R(z, \xi_2) \) converges to only

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L(\xi_1, \xi_2) and its \Gamma\text{-images. Thus } \xi_2 \text{ is an intransitive point. Similarly all the hyperbolic fixed points of } \Gamma \text{ are intransitive points under } \Gamma.

In the proofs of the following lemmas and a theorem, we only consider Fuchsian groups acting on D. But the result is true for Fuchsian groups acting on H.

Lemma 1. Let L(\xi_1, \xi_2) be an intransitive geodesic line under \Gamma. If L(\xi_1, \xi_2) converges to some L(\theta_1, \theta_2), then L(\theta_1, \theta_2) is an intransitive geodesic line under \Gamma.

Proof. Assume that L(\theta_1, \theta_2) is a transitive geodesic line. Then, for an arbitrary geodesic line L(\eta_1, \eta_2) and for all \epsilon > 0, there exists an element \gamma \in \Gamma such that |\gamma(\eta_1) - \eta_1| + |\gamma(\eta_2) - \eta_2| < \epsilon. Since each element of \Gamma maps S to S continuously, there exists \delta > 0 such that, for |\theta_1 - \theta_1'| + |\theta_2 - \theta_2'| < \delta, |\gamma(\theta_1') - \eta_1| + |\gamma(\theta_2') - \eta_2| < \epsilon. For this \delta, there exists an element \beta \in \Gamma such that |\beta(\xi_1) - \theta_1| + |\beta(\xi_2) - \theta_2| < \delta since L(\xi_1, \xi_2) converges to L(\theta_1, \theta_2). Hence we have |\gamma \beta(\xi_1) - \eta_1| + |\gamma \beta(\xi_2) - \eta_2| < \epsilon. This shows that L(\xi_1, \xi_2) converges to an arbitrary geodesic line L(\eta_1, \eta_2). This contradicts the assumption of L(\xi_1, \xi_2). q.e.d.

Lemma 2. If the geodesic ray R(z, \xi) converges to an arbitrary transitive geodesic line, then \xi is a transitive point under \Gamma.

Proof. We take a transitive geodesic line L(\theta_1, \theta_2). For an arbitrary geodesic line L(\eta_1, \eta_2) and all \epsilon > 0, there exists
an element \( \gamma \in \Gamma \) such that \(|\gamma(\theta_1) - \eta_1| + |\gamma(\theta_2) - \eta_2| < \varepsilon/2\). Since \(L(\gamma(\theta_1), \gamma(\theta_2))\) is also a transitive geodesic line, there exists an element \( \beta \in \Gamma \) such that \(|\beta(z) - \gamma(\theta_1)| + |\beta(\zeta) - \gamma(\theta_2)| < \varepsilon/2\). Hence we have \(|\beta(z) - \eta_1| + |\beta(\zeta) - \eta_2| < \varepsilon\). This shows that \( \zeta \) is a transitive point under \( \Gamma \). q.e.d.

Using above two lemmas, we prove the following theorem.

**Theorem 2.** Both \( \zeta_1 \) and \( \zeta_2 \) are intransitive points under \( \Gamma \) if and only if \( L(\zeta_1, \zeta_2) \) is an intransitive geodesic line under \( \Gamma \).

**Proof.** The sufficient condition is clear from the definitions.

First, we assume that at least one of \( \zeta_1 \) and \( \zeta_2 \) is a hyperbolic fixed point of \( \Gamma \), say \( \zeta_1 \). By \( \zeta'_1 \) we denote another fixed point of the hyperbolic element which fixes \( \zeta_1 \). We take \( z \in L(\zeta_1, \zeta_2) \cap D \). The geodesic ray \( R(z, \zeta_1) \) converges to only \( L(\zeta'_1, \zeta_1) \) and its \( \Gamma \)-images. Hence \( R(z, \zeta_2) \) must converge to an arbitrary transitive geodesic line, if \( L(\zeta_1, \zeta_2) \) is a transitive geodesic line. Therefore, by Lemma 2, \( \zeta_2 \) is a transitive point. This is contradiction. Hence \( L(\zeta_1, \zeta_2) \) is an intransitive geodesic line.

Next, we assume that neither \( \zeta_1 \) nor \( \zeta_2 \) is a hyperbolic fixed point. Take an arbitrary hyperbolic fixed point \( \zeta_3 \). By the above argument, \( L(\zeta_3, \zeta_2) \) is an intransitive geodesic line. So, by Lemma 1, \( R(z, \zeta_2) \) converges to only intransitive geodesic lines. Hence \( R(z, \zeta_1) \) converges to an arbitrary transitive
geodesic line, if $L_1(\zeta_1,\zeta_2)$ is a transitive geodesic line. This means $\zeta_1$ is a transitive point by Lemma 2. Therefore $L_1(\zeta_1,\zeta_2)$ is an intransitive geodesic line. q.e.d.

2. Cutting sequences.

In the following sections, by $G$ we denote the modular group. We consider the Farey tesselation $F$, the tesselation of $H$ by images of the imaginary axis under $G$. Each tesselera of $F$ is a non-euclidean triangle whose vertices are all on $\hat{\mathbb{R}}$. An arbitrary oriented geodesic line $L(x,y)$ is divided into oriented segments by the triangles of $F$. We label each oriented segment either $R$ or $L$ according as two sides of the triangle which the segment crosses meet to the right or left of the segment. If $L(x,y)$ starts from a vertex of some triangle or ends in a vertex of some triangle, we may label the segment $R$ or $L$ freely. We arrange the letters $R$ and $L$ as according to the order of the directed segments of $L(x,y)$. If $R$'s (or $L$'s) are successively arranged n times, we write $R^n$ (or $L^n$). In this way, we associate a sequence $\ldots L^n_0 R^n_1 L^n_2 R^n_3 \ldots$ to the directed geodesic line $L(x,y)$. Series [7] called it the cutting sequence of $L(x,y)$. If $L(x,y)$ starts from a vertex of some triangle, then the cutting sequence is finite on the left side. If $L(x,y)$ ends in a vertex of some triangle, then the cutting sequence is finite on the right side. Since each element of $G$ is orientation preserving, labels $R$ and $L$ are invariant under $G$. For simplicity, we define numbers of even order always denote
L and numbers of odd order always denote R and we write the cutting sequence $\langle \cdots n_{-1}, n_0, n_1, n_2, \cdots \rangle$. Series [7] showed the following theorem.

**Theorem A.** Let $x$ be in $[-1,0)$ and $y$ be in $[1,\infty)$. Then the cutting sequence of $L(x,y)$ is of the form $\langle \cdots n_{-1}, |n_0, n_1, \cdots \rangle$, where the symbol $|$ corresponds to the position where $L(x,y)$ and the imaginary axis cross, if and only if $x = [-[0, n_{-1}, n_{-2}, \cdots]]$ and $y = [n_0, n_1, n_2, \cdots]$.

3. Proof of Theorem 1.

The modular group $G$ is generated by $\tau(z) = -1/z$ and $\sigma(z) = z + 1$. If $x = -[n_0, n_1, n_2, \cdots], n_0 \neq 0$ then $\tau(x) = [1, n_0, n_1, \cdots]$. If $x = -[0, n_1, n_2, \cdots]$ then $\tau(x) = [n_1, n_2, \cdots]$. If $x = [0, n_1, n_2, \cdots]$ then $\sigma(x) = [1, n_1, n_2, \cdots]$. Hence we consider only the case $x > 1$. It is well-known (e.g. [1]) that the arbitrary directed geodesic line except for $\{L(g(0), g(\infty)) | g \in G\}$ is equivalent under $G$ to some directed geodesic line $L(\theta_1, \theta_2)$ where $\theta_1$ is in $[-1, 0)$ and $\theta_2$ is in $[1, \infty)$. Since the point $-1$ is an intransitive point, Theorem 2 implies that $L(-1, x)$ is a transitive geodesic line if and only if $x$ is a transitive point. From the above fact, we prove the following theorem for the proof of Theorem 1.

**Theorem 1'.** Let the continued fraction of $x$ be of the form $[n_0, n_1, n_2, \cdots], n_0 \neq 0$. Then the directed geodesic line
L(-1, x) converges to an arbitrary directed geodesic line
\( L(\theta_1, \theta_2), \theta_1 \in [-1, 0) \) and \( \theta_2 \in [1, \infty) \) if and only if, for an arbitrary finite sequence \( a_0, a_1, \ldots, a_m \), where \( a_i \) is a positive integer, there exists \( k \) such that \( n_k = a_0, n_{k+1} = a_1, \ldots, n_{k+m} = a_m \).

Remark. The later condition implies the condition that, for an arbitrary finite sequence \( a_0, a_1, \ldots, a_m \) and for an arbitrary integer \( i, 0 \leq i \leq m \), there exists \( u \) such that \( n_{2u-i} = a_0, \ldots, n_{2u} = a_i, n_{2u+1} = a_{i+1}, \ldots, n_{2u+m-1} = a_m \) (see [1]).

Proof. Since irrational numbers are dense in \( \mathbb{R} \), it is sufficient to consider the case that \( \theta_1 \) and \( \theta_2 \) are irrational. Set \( \theta_1 = [-0, a_{-1}, a_{-2}, \ldots] \) and \( \theta_2 = [a_0, a_1, a_2, \ldots] \). From the theory of Diophantine approximations, we see, for an arbitrary \( \varepsilon > 0 \), there exist positive integers \( t \) and \( s \) such that

\[
|\theta_1 + [0, a_{-1}, a_{-2}, \ldots, a_{-t}, \omega_1]| < \varepsilon/2 \quad \text{and} \\
|\theta_2 - [a_0, a_1, a_2, \ldots, a_s, \omega_2]| < \varepsilon/2,
\]

where \( \omega_1 \) and \( \omega_2 \) are arbitrary numbers greater than 1. We consider the finite sequence \( a_{-t}, \ldots, a_{-1}, a_0, a_1, \ldots, a_s \). By the assumption and the fact we remark, there exists \( u \) such that \( n_{2u-t} = a_{-t}, \ldots, n_{2u-1} = a_{-1}, n_{2u} = a_0, n_{2u+1} = a_1, \ldots, n_{2u+s} = a_s \). Hence the cutting sequence of \( L(-1, x) \) is of the form

\(<1, |n_0, n_1, \ldots, n_{2u-t-1}, a_{-t}, \ldots, a_{-1}, a_0, a_1, \ldots, a_s, n_{2u+s+1}, \ldots>\).

There exists an element \( g \in \mathcal{G} \) which maps the side of the tessera which the segment corresponding to \( a_{-1}, a_0 \) crosses to the
imaginary axis. Hence the cutting sequence of \( g(L(-1,x)) \)
\[ = L(g(-1), g(x)) \]
is of the form
\[ <1, n_0, n_1, \ldots, n_{2u-t-1}, a_{-t}, \ldots, a_{-1}, |a_0, a_1, \ldots, a_s, n_{2u+s+1}, \ldots>. \]

By Theorem A, we have
\[ g(-1) = [-0, a_{-1}, \ldots, a_{-t}, n_{2u-t-1}, \ldots, n_1, 1] \quad \text{and} \]
\[ g(x) = [a_0, a_1, \ldots, a_s, n_{2u+s+1}, \ldots]. \]

Therefore we have
\[ |g(-1) - \theta_1| + |g(x) - \theta_2| < \varepsilon. \]

This shows that \( L(-1,x) \) converges to \( L(\theta_1, \theta_2) \).

To show the converse direction, we follow the above argument conversely.

q.e.d.

4. An application.

Let \( x = [n_0, n_1, n_2, \ldots] \) be an irrational number. We call \( x \) of constant type if there exists a constant \( M \) such that \( n_1 < M \) for all \( i \) (see [3]). By Theorem 1, numbers of constant type are intransitive under \( G \). We assume that \( n_0 > 0 \). We consider the directed geodesic line \( L(-x,x) \). The cutting sequence of \( L(-x,x) \) is of the form
\[ <\cdots, n_2, n_1, 2n_0, n_1, n_2, \ldots>. \]

We set the element of \( G \)
\[ g(z) = \frac{rz+s}{qz+p}. \]

The geodesic line \( g(L(-x,x)) = L(g(-x), g(x)) \) is a semicircle whose center is in \( \hat{R} \), and whose diameter is
\[ |g(x) - g(-x)| = \frac{2x}{q^2|x-p/q||x+p/q|}. \]
On the other hand, the cutting sequence $\cdots n_2, n_1, 2n_0, n_1, n_2, \cdots$ implies that $g(L(-x,x))$ cuts at most $2M$ axes which are parallel to imaginary axis and whose endpoints are integers. Hence we have

$$|g(x)-g(-x)| < 2M+2.$$  

Therefore, this inequality is satisfied if and only if the inequality

$$|x-p/q| > c/q^2,$$

where $c$ is a constant which is independent of $p$ and $q$, is satisfied.

Next, we consider the Riemann surface $H/G$. The fundamental region of the modular group is $F = \{ z = x+iy | 0 \leq x < 1, \quad x^2+y^2 \geq 1 \ (0 \leq x \leq 1/2), \quad (x-1)^2+y^2 > 1 \ (1/2 < x < 1) \}$. We identify the Riemann surface $H/G$ with this fundamental region.

By $\pi$ we denote the natural projection from $H$ to $H/G$. All the elements of the set $\{ L(g(-x),g(x)) | g \in G \}$ exist below the line $y = M+1$ if and only if the geodesic line $\pi(L(-x,x))$ on $H/G$ is in $F \cap \{ z = x+iy | y \leq M+1 \}$. Hence we conclude the following theorem.

**Theorem 3.** The following three conditions are equivalent.

1) $x$ is of constant type.

2) $|x-p/q| > c/q^2$ for all integers $p$ and $q$ which are relatively prime numbers, where $c$ is a constant which is independent of $p$ and $q$.

3) $\pi(L(-x,x))$ is in some compact set in $H/G$. 

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Remark. The equivalence of 1) and 11) has been already proved by other method (e.g. [3]).

References.


