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On puncture variation

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1. Introduction.

Let Δ be the unit disc and let D ($\neq \mathbf{C}$) be a plane domain containing the origin. Set $D_p = D\setminus\{p\}$ for $p \in D\setminus\{0\}$. Then there exists unique holomorphic universal covering $f_p : \Delta \to D_p$ satisfying

$$f_p(0) = 0, \quad f_p'(0) > 0.$$

Our aim is to derive the variation of the covering f_p by moving the puncture p in the domain D. Such a variation is called a puncture variation. Theorem 4.2. is the main result of this paper which gives explicitly the variation of f_p. As a corollary we have a puncture variation of the Poincaré metric. To obtain the formula we use quasiconformal mappings and apply a well-known representation theorem for quasiconformal maps with small dilatation.

2. Construction of $f_{p+\varepsilon}$ from f_p.

For sufficiently small $\rho\in R$ let $N = \{z \mid 0<|z-p|<e^{\rho}\}$ be a punctured disc contained in $f_p(\Delta)$ with $0\notin N$. Let Δ_0 be a fixed component of $f^{-1}(N)$. Note that Δ_0 does not contain the origin. Let Γ be the covering group of f_p. The following Lemma is well known.

LEMMA 2.1. There exist a parabolic element $\beta\in\Gamma$ and a Möbius transformation A with the following properties,

(1) A maps the upper half-plane onto Δ,

(2) $A(\infty)\in\partial\Delta$ is the fixed point of β and $A^{-1}\circ\beta\circ Az = z+1$,

(3) Δ_0 is simply connected and contains a disc $A(U_c)$ with $U_c = \{z\in C \mid \operatorname{Im} z>c\}$ $(c>0)$, and

(4) two points z_1 and z_2 of Δ_0 are equivalent under Γ if and only if $z_2 = \beta^n(z_1)$ for some integer n.

PROOF. See Kra [3, p.52] or Ahlfors [1, Lemma 1] where more general Kleinian case is considered. q.e.d

Let Γ_0 be the cyclic subgroup of Γ generated by $\beta\in\Gamma$. Expressing $f_p^{-1}(N)$ as a disjoint union of the components, we have

$$f_p^{-1}(N) = \bigcup_{\gamma\in\Gamma/\Gamma_0} \gamma\Delta_0 \tag{2.1}$$

where Γ/Γ_0 denotes the set of left cosets. Let $\Pi: L\to N$ be a universal covering given by

$$\Pi(z) = p + e^{z+\rho},$$

where L is the left half-plane $\{z \mid \operatorname{Re} z<0\}$. By the theory of covering surface we can find a conformal map $\varphi:\Delta_0\to L$ such that

$$\varphi\circ\beta = \varphi+2\pi i \tag{2.2}$$

and

$$f_p = \Pi \circ \varphi \quad \text{on } \Delta_0 . \tag{2.3}$$

LEMMA 2.2. For $\varepsilon \in \mathbf{C}$ small, there exists a quasiconformal map $\psi_\varepsilon : L \to \mathbf{C}$ such that

$$\psi_\varepsilon(z+2\pi i) = \psi_\varepsilon(z)+2\pi i \quad \text{on } L \tag{2.4}$$

and

$$\varepsilon + \Pi \circ \psi_\varepsilon = \Pi \quad \text{on } \partial L \tag{2.5}$$

with complex dilatation

$$-\varepsilon e^{\overline{z}-\rho} + O(\varepsilon^2) \tag{2.6}$$

for $z \in L$. The estimate is uniform for $z \in L$.

PROOF. Taking a branch of the logarithm, we set

$$\psi_\varepsilon(z) = z + \ln\,(1-\varepsilon e^{\overline{z}-\rho}).$$

It is easy to see that this is a desired quasiconformal map. q.e.d.

Define a map $\tilde{f} : \Delta \to D_{p+\varepsilon}$ by

$$\tilde{f}(z) = \begin{cases} f_p(z) & , z \in f_p^{-1}(N) \\ \varepsilon + \Pi \circ \psi_\varepsilon \circ \varphi \circ \gamma^{-1}(z) & , z \in \gamma\Delta_0 \quad (\gamma \in \Gamma/\Gamma_0). \end{cases}$$

It is seen from (2.1)-(2.5) that $\tilde{f}$ is a well-defined (topological) covering of $D_{p+\varepsilon}$. Let g_μ denote the quasiconformal automorphism of Δ with complex dilatation μ which is holomorphic near the origin and satisfies $g_\mu(0) = 0$, $g_\mu'(0) > 0$.

LEMMA 2.3. We have the identity

$$f_{p+\varepsilon} = \tilde{f}\circ g_{\varepsilon\mu}^{-1} \quad \text{on } \Delta$$

where the complex dilatation μ is given by

$$\mu(z) = \begin{cases} 0 & , \ z\in f_p^{-1}(N) \\ \varepsilon^{-1}(\varphi\circ\gamma^{-1})^{*}\mu_{\psi_\varepsilon}(z) & , \ z\in\gamma\Delta_0 \quad (\gamma\in\Gamma/\Gamma_0) \ . \end{cases} \tag{2.7}$$

Here, $\varphi^{*}\mu$ denotes as usual the pull-back $\mu\circ\varphi\cdot\overline{\varphi'}/\varphi'$ of the Beltrami coefficient μ.

PROOF. Computing the complex dilatation we have $\mu_{\tilde{f}\circ g_{\varepsilon\mu}^{-1}} = 0$ a.e. on Δ. Hence $\tilde{f}\circ g_{\varepsilon\mu}^{-1}$ is a holomorphic covering of $D_{p+\varepsilon}$ such that

$$\tilde{f}\circ g_{\varepsilon\mu}^{-1}(0) = 0, \ (\tilde{f}\circ g_{\varepsilon\mu}^{-1})'(0)>0.$$

Since these conditions determine a holomorphic covering uniquely, we conclude that $f_{p+\varepsilon} = \tilde{f}\circ g_{\varepsilon\mu}^{-1}$. q.e.d.

3. Integral representation of the variation.

Let f_μ be the quasiconformal automorphism of Δ with complex dilatation μ which leaves 0 and 1 fixed. The following perturbation formula is well known [2, p.105].

LEMMA 3.1. For $\varepsilon\in C$ small and $\zeta\in\Delta$, $f_{\varepsilon\mu}$ is given by

$$f_{\varepsilon\mu}(\zeta) = \zeta+\dot{f}(\zeta) + O(\varepsilon^2)$$

where

$$\dot{f}(\zeta) = -\frac{\varepsilon}{\pi}\iint_\Delta \mu(z)R(z,\zeta)dxdy + \frac{\bar{\varepsilon}}{\pi}\iint_\Delta \overline{\mu(z)}\zeta^2\,\overline{R(z,1/\bar{\zeta})}dxdy$$

and

$$R(z,\zeta) = \frac{\zeta(\zeta-1)}{z(z-1)(z-\zeta)} .$$

The estimate is uniform for compact subsets of Δ.

It is convenient for our purpose to have a lemma with different normalization. The next lemma is a useful perturbation formula for g_μ. Recall that μ vanishes near the origin and that $g_\mu(0)=0$ and $g_\mu'(0)>0$.

LEMMA 3.2. For $\varepsilon\in C$ small and $\zeta\in\Delta$, $g_{\varepsilon\mu}$ is given by

$$g_{\varepsilon\mu}(\zeta) = \zeta+\dot{g}(\zeta)+O(\varepsilon^2)$$

where

$$\dot{g}(\zeta) = -\frac{\varepsilon\zeta}{2\pi}\iint_\Delta \mu(z)Q(z,\zeta)dxdy + \frac{\overline{\varepsilon\zeta}}{2\pi}\iint_\Delta \overline{\mu(z)Q(z,1/\overline{\zeta})}dxdy$$

and

$$Q(z,\zeta) = \frac{z+\zeta}{z^2(z-\zeta)}.$$

The estimate is uniform for compact subsets of Δ.

PROOF. Observe that

$$\dot{g}(\zeta) = \dot{f}(\zeta) + \frac{1}{2}\zeta(\overline{\dot{f}'(0)}-\dot{f}'(0)).$$

LEMMA 3.1. yields

$$\dot{f}'(0) = \frac{\varepsilon}{\pi}\iint_\Delta \frac{\mu(z)}{z^2(z-1)}dxdy - \frac{\overline{\varepsilon}}{\pi}\iint_\Delta \frac{\overline{\mu(z)}}{\overline{z}(\overline{z}-1)}dxdy.$$

Combining these identities, we obtain the Lemma. q.e.d.

From Lemmas 2.3 and 3.2 we have, for $\zeta\notin f_p^{-1}(N)$,

$$f_{p+\varepsilon} = f_p(\zeta) + \zeta f_p{}'(\zeta)(\varepsilon I(\zeta)+\bar{\varepsilon}J(\zeta)) + O(\varepsilon^2) \qquad (3.1)$$

where

$$I(\zeta) = \frac{1}{2\pi}\iint_{f_p^{-1}(N)} \mu(z)Q(z,\zeta)dxdy$$

and

$$J(\zeta) = -\overline{I(1/\bar{\zeta})}. \qquad (3.2)$$

Since (2.1) is a disjoint union, $I(\zeta)$ is expressed as a series of the form

$$I(\zeta) = \sum_{\gamma\in\Gamma/\Gamma_0} I_\gamma(\zeta) \qquad (3.3)$$

where

$$I_\gamma(\zeta) = \frac{1}{2\pi}\iint_{\gamma\Delta_0} \mu(z)Q(z,\zeta)dxdy.$$

4. Evaluation of $I_\gamma(\zeta)$.

By (2.6) and (2.7) we have

$$\begin{aligned} I_\gamma(\zeta) &= \frac{1}{2\pi}\iint_{\Delta_0} \gamma^*\mu(z)\gamma^*Q(z,\zeta)dxdy \\ &= \frac{1}{2\pi}\iint_{\Delta_0} \varepsilon^{-1}\varphi^*\mu_{\psi_\varepsilon}(z)\gamma^*Q(z,\zeta)dxdy \\ &= \frac{1}{2\pi}\iint_L \varepsilon^{-1}\mu_{\psi_\varepsilon}(z)(\gamma\circ\varphi^{-1})^*Q(z,\zeta)dxdy \\ &= -\frac{1}{2\pi}\iint_{x\leq 0} e^{\bar{z}-\rho}(\gamma\circ\varphi^{-1})^*Q(z,\zeta)dxdy + O(\varepsilon), \end{aligned}$$

where $\gamma^*Q(z,\zeta) = Q(\gamma(z),\zeta)(\frac{d\gamma}{dz})^2$ is the pull-back of Q considered as a quadratic differential of z. Therefore,

$$I_\gamma(\zeta) = I + O(\varepsilon) \tag{4.1}$$

where

$$I = -\frac{1}{2\pi}\iint_{x\leq 0} e^{\bar{z}-\rho}(\gamma\circ\varphi^{-1})^{*}Q(z,\zeta)dxdy.$$

Our task is to evaluate the double integral I by using the calculus of residues. For convenience we introduce the functions u and w with the following properties,

(1) $\varphi = u\circ w$,

(2) $w:\Delta\to L$ is a Möbius transformation onto L such that $w\circ\beta = w+2\pi i$, and

(3) $u:w(\Delta_0)\to L$ is a conformal surjection such that $u(z+2\pi i) = u(z)+2\pi i$.

Obviously, such u and w exist but not uniquely. We fix once and for all a choice of w.

LEMMA 4.1. For fixed $\zeta\notin f_p^{-1}(N)$,

$$(\gamma\circ\varphi^{-1})^{*}Q(z,\zeta) = O(z^{-4}) \quad \text{as } z\to\infty,\ z\in L.$$

PROOF. Setting $\gamma_1 = \gamma\circ w^{-1}$ and $u_1 = u^{-1}$, we have

$$\gamma\circ\varphi^{-1} = \gamma_1\circ u_1$$

where $\gamma_1:L\to\Delta$ is a Möbius transformation and $u_1:L\to L$ is holomorphic. Clearly,

$$\gamma_1'(z) = O(z^{-2}) \quad (z\to\infty). \tag{4.2}$$

On the other hand, by expanding the function $u_1(z)-z$, which is periodic with period $2\pi i$, in a Fourier series, it is not hard to see that

$$u_1(z) = z+O(1) \text{ and } u_1'(z) = O(1) \quad (z\to\infty), \tag{4.3}$$

since $u_1(z)$ is analytic on ∂L and u_1 maps L into itself. (4.2) and (4.3) show that $(\gamma\circ\varphi^{-1})'(z) = O(z^{-2})$ $(z\to\infty)$. This immediately gives the Lemma. q.e.d.

Cauchy's integral theorem and Lemma 4.1. imply that the integral

$$\int_{-\infty}^{\infty} e^{-z-\rho}(\gamma\circ\varphi^{-1})^*Q(z,\zeta)dy$$

is independent of x=Re z. Thus

$$I = -\frac{1}{2\pi}\int_{-\infty}^{0} e^{2x}dx\int_{-\infty}^{\infty} e^{-z-\rho}(\gamma\circ\varphi^{-1})^*Qdy$$

$$= -\frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-iy-\rho}(\gamma\circ\varphi^{-1})^*Q(iy,\zeta)dy$$

$$= -\frac{1}{4\pi i}\int_{\partial L} e^{-z-\rho}(\gamma\circ\varphi^{-1})^*Q(z,\zeta)dz$$

$$= -\frac{1}{4\pi i}\int_{\ell} \frac{(\gamma\circ w^{-1})^*Q(z,\zeta)}{u'(z)e^{u(z)+\rho}}\,dz$$

where ℓ is a vertical line contained in $w(\Delta_0)$. Since the function $f_p\circ w^{-1}(z)$ is periodic with period $2\pi i$ on L, it is of the form $f_p\circ w^{-1}(z) = F(e^z)$ where $F(z)$ is regular in Δ with $F(0)=p$. Differentiating both sides of the identity $F(e^z) = p + e^{u(z)+\rho}$, we have

$$u'(z)e^{u(z)+\rho} = F'(e^z)e^z.$$

Hence

$$I = -\frac{1}{4\pi i}\int_{\ell} \frac{(\gamma\circ w^{-1})^*Q}{F'(e^z)e^z}\,dz.$$

By noting the estimates

$$(\gamma \circ w^{-1})^* Q = O(z^{-4}), \quad (z \to \infty)$$

and

$$\frac{1}{Ce^z} - \frac{1}{F'(e^z)e^z} = O(1), \quad (z \to \infty)$$

with $C = F'(0)$, a standard application of Cauchy's integral theorem yields

$$I = -\frac{1}{4\pi iC}\int_{\ell} e^{-z}(\gamma \circ w^{-1})^* Q(z,\zeta)dz.$$

Although this integral can be evaluated by computing the residues in the right half-plane determined by ℓ, it is easier to evaluate the integral by changing the variable z to $w \circ \gamma^{-1}(z)$. Thus

$$I = -\frac{1}{4\pi iC}\int_{\gamma(h)} \frac{e^{-w \circ \gamma^{-1}(z)}}{(w \circ \gamma^{-1})'(z)} Q(z,\zeta)dz$$

where h is a circle in Δ which is tangent to $\partial\Delta$ at the fixed point of β. Denoting the residue of the integrand at z by $\mathrm{Res}(z)$, we have

$$I = \frac{1}{2C}[\mathrm{Res}(\zeta) + \mathrm{Res}(0) + \mathrm{Res}(\infty)].$$

Observe that $w \circ \gamma^{-1}(z)$ is of the form

$$w \circ \gamma^{-1}(z) = \frac{t\alpha + \bar{t}z}{\alpha - z}, \quad |\alpha| = 1, \ \mathrm{Re}\ t < 0.$$

After elementary calculations, we obtain

$$I = \frac{1}{C\zeta}\left\{\frac{e^{-iu_\gamma}}{(w \circ \gamma^{-1})'(\zeta)}\left[e^{-w \circ \gamma^{-1}(\zeta) + iu_\gamma} + (w \circ \gamma^{-1}(\zeta) - iu_\gamma)\frac{\sinh t_\gamma}{t_\gamma}\right.\right.$$

$$-\cosh t_\gamma\Big] - \zeta e^{-iu_\gamma}\sinh t_\gamma\Big\} \tag{4.4}$$

with $w\circ\gamma^{-1}(0) = t_\gamma + iu_\gamma$ (t_γ, $u_\gamma \in \mathbf{R}$). Since

$$w\circ\gamma^{-1}(1/\bar{\zeta}) = -\overline{w\circ\gamma^{-1}(\zeta)},\ \zeta\in\Delta,$$

identities (3.1)-(3.3), (4.1) and (4.4) give us the following final form of the variation of f_p.

THEOREM 4.2. For sufficiently small $\varepsilon\in\mathbf{C}$, the universal covering $f_{p+\varepsilon}$ of $D_{p+\varepsilon}$ is given by

$$f_{p+\varepsilon}(z) = f_p(z) + f_p'(z)\left[\frac{\varepsilon}{C}I_1 - \frac{\bar{\varepsilon}}{\bar{C}}I_2\right] + O(\varepsilon^2),\ z\in\Delta$$

where

$$I_1 = \sum_{\gamma\in\Gamma/\Gamma_0}\left\{\frac{e^{-iu_\gamma}}{(w\circ\gamma^{-1})'(z)}\left[e^{-w\circ\gamma^{-1}(z)+iu_\gamma} + (w\circ\gamma^{-1}(z)-iu_\gamma)\frac{\sinh t_\gamma}{t_\gamma}\right.\right.$$

$$\left.\left. - \cosh t_\gamma\right] - ze^{-iu_\gamma}\sinh t_\gamma\right\},$$

and

$$I_2 = \sum_{\gamma\in\Gamma/\Gamma_0}\left\{\frac{e^{iu_\gamma}}{(w\circ\gamma^{-1})'(z)}\left[e^{w\circ\gamma^{-1}(z)-iu_\gamma} - (w\circ\gamma^{-1}(z)-iu_\gamma)\frac{\sinh t_\gamma}{t_\gamma}\right.\right.$$

$$\left.\left. - \cosh t_\gamma\right] - ze^{iu_\gamma}\sinh t_\gamma\right\}$$

with $w\circ\gamma^{-1}(0) = t_\gamma + iu_\gamma$ (t_γ, $u_\gamma\in\mathbf{R}$). The constant C denotes the derivative $F'(0)$ of the function F satisfying the identity $f_p\circ w^{-1}(z) = F(e^z)$. The estimate is uniform as long as z stays in compact subsets of Δ.

Let $\lambda_p(z)|dz|$ be the Poincaré metric of the domain D_p. By definition $\lambda_p(z)$ satisfies

$$\lambda_p(f_p(z))|f_p'(z)| = \frac{1}{1-|z|^2}, \quad z\in\Delta.$$

In particular, we have $\lambda_p(0) = 1/f_p'(0)$. Theorem 4.2. easily gives the following

COROLLARY. For sufficiently small $\varepsilon\in C$, $\lambda_{p+\varepsilon}(0)$ is given by

$$\ln \lambda_{p+\varepsilon}(0) = \ln \lambda_p(0) + 2\mathrm{Re}\left\{\frac{\varepsilon}{C}\sum_{\gamma\in\Gamma/\Gamma_0} e^{-iu_\gamma}\left(\cosh t_\gamma - \frac{\sinh t_\gamma}{t_\gamma}\right)\right\}$$

$$+ O(\varepsilon^2).$$

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