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Uniformization, automorphic forms and accessory parameters*

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Let X be a compact Riemann surface of genus $p \geq 2$. Then X can be realized in many ways; for example:

(I) X is the Riemann surface of an algebraic function field $K(X) \cong \mathbb{C}(z)[w]$, where z and w satisfy an irreducible polynomial equation P over $\mathbb{C}$.

(II) X can be represented (in an essentially unique way) by a Fuchsian group Γ acting on the upper half-plane U. The group Γ can be chosen to be isomorphic to the fundamental group of X and hence generated by $A_j, B_j \in PSL(2,\mathbb{R})$, $j = 1,2,\cdots,p$, with

$$A_1 \circ B_1 \circ A_1^{-1} \circ B_1^{-1} \circ \cdots \circ A_p^{-1} \circ B_p^{-1} = I.$$

(III) X can be represented (in many ways) by a Schottky group G; that is, by a free group on p generators $C_j \in PSL(2,\mathbb{C})$, $j = 1,2,\cdots,p$.

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It is a difficult and important problem in function theory to understand the relations between the various representations of X; for example, between P, Γ and G.

As an illustration of the type of solution one would like, consider the classical uniformization of a surface X of genus 1 by an elliptic group Γ generated by the two translations

$$z \mapsto z+1, \quad z \mapsto z+\tau \qquad (\mathrm{Im}\ \tau > 0).$$

Then the algebraic equation for $X = \mathbb{C}/\Gamma$ is

$$w^2 = (z-e_1)(z-e_2)(z-e_3),$$

where

$$e_1 = \wp(\tfrac{1}{2}), \quad e_2 = \wp(\tfrac{\tau}{2}), \quad e_3 = \wp(\tfrac{1+\tau}{2}),$$

and

$$\wp(z) = \frac{1}{z^2} + \sum_{\gamma\varepsilon\Gamma-\{I\}} \left(\frac{1}{\gamma(z)^2} - \frac{1}{\gamma(0)^2}\right), \quad z \ \varepsilon\ \mathbb{C}.$$

The torus X can also be represented by a cyclic loxodromic group with generator

$$z \mapsto \lambda z, \quad 0 < |\lambda| < 1.$$

Again, the relation between τ and λ is well known. We have

$$\lambda = e^{2\pi i\tau}.$$

§1. Riemann surfaces represented by Kleinian groups (the fundamental problem).

Let Γ be a finitely generated non-elementary Kleinian group with region of discontinuity Ω and limit set Λ. The most important result and the starting point of all investigations concerning such groups is

AHLFORS' FINITENESS THEOREM ([1]). *The orbit space Ω/Γ is of finite analytic type; that is, Ω/Γ has finitely many components, each is a compact surface punctured at finitely many points, and the canonical projection $\Omega \to \Omega/\Gamma$ is ramified over finitely many points.*

It is quite easy to conclude that each component of Ω (thus also of Ω/Γ) carries a unique complete Riemannian metric $\lambda(z)|dz|$ of constant negative curvature -1. As a matter of fact, there is a quantitative improvement of Ahlfors' result:

BERS' AREA INEQUALITY ([4]). *If Γ is generated by N elements, then*

$$\text{Area } (\Omega/\Gamma) \leq 4\pi(N-1).$$

PROBLEM. *Assume that we know all the topological properties of the cover $\Omega \to \Omega/\Gamma$, and the algebraic properties of the group Γ. Determine the equations of the algebraic curves represented by Ω/Γ.*

We are interested in recovering the complex structure of Ω/Γ from Γ. Thus, in particular, if a component of Ω/Γ is

a punctured sphere (here $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$):

$$X = \hat{\mathbb{C}} - \{\lambda_1, \cdots, \lambda_n\},$$

then ($n \geq 3$ and) we would like, as part of the solution of our problem, a formula for determining either the covering map of X or for evaluating the punctures λ_j, $1 \leq j \leq n$.

§2. Spaces of cusp forms and Poincaré series (the basic tools).

We fix an integer $q \geq 2$, and consider the space of cusp forms for Γ of weight $-2q$ (the space of q-forms, for short):

$$\mathbb{A}_q(\Gamma) = \{\phi \text{ holomorphic on } \Omega;\ (\phi\circ\gamma)(\gamma')^q = \phi \text{ for all } \gamma \in \Gamma,$$

and

$$\iint_{\Omega/\Gamma} \lambda^{2-q}(z)|\phi(z)dz\, d\bar{z}| < \infty\}.$$

If Δ is a Γ-invariant union of components of Ω, then we define

$$\mathbb{A}_q(\Gamma,\Delta) = \{\phi \in \mathbb{A}_q(\Gamma);\ \phi \equiv 0 \text{ off } \Delta\}.$$

In particular,

$$\mathbb{A}_q(\Gamma,\Omega) = \mathbb{A}_q(\Gamma).$$

It is an immediate consequence of Ahlfors' finiteness theorem and the Riemann-Roch theorem (see, for example, [11, pp. 261-269]) that $\mathbb{A}_q(\Gamma)$ is finite dimensional. The dimension of this space depends only on the topological nature of the covering $\Omega \to \Omega/\Gamma$.

Let ϕ and ψ be two non-trivial elements of $\mathbb{A}_q(\Gamma)$. Assume that Δ is a Γ-invariant union of components and that neither ϕ nor

ψ vanishes identically on any component of Δ. Then ϕ/ψ is a meromorphic automorphic function on Δ (its projection to Δ/Γ extends to all the punctures) that is non-trivial on any component of Δ.

The best known method for constructing cusp forms is via Poincaré series or relative Poincaré series. Let f be a holomorphic function on Ω. Let Γ_0 be a subgroup of Γ and assume that f is Γ_0-invariant in the sense that $(f\circ\gamma)(\gamma')^q = f$ for all $\gamma \in \Gamma_0$. Then we can define the <u>relative Poincaré series of</u> f by the formula

$$\Theta f = \Theta_{\Gamma_0\backslash\Gamma} f = \sum_{\gamma\in\Gamma_0\backslash\Gamma} (f\circ\gamma)(\gamma')^q;$$

the summands always make sense (that is, $(f\circ\gamma)(\gamma')^q$ is independent of the choice of representative γ for the coset $\Gamma_0\gamma$ of Γ); however, the sum might not converge. There are no convergence problems for Γ_0 the trivial group (in which case $\Theta_{\Gamma_0\backslash\Gamma} f$ is written $\Theta_\Gamma f$ and called the <u>Poincaré series of</u> f) and f in $L^1(\Omega)$ with respect to $\lambda(z)^{2-q}|dz\, d\bar{z}|$.

More generally: if $\Gamma_0 \subset \Gamma$, then $\Omega_0 \supset \Omega$ and $\lambda_0 \leq \lambda$. Let ω be a fundamental domain for Γ. We may choose $\omega \subset \omega_0$ (the subscripted quantities refer, of course, to the group Γ_0), and as a matter of fact we take

$$\omega_0 = \bigcup_{\gamma\in\Gamma_0\backslash\Gamma} \gamma(\omega)$$

(for some choice of coset representatives). Then for $f \in \mathcal{A}_q(\Gamma_0)$,

$$\iint_\omega \lambda^{2-q}(z)|(\Theta f)(z)||dz\, d\bar{z}| \le \sum_{\gamma\in\Gamma_0\backslash\Gamma} \iint_\omega \lambda(z)^{2-q}|f(\gamma z)\gamma'(z)^q dz\, d\bar{z}|$$

$$= \sum_{\gamma\in\Gamma_0\backslash\Gamma} \iint_{\gamma(\omega)} \lambda(z)^{2-q}|f(z)dz\, d\bar{z}| = \iint_{\omega_0} \lambda(z)^{2-q}|f(z)dz\, d\bar{z}|$$

$$\le \iint_{\omega_0} \lambda_0(z)^{2-q}|f(z)dz\, d\bar{z}|.$$

We have shown that

$$\Theta_{\Gamma_0\backslash\Gamma} : \mathbb{A}_q(\Gamma_0) \to \mathbb{A}_q(\Gamma)$$

is a bounded linear operator of norm less than or equal to one.

It is not hard to see that each element $\phi \in \mathbb{A}_q(\Gamma)$ is also bounded in the sense that $\lambda^{-q}\phi \in L^\infty(\Omega)$. It follows that $\mathbb{A}_q(\Gamma)$ becomes a Hilbert space with the <u>Petersson inner product</u>

$$\langle\phi,\psi\rangle = i\iint_{\Omega/\Gamma} \lambda(z)^{2-2q}\phi(z)\overline{\psi(z)}\, dz\, d\bar{z},\ \phi,\psi \in \mathbb{A}_q(\Gamma).$$

An element of $\mathbb{A}_q(\Gamma)$ is holomorphic on Ω and satisfies the cusp conditions (vanishes at parabolic fixed points corresponding to punctures). The Petersson inner product is defined for the wider class of functions ϕ,ψ that are Γ-invariant, have finitely many inequivalent simple poles and are holomorphic at the cusps—as long as $\phi\psi$ has only simple poles and satisfies the cusp condition.

§3. <u>Eichler cohomology</u>.

The dimension of the Banach space $\mathbb{A}_q(\Gamma)$ depends only on topological data. We discuss next a vector space associated to Γ that depends only on algebraic data. When these two spaces

have the same dimension, we will be able to obtain interesting consequences. Let

$$\Pi_{2q-2} = \{p \in \mathbb{C}[z];\ \deg p \leq 2q-2\}.$$

The group Γ acts on the right (Eichler action) on Π_{2q-2} by

$$(p\cdot\gamma)(z) = p(\gamma z)\gamma'(z)^{1-q},\ p \in \Pi_{2q-2},\ \gamma \in \Gamma,\ z \in \mathbb{C}.$$

A mapping $\chi : \Gamma \to \Pi_{2q-2}$ is a cocyle if

$$\chi(\gamma_1 \circ \gamma_2) = \chi(\gamma_1)\cdot\gamma_2 + \chi(\gamma_2),\ \text{all } \gamma_1, \gamma_2 \in \Gamma;$$

it is a coboundary if there exists a $p \in \Pi_{2q-2}$ such that

$$\chi(\gamma) = p\cdot\gamma - p, \quad \text{all } \gamma \in \Gamma.$$

The cohomology space $H^1(\Gamma,\Pi_{2q-2})$ is defined as the vector space of cocycles modulo coboundaries. A cohomology class is parabolic if it can be represented by a cocycle that restricts to a coboundary on each parabolic cyclic subgroup of Γ. The subspace of parabolic cohomology classes is denoted by $PH^1(\Gamma,\Pi_{2q-2})$.

It is quite clear that $PH^1(\Gamma,\Pi_{2q-2})$ is a finite dimensional vector space and that it depends only on the presentation of the group Γ (we always assume that the presentation of Γ specifies which elements of Γ are parabolic and which of these parabolic elements correspond to punctures on Ω/Γ).

Ahlfors [1] and Bers [4] used with great success the injective conjugate linear map

$$(3.1) \qquad \beta^* : \mathcal{A}_q(\Gamma) \to PH^1(\Gamma,\Pi_{2q-2})$$

that we proceed to define. Let $a_1,\cdots,a_{2q-1}$ be 2q-1 distinct

points in $\hat{\mathbb{C}}$. Let $b \in \hat{\mathbb{C}} - \{a_1, \cdots, a_{2q-1}\}$. Define

$$(3.2) \qquad f(b,z) = \frac{-1}{2\pi} \frac{1}{z-b} \prod_{j=1}^{2q-1} \frac{b-a_j}{z-a_j}, \quad z \in \mathbb{C}^{(1)},$$

and

$$(3.3) \qquad \phi(b,z) = \sum_{\gamma \in \Gamma} f(b,\gamma z)\gamma'(z)^q, \quad z \in \Omega.$$

If a_j and b are points of Λ, then $\phi(b,\cdot) \in \mathbb{A}_q(\Gamma)$. Otherwise, the series converges but may have simple poles at appropriate points in Ω.

Now let $\psi \in \mathbb{A}_q(\Gamma)$. Define

$$F(z) = \langle \phi(z,\cdot), \psi \rangle, \quad z \in \mathbb{C}.$$

Then F is a continuous function on $\mathbb{C}$; and

$$\chi(\gamma)(z) = F(\gamma z)\gamma'(z)^{1-q} - F(z), \quad z \in \mathbb{C}, \ \gamma \in \Gamma,$$

defines a cocycle whose cohomology class is precisely $\beta^*(\psi)$.

If $\dim \mathbb{A}_q(\Gamma) = \dim PH^1(\Gamma, \Pi_{2q-2})$, then β^* is surjective and hence it establishes an isomorphism between a space defined analytically and another space defined purely algebraically. It should be noted that surjectivity of β^* is a consequence of topological and algebraic data—and not the complex structure of the surfaces Ω/Γ. It is known that the map β^* is an isomorphism for a geometrically finite function group (all $q \geq 2$), [20], [22], and for Γ an arbitrary torsion free geometrically finite group for $q=2$, [24], [18]. For all "interesting groups" (except degenerate groups), it appears that β^* is an isomorphism.

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(1) See [13, §0.2] regarding conventions for parameters such as a_j or b becoming ∞.

§4. Poincaré series of rational functions.

Let Λ_q denote the union of the limit set Λ of Γ with the set of fixed points $z_0 \in \Omega$ whose stabilizers in Γ have order ν with $q-1 \not\equiv 0 (\mathrm{mod}\ \nu)$. Let S be a finite subset of Λ_q. We define $\mathcal{R}_q(S)$ to be the space of rational functions f that satisfy

(1) f is holomorphic on $\hat{\mathbb{C}} - S$,

(2) all the poles of f are simple, and

(3) $f(z) = O(|z|^{-2q})$, $z \to \infty$, if $\infty \notin S$,
$f(z) = O(|z|^{-(2q-1)})$, $z \to \infty$, if $\infty \in S$.

THEOREM ([13]). *Let* $a_1, \cdots, a_{2q-1}$ *be* $(2q-1)$ *distinct points in* Λ_q. *Let* $\gamma_0 = I, \gamma_1, \cdots, \gamma_N$ *be generators for* Γ. *If*

(4.1) $$S = \{\gamma_j(a_k);\ k=1,\cdots,2q-1,\ j=0,\cdots,N\},$$

then

$$\Theta(\mathcal{R}_q(S)) = \mathcal{A}_q(\Gamma).$$

The above theorem yields a finite set of functions whose Poincaré series span the space of cusp forms for Γ. Our result is a quantitative version of earlier theorems of Bers [5], [6]. We would like to select a finite set of functions whose Poincaré series form a basis for $\mathcal{A}_q(\Gamma)$. We proceed to describe how to accomplish this for an interesting class of groups.

Assume that the map β^* *of* (3.1) *is surjective*. Let $d = \dim \mathcal{A}_q(\Gamma) = \dim PH^1(\Gamma, \Pi_{2q-2})$. Let $\chi_1, \cdots, \chi_{d+2q-1}$ be a basis for the space of parabolic cocycles for the group Γ. The

computation of such a basis involves a lot of work; but is determined completely by the presentation of the Kleinian group Γ. See [13] for computations of bases for spaces of cocycles for Schottky, quasi-Fuchsian, and $\mathbb{Z}_2$-extensions of quasi-Fuchsian groups.

Now let $S \subset \Lambda_q$ be a finite set consisting only of

(1) fixed points of loxodromic elements, and/or

(2) fixed points of parabolic elements that represent at least one puncture on Ω/Γ, and/or

(3) points a in $\hat{\mathbb{C}}$ $(= \Omega \cup \Lambda)$ with the property that the maximal elliptic subgroup of Γ stabilizing a has order ν with $q-1 \not\equiv 0 (\mathrm{mod}\ \nu)$.

Points satisfying (1), (2) or (3) above will be called q-<u>uniqueness</u> points. (Note that this is a slight departure from the terminology of [13], where q-uniqueness points were defined as those satisfying (1) or (3).)

Let $n_o = |S|$. If $n_o \leq 2q-1$, then $\mathcal{R}_q(S)$ is empty. So assume that $n_o = 2q-1+n$ with $n > 0$. Select $2q-1$ distinct points $a_1, \cdots, a_{2q-1} \in S$. Label the remaining points $a_{2q}, \cdots, a_{2q-1+n}$.

The functions $f(a_{2q-1+j}, \cdot)$, $1 \leq j \leq n$, defined by (3.2), form a basis for $\mathcal{R}_q(S)$; and the functions $\phi(a_{2q-1+j}, \cdot)$, $1 \leq j \leq n$, defined by (3.3), span $\oplus(\mathcal{R}_q(S))$.

Each of the points in S is the fixed point of an

appropriate non-trivial element of Γ. We assume that a_j is fixed by $A_j \in \Gamma - \{I\}$, $1 \le j \le 2q-1+n$. Next for each cocycle χ_i, $1 \le i \le d+2q-1$, we construct a function F_i on S as follows:

$$F_i(a_j) = \chi_i(A_j)(a_j)[A_j'(a_j)^{1-q} - 1]^{-1},$$

if A_j is not parabolic; and if A_j is parabolic,

$$F_i(a_j) = \frac{\chi_i(A_j)'(a_j)}{(1-q)A_j''(a_j)} .$$

THEOREM ([13]). <u>If</u> β^* <u>is surjective</u>, <u>then the dimension of</u> $\Theta(\mathcal{R}_q(S))$ <u>is the rank of the</u> $(2q-1+d) \times (2q-1+n)$ <u>matrix</u>

$$(F_i(a_j))_{\substack{1 \le i \le 2q-1+d \\ 1 \le j \le 2q-1+n}}$$

<u>minus</u> $(2q-1)$.

<u>Proof</u>. Our theorem is a restatement, in slightly different language, of Theorem 4.5 of [13].

REMARK. Let $a_1,\cdots,a_{2q-1}$ be q-uniqueness points and let $\gamma_0 = I, \gamma_1,\cdots,\gamma_N$ generate Γ. The first theorem of this section guarantees that $\Theta(\mathcal{R}_q(S)) = \mathcal{A}_q(\Gamma)$ for S defined by (4.1). Surjectivity of β^* is not needed for this part. The second theorem of this section shows how to select (using a finite algorithm) a subset S_0 of S so that Θ establishes an isomorphism between $\mathcal{R}_q(S_0)$ and $\mathcal{A}_q(\Gamma)$. The subset $S_0 \subset S$ can be chosen to contain 2q-1 arbitrary points of S; for example, $a_1,\cdots,a_{2q-1}$. If Δ is a Γ-invariant union of components of Ω,

one would like to select S_0 so that Θ maps $R_q(S_0)$ isomorphically onto $\mathbb{A}_q(\Gamma,\Delta)$. This is a more difficult problem and very little is known about it. See [13, §0.5 and §9].

§5. The dual space of $H^1(\Gamma,\Pi_{2q-2})$.

Let $A \in \Gamma$ be loxodromic with fixed points α (attractive) and β (repulsive). We define a linear functional

$$\ell_A : H^1(\Gamma,\Pi_{2q-2}) \to \mathbb{C}$$

as follows. Let χ be a cocycle representing a cohomology class in $H^1(\Gamma,\Pi_{2q-2})$. We expand $\chi(A) \in \Pi_{2q-2}$ as

$$\chi(A)(z) = \sum_{j=0}^{2q-2} a_j(\alpha-\beta)^{-j}(z-\alpha)^j(z-\beta)^{2q-2-j}, \quad z \in \mathbb{C}, \tag{5.1}$$

and we define

$$\ell_A(\chi) = a_{q-1}. \tag{5.2}$$

It is easy to verify that ℓ_A depends only on the cohomology class of χ.

THEOREM ([21]). <u>The linear functionals</u>

$$\{\ell_A;\ A \in \Gamma \text{ and } A \text{ is loxodromic}\}$$

<u>span the dual space of</u> $H^1(\Gamma,\Pi_{2q-2})$.

For elliptic $A \in \Gamma$, (5.1) and (5.2) define the zero linear functional on $H^1(\Gamma,\Pi_{2q-2})$. For parabolic $A \in \Gamma$ with fixed point $a \in \mathbb{C}$, the linear functional ℓ_A is defined by

$$\ell_A(\chi) = \chi(A)(a), \quad \chi \in H^1(\Gamma,\Pi_{2q-2}). \tag{5.3}$$

We have shown in [12] that $\ell_A(\chi) = 0$ for all parabolic $A \in \Gamma$ if and only if $\chi \in PH^1(\Gamma, \Pi_{2q-2})$.

Let us consider the case q=2. Every non-trivial element $A \in \Gamma$ has a simple <u>eigenvector</u> e_A corresponding to the eigenvalue 1 for the action of A on Π_2. If A fixes α and β with $\alpha \neq \beta$, then

$$e_A(z) = (z-\alpha)(z-\beta), \; z \in \mathbb{C};$$

while for parabolic A with fixed point α,

$$e_A(z) = (z-\alpha)^2.$$

Three elements of Γ with distinct fixed points are said to be <u>independent</u> provided their eigenvectors are linearly independent. If A_j, j=1,2,3, are three parabolic elements of Γ with distinct fixed points, then they are independent. If A_j, j=1,2,3, are three dependent elements with distinct fixed points, then at least one of these must not be parabolic; say A_1 is <u>not</u> parabolic. It is easy to show that in this case A_1, A_2, $A_1 \circ A_3$ are independent.

Let $A_1, A_2, \cdots, A_N$ be generators for Γ. Assume without loss of generality that these N elements have distinct fixed points. If not all the A_j are parabolic, assume that A_1 is loxodromic or elliptic. If A_1, A_2, A_j, $3 \leq j \leq N$, are dependent, replace A_j by $A_1 \circ A_j$. Thus we may assume that A_1, A_2, A_j, $3 \leq j \leq N$, are independent. Let

$$S = \{A_1, \cdots, A_N, A_1 \circ A_2, \cdots, A_1 \circ A_N, A_2 \circ A_3, \cdots, A_2 \circ A_N\}. \tag{5.4}$$

THEOREM. <u>If</u> $\chi \in H^1(\Gamma,\Pi_2)$ <u>and</u> $\ell_B(\chi) = 0$ <u>for all</u> $B \in S$, <u>then</u> $\chi = 0$.

REMARK. The linear functional defined for parabolic A by (5.3) is, in a certain sense, the limit of the linear functionals defined by (5.1) and (5.2) for loxodromic or elliptic A with β approaching α (for example). This observation (in its precise formulation) is useful in the study of degeneration of Kleinian groups and will be pursued elsewhere.

§6. Cusp forms associated to loxodromic elements.

Let $A \in \Gamma$ be loxodromic with attractive fixed point α and repulsive fixed point β. The function

$$g(z) = \frac{(\alpha-\beta)^q}{(z-\alpha)^q(z-\beta)^q}, \quad z \in \hat{\mathbb{C}},$$

is automorphic with respect to $\Gamma_0 = \langle A\rangle$, the cyclic subgroup generated by A. We can thus form the <u>relative Poincaré series associated with the element</u> A:

$$\phi_A = \sum_{\gamma\in\Gamma_0\backslash\Gamma} (g\circ\gamma)(\gamma')^q.$$

It is easy to show that $\phi_A \in \mathbb{A}_q(\Gamma)$. The basic results concerning these relative Poincaré series are summarized below.

THEOREM ([14]). <u>For all</u> $\psi \in \mathbb{A}_q(\Gamma)$,

$$\beta^*(\psi) = -\frac{1}{2\pi}\langle\phi_A,\psi\rangle.$$

THEOREM ([14]). <u>The cusp forms</u>

$$\{\phi_A;\ A \in \Gamma,\ A \text{ is loxodromic}\}$$

<u>span</u> $\mathcal{A}_q(\Gamma)$.

We consider next the generators for Γ discussed in §5.

THEOREM. (<u>Assume</u> $q=2$.) <u>The cusp forms</u>

$$\{\phi_B;\ B \in S \text{ as defined by (5.4) and } B \text{ is loxodromic}\}$$

<u>span</u> $\mathcal{A}_2(\Gamma)$.

The above result is a consequence of the first theorem of this section and the last theorem of the previous section.

Assume now that β^* is surjective. Let $A_1,\cdots,A_m$ be an arbitrary collection of loxodromic elements of Γ. Let $x_1,\cdots,x_{d+2q-1}$ be a basis for the space of parabolic cocycles for Γ as in §4.

THEOREM ([14]). <u>If</u> β^* <u>is surjective, then the dimension of the linear span of</u>

$$\{\phi_{A_1},\cdots,\phi_{A_m}\}$$

<u>in</u> $\mathcal{A}_q(\Gamma)$ <u>is the same as the rank of the</u> $(d+2q-1) \times m$ <u>matrix</u>

$$(\ell_{A_i}(x_j))_{\substack{1 \le j \le 2q-1+d \\ 1 \le i \le m}}$$

REMARKS. (1) The reader should observe the analogies as well as the differences between our results on Poincaré series (§4) and our results on relative Poincaré series (§6).

(2) Special case of our results have been obtained previously by Hejhal [9], Wolpert [26] and Katok [10].

§7. Holomorphic forms associated to parabolic fixed points.

Let $A \in \Gamma$ be parabolic with fixed point $a \in \mathbb{C}$. There are two interesting subgroups of Γ associated to the point a:

$$P_a = \{\gamma \in \Gamma;\ \gamma \text{ is parabolic or the identity and } \gamma(a) = a\},$$

and

$$\Gamma_a = \{\gamma \in \Gamma;\ \gamma(a) = a\}.$$

The group P_a is isomorphic to either $\mathbb{Z}$ or $\mathbb{Z} \oplus \mathbb{Z}$; while $\Gamma_a/P_a \cong \mathbb{Z}_n$ with $n = 1$ or 2 if $P_a \cong \mathbb{Z}$, and $n = 1,2,3,4$ or 6 if $P_a \cong \mathbb{Z} \oplus \mathbb{Z}$. We say that a is q-admissible if $q \equiv 0 \pmod{n}$.

The function

$$g(z) = \frac{1}{(z-a)^{2q}}, \qquad z \in \hat{\mathbb{C}},$$

is automorphic with respect to the subgroup P_a of Γ. It is not a cusp form if a represents a puncture on $X_a = (\hat{\mathbb{C}}-\{a\})/P_a$. (Its projection to X_a has a pole of order q at the puncture.) The function g is automorphic with respect to Γ_a if and only if a is q-admissible. If we form (for q-admissible a)

$$\psi_a = \sum_{\gamma \in \Gamma_a \backslash \Gamma} (g \circ \gamma)(\gamma')^q,$$

then ψ_a converges for $q \geq 3$ and also for $q=2$ provided that a is cusped (see [12] for the definition). The automorphic form

ψ_a is holomorphic on Ω and at all parabolic fixed points representing punctures on Ω/Γ. If a does not represent a puncture (for example, if $P_a \cong \mathbb{Z} \oplus \mathbb{Z}$), then $\psi_a \in \mathcal{A}_q(\Gamma)$. The automorphic form ψ_a is non-trivial if a is accessible [12, p. 57]. Much remains to be studied about the function ψ_a. See [2], [23] and [12].

REMARK. The automorphic form ψ_a is the limiting case of the automorphic form ϕ_A (of §6). This connection will be explored elsewhere.

§8. Uniformizations of punctured spheres.

Let Γ be a torsion free Fuchsian group operating on the upper half-plane $\mathbb{U}$ such that

$$X = U/\Gamma = \hat{\mathbb{C}} - \{\lambda_1, \cdots, \lambda_{n+3}\}, \quad n \geq 0; \tag{8.1}$$

that is, Γ is finitely generated of the first kind of signature $(0, n+3; \infty, \cdots, \infty)$. Let $\pi : U \to X$ be a holomorphic universal covering map. We would like a formula for computing π and the punctures λ_j, $1 \leq j \leq n+3$.

Assume for the moment that n=0. In this case the group can be constructed. One can take Γ, for example, to be the group generated by $A = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. (Thus ∞, 0 and 2 are a maximal set of inequivalent parabolic fixed points.) The computation of π is also easy. We can work with q=2. We claim that $\pi = \dfrac{\psi_\infty}{\psi_0}$, for example. (Note that $\psi_\infty(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \gamma'(z)^2$, $z \in U$.)

Since neither ψ_∞ nor ψ_0 vanish on U, π is holomorphic and non-zero on U. Since $\psi_\infty(\infty) \neq 0$ and $\psi_0(\infty) = 0$, $\pi(\infty) = \infty$. Similarly $\pi(0) = 0$. It is not known whether $\pi(2)$ can be computed by a finite algorithm. It is clear, however, that $\pi(2) \in \mathbb{C}^* = \mathbb{C} - \{0\}$.

Assume next that n=1. Let E_1, E_2, E_3, E_4 be parabolic elements of Γ that generate the group and satisfy

$$E_1 \circ E_2 \circ E_3 \circ E_4 = I.$$

Let a_j be the fixed point of E_j for j = 1,2,3 and 4. Let $b_1 = E_1(a_2)$, $b_2 = E_2(a_1)$. We work, as before, with q=2. Normalize at a_1, a_2, a_3 and use the rational functions defined by (3.2). The two cusp forms $\phi(b_1,\cdot)$ and $\phi(b_2,\cdot)$ (when restricted to U) are linearly independent over $\mathbb{R}$, but linearly dependent over $\mathbb{C}$. A formula for the covering map is now $\pi = \dfrac{\phi(b_1,\cdot)}{\psi_{a_1}}$. It follows that $\pi(a_1) = 0$. It would be interesting to compute $\pi(a_j)$ for j = 2,3,4.

Similarly, it is not known how to compute the map π (explicitly, as above) for n > 1.

§9. Accessory parameters for punctured spheres.

We proceed to discuss the inverse of the problem treated in §8. Normalize X of (8.1) so that $\lambda_{n+1} = 0$, $\lambda_{n+2} = 1$ and $\lambda_{n+3} = \infty$. Let f be a local inverse of π and let ϕ be the Schwarzian derivative of f. It is easy to see that ϕ is holomorphic (single-valued) on X and, in fact, ϕ extends to be a rational function on $\hat{\mathbb{C}}$ with residues $\frac{1}{2}$ at each of the punctures

(when $\phi(z)dz^2$ is regarded as a quadratic differential on $\hat{\mathbb{C}}$). Hence (we will call ϕ the <u>uniformizing connection</u> for X)

$$\phi(z) = \frac{1}{2}\frac{z^2-z+1}{z^2(z-1)^2} + \sum_{j=1}^{n} \frac{\lambda_j(\lambda_j-1)}{z(z-1)(z-\lambda_j)}\left[\frac{1}{2}\frac{1}{z-\lambda_j} + c_j\right],\ z \in \hat{\mathbb{C}},$$

where $c_1,\cdots,c_n$ are constants known as the <u>accessory parameters</u> of the uniformization. The determination of these constants is very difficult. Knowing the constants implies knowing the covering group (as a monodromy group of the Schwarzian differential equation). Hence it is useful to investigate the behavior of the accessory parameters.

Let us define

$$M^n = \{\lambda = (\lambda_1,\cdots,\lambda_n) \in \mathbb{C}^n;\ \lambda_j \neq 0,1 \text{ and } \lambda_j \neq \lambda_k \text{ for } j \neq k\}.$$

Set ϕ_λ to be the uniformizing connection for $X_\lambda = \mathbb{C} - \{0,1,\lambda_1,\cdots,\lambda_n\}$ and $c(\lambda) = (c_1,\cdots,c_n)$ the vector of accessory parameters for X_λ.

THEOREM ([15]). <u>The mapping</u>

$$M^n \ni \lambda \mapsto c(\lambda) \in \mathbb{C}^n$$

<u>is real</u> (<u>but not complex</u>) <u>analytic</u>.

THEOREM ([15]). <u>Let</u> $\lambda^{(m)} \in M^n$, $m = 1,2,3,\cdots$, <u>and assume that the sequence</u> $\{\lambda^{(m)}\}$ <u>converges to a boundary point</u> $\lambda^{(\infty)}$ <u>of</u> M^n. <u>Then</u> $\lim_{m\to\infty} \phi_{\lambda^{(m)}} = \phi_{\lambda^{(\infty)}}$ <u>uniformly on compact subsets</u>.

REMARKS. (1) The above two theorems are also valid for most uniformizations involving torsion.

(2) The second of the above theorems is not as innocent as it appears. There are examples of covering maps π_m of $X_{\lambda(m)}$ (involding groups with torsion), such that $\pi_m \to 0$ uniformly on compact subsets of U, yet $\phi_{\lambda(m)} \to \phi_{\lambda(\infty)}$ as in the second theorem.

Note that $X_{\lambda(\infty)}$ is $\mathbb{C} - \{0,1\}$ punctured at the distinct points in $\{\lambda_1^{(\infty)},\cdots,\lambda_n^{(\infty)}\}$. (This latter set may, of course, be empty.)

We end with a

CONJECTURE (Thompson [25]). <u>Let $\overline{\mathbb{Q}}$ be the field of algebraic numbers. If $\lambda_j \in \overline{\mathbb{Q}}$, $j = 1,\cdots,n$, then so do the accessory parameters c_j, $1 \le j \le n$.</u>

Flimsy evidence towards the conjecture is provided by the remarkable

THEOREM (Belyi [3]). Let $\lambda_1,\cdots,\lambda_n \in \overline{\mathbb{Q}}$, <u>then there exist points $\mu_1,\cdots,\mu_m \in \overline{\mathbb{Q}}$ such that the accessory parameters for $X^* = \hat{\mathbb{C}} - \{\lambda_1,\cdots,\lambda_n,\mu_1,\cdots,\mu_m\}$ are algebraic.</u>

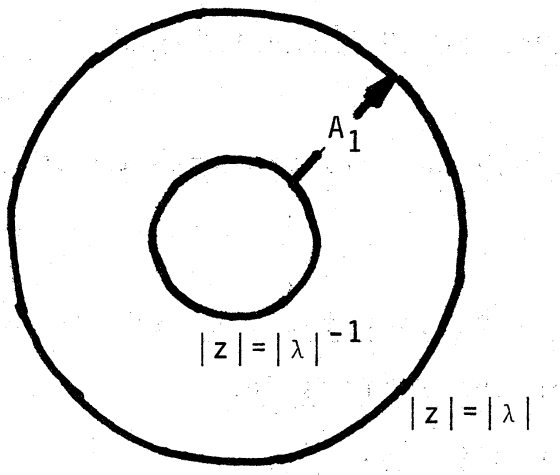
Belyi's theorem yields much more. The covering group for X^* is a subgroup of $PSL(2,\mathbb{Z})$. We will not speculate where this could lead; but, there are many interesting dreams.

§10. <u>Hyperelliptic Schottky groups (an illustrative example)</u>.

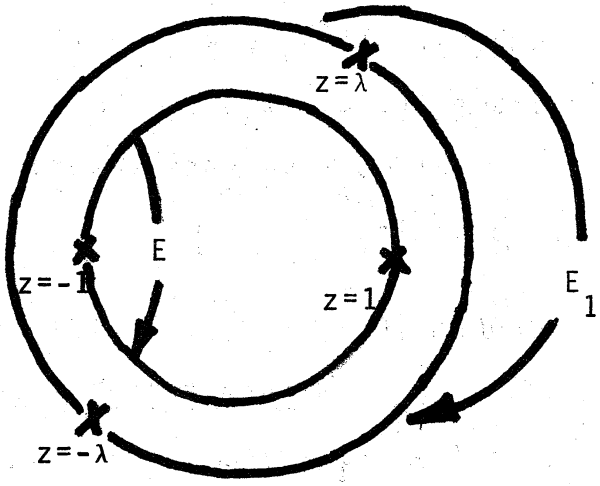
We shall study automorphic forms (for q=2) for a Schottky group Γ that admits a hyperelliptic involution; specifically, we consider a Schottky group Γ on $p \ge 2$ free generators $A_1,\cdots,A_p$ having the following additional property:

(*) There exists an element $E \in PSL(2,\mathbb{C})$ such that $E^2 = I$ and $E \circ A_j = A_j^{-1} \circ E$ for $j = 1,\cdots,p$.

Our first task is to construct an example of such a group. We will let G be the group generated by Γ and E; then $[G:\Gamma] = 2$ and it will turn out that G represents a sphere with 2p+2 ramification points of order 2. We start with the construction for p=1. We may take $E(z) = \frac{1}{z}$, and $A_1(z) = \lambda^2 z$ with $|\lambda|$ large. We let $E_1 = E \circ A_1$. We let $\Gamma_1 = \langle A_1 \rangle$, $G_1 = \langle E, E_1 \rangle = \langle E, A_1 \rangle$.



Fundamental domain ω_1 for Γ_1.



Fundamental domain $\tilde{\omega}_1$ for G_1.

For $A \in PSL(2,\mathbb{C})$ having two fixed points distinct from 0 and ∞, the condition $E \circ A = A^{-1} \circ E$ is equivalent to E interchanging the fixed points of A. If such an A has fixed points $\alpha \neq \pm 1$ (and $\frac{1}{\alpha}$) and multiplier $K \neq \alpha^{\pm 2}$, it is of the form

$$A = K^{1/2}(\alpha - \frac{1}{\alpha})\begin{pmatrix} \alpha - \frac{K}{\alpha} & K - 1 \\ 1-K & K\alpha - \frac{1}{\alpha} \end{pmatrix},$$

and the elliptic element $E \circ A$ of order 2 is of the form

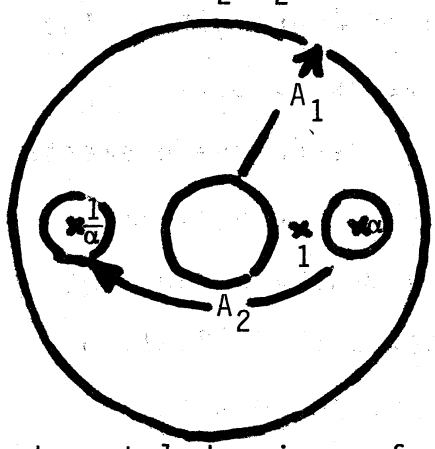
$$iK^{1/2}(\alpha - \frac{1}{\alpha})\begin{pmatrix} 1-K & K\alpha - \frac{1}{\alpha} \\ \alpha - \frac{K}{\alpha} & K-1 \end{pmatrix}.$$

The fixed point of $E \circ A$ are

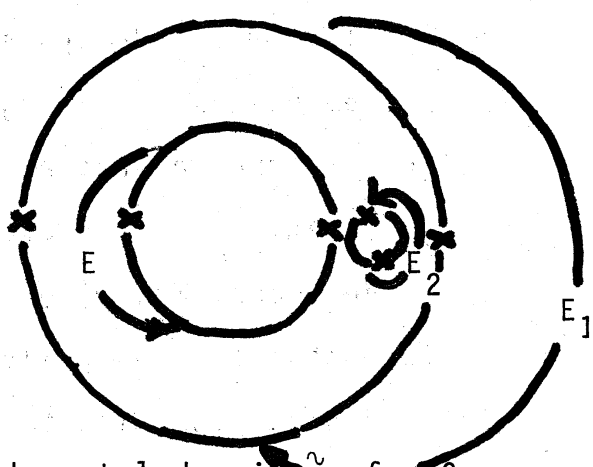
$$\frac{1 - K \pm K^{1/2}(\alpha - \frac{1}{\alpha})}{\alpha - \frac{K}{\alpha}} ;$$

these are close to α for large $|K|$ and close to $\frac{1}{\alpha}$ for small $|K|$.

Choose $\alpha = \alpha_2 \in \tilde{\omega}_1$ (thus α_2 and $\frac{1}{\alpha_2} \in \omega_1$). For $|K|$ sufficiently large the fixed points of $E \circ A$ will lie in $\tilde{\omega}_1$ and hence we can choose a circle through the fixed points of $E \circ A$ to lie entirely in $\tilde{\omega}_1$. It follows by use of the Klein-Maskit combination theorem [19] that the group generated by G_1 and $E \circ A$ is Kleinian, has a connected region of discontinuity and represents a sphere with 6 ramification points of order 2. Similarly for sufficiently large $|K|$, A maps the interior of a small circle about α onto the exterior of a small circle about $\frac{1}{\alpha}$. It follows that the group generated by Γ_1 and A will be a Schottky group of genus 2. We have defined A_2 and $E_2 = E \circ A_2$, and hence Γ_2 and G_2. It is clear that $[G_2 : \Gamma_2] = 2$.



Fundamental domain ω_2 for Γ_2.



Fundamental domain $\tilde{\omega}_2$ for G_2.

It is now clear how to proceed by induction to construct the groups $\Gamma = \Gamma_p$ and $G = G_p$.

To construct cusp forms for G we work with q=2. We consider the p+1 generators $E = E_o, E_1, \cdots, E_p$ of G. If E, E_1, E_j, $2 \le j \le p$, are dependent, we replace E_j by $E \circ E_j = A_j$. Let B be loxodromic. The relative Poincaré series associated with B as an element of Γ will be denoted (as usual) by ϕ_B; while the corresponding series with respect to the group G will be denoted by $\tilde{\phi}_B$. Note that

$$\tilde{\phi}_B(z) = \phi_B(z) + \phi_B(Ez)E'(z)^2, \quad z \in \Omega$$

(both Γ and G have the same region of discontinuity Ω). We have shown that the following 2p-1 relative Poincaré series span $\mathbb{A}_2(G)$:

$$(10.1)\begin{cases} \tilde{\phi}_{A_j}, \ j = 1,2,\cdots,p, \\ \tilde{\phi}_{E_1 \circ E_j}, \text{ if } j = 2,3,\cdots,p \text{ and } E, E_1, E_j \text{ are independent,} \\ \tilde{\phi}_{E_1 \circ E \circ E_j} = \tilde{\phi}_{E_1 \circ A_j}, \text{ if } j = 2,3,\cdots,p \text{ and } E, E_1, E_j \text{ are dependent.} \end{cases}$$

The mapping E induces the hyperelliptic involution on Ω/Γ and $\mathbb{A}_2(G)$ is the 1-eigenspace of the action of E on $\mathbb{A}_2(\Gamma)$. The 2p+2 fixed points $z_1, \cdots, z_{2p+2}$ of the elements $E, E_1, \cdots, E_p$ project to the Weierstrass points on Ω/Γ. Hence the elements of $\mathbb{A}_2(G)$ vanish to even order at each of the points $z_1, \cdots, z_{2p+2}$. Let us denote by $\tilde{\phi}_1, \cdots, \tilde{\phi}_{2p-1}$ the basis for $\mathbb{A}_2(G)$ given by (10.1). We claim that for any collection $z_1, \cdots, z_n$, $1 \le n \le 2p-1$, of lifts of distinct Weierstrass points on Ω/Γ, the matrix

$$(10.2) \qquad (\tilde{\phi}_j(z_k))_{\substack{1 \le j \le 2p-1 \\ 1 \le k \le n}}$$

has rank n. To see this, let us represent the surface Ω/Γ (which we have shown to be hyperelliptic) by the algebraic equation $w^2 = \prod_{j=1}^{2p+2} (z-e_j)$, where $e_1,\cdots,e_{2p+2}$ are distinct complex numbers. Here z is a two sheeted cover, $z : \Omega/\Gamma \to \hat{\mathbb{C}}$, and the e_j's are the values of the function z at the Weierstrass points. A basis for the quadratic differentials on Ω/Γ that are invariant under the hyperelliptic involution is given by

$$\frac{z^j(dz)^2}{w^2}, \quad j = 0,\cdots,2p-2.$$

It follows that (after relabeling) the rank of the matrix (10.2) is the same as the rank of the matrix

$$(e_k^j)_{\substack{0 \le j \le 2p-2 \\ 1 \le k \le n}} .$$

This latter matrix is known to have rank n (see, for example, [7, p. 304]). Let $1 \le i_1 < i_2 < \cdots < i_{2p-2} \le 2p+2$ be any collection of integers. Consider the cusp form

$$\tilde{\phi}_{i_1 i_2 \cdots i_{2p-2}}(z) = \det \begin{bmatrix} \tilde{\phi}_1(z) & \tilde{\phi}_2(z) & \cdots \tilde{\phi}_{2p-1}(z) \\ \tilde{\phi}_1(z_{i_1}) & \tilde{\phi}_2(z_{i_1}) & \cdots \tilde{\phi}_{2p-1}(z_{i_1}) \\ \vdots & & \vdots \\ \tilde{\phi}_1(z_{i_{2p-2}}) & \tilde{\phi}_2(z_{i_{2p-2}}) & \cdots \tilde{\phi}_{2p-1}(z_{i_{2p-2}}) \end{bmatrix}.$$

The rank condition on (10.2) shows that the above cusp form is

not identically zero. It vanishes at each z_{i_j} to order at least two. Hence it vanished precisely at these 2p-2 Weierstrass points to order 2 (and points equivalent to these points under Γ) and <u>at no other points</u>. Thus the function

$$f(z) = \frac{\tilde{\phi}_{12\cdots(2p-2)}}{\tilde{\phi}_{23\cdots(2p-1)}}$$

is a two sheeted cover of $\hat{\mathbb{C}}$ (it vanishes at z_1 and has a pole at z_{2p-1}), and the branch values

$$e_j = f(z_j),\ j = 1,\cdots,2p+2$$

are expressible in terms of relative Poincaré series.

REMARK. B. Maskit has suggested the following simpler constructions of more restrictive classes Schottky groups Γ with symmetries. For $j = 1,\ldots,p$ choose real numbers $a_j > 0$ and $r_j > 0$ such that

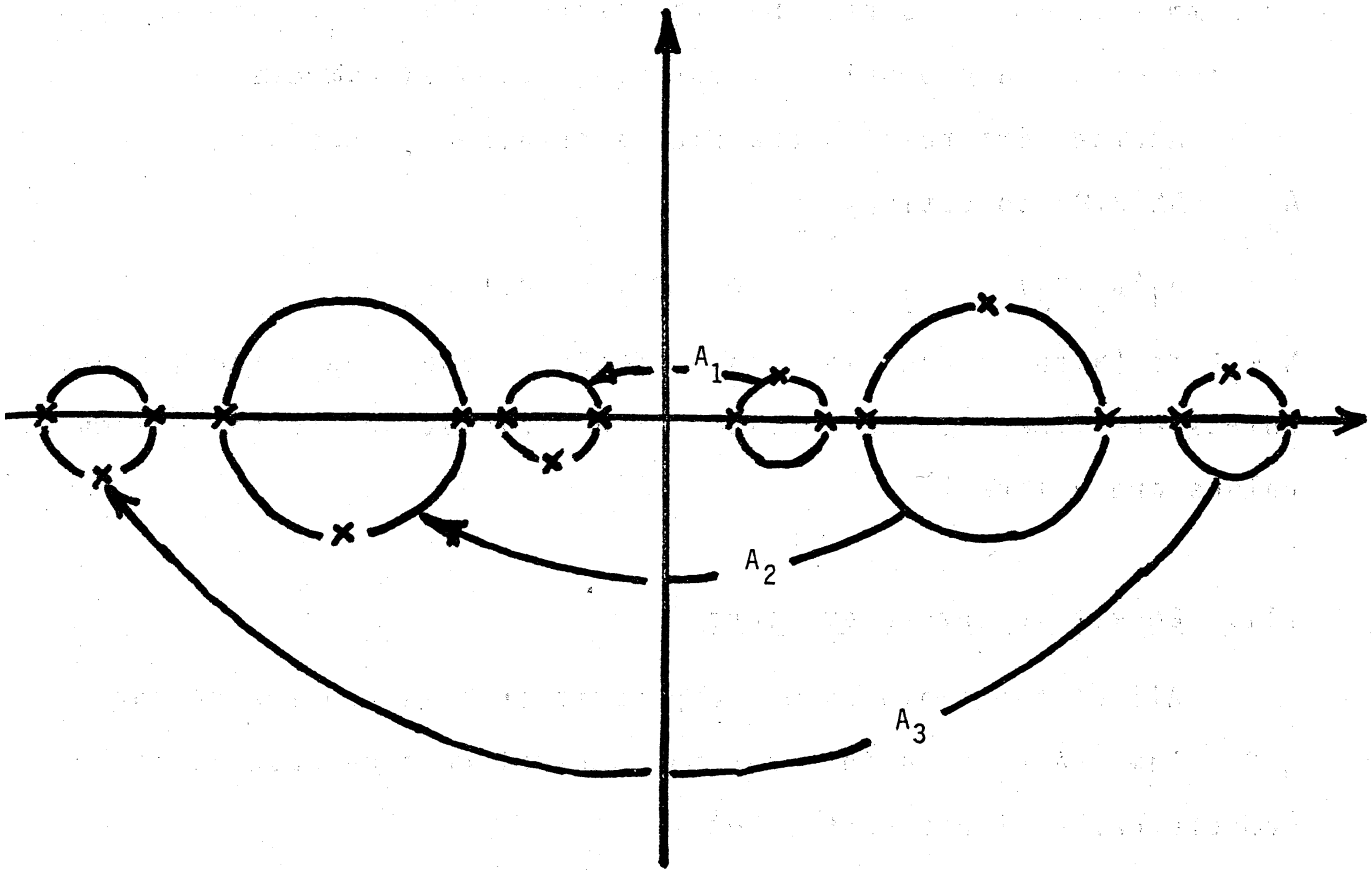
$$r_1 < a_1 \text{ and } a_j + r_j < a_{j+1} - r_{j+1} \text{ for } j = 1,\ldots,p-1.$$

Let c_j be the circle with radius r_j and center a_j. Let $A_j \in PSL(2,\mathbb{C})$ satisfy

$$A_j(a_j \pm r_j) = -a_j \pm r_j,\ A_j(a_j + ir_j) = -a_j - ir_j.$$

A simple calculation shows that $A_j(a_j) = \infty$. It follows that A_j takes the interior of the circle c_j onto the exterior of the circle $-c_j$ (with center $-a_j$ and radius r_j) and that A_j preserves the real axis. We conclude that the group Γ generated by $A_1,\ldots,A_p$ is a Schottky group on p free generators. A fundamental domain ω for Γ consists of the intersection of the exteriors of $c_1,\ldots,c_p,-c_1\ldots,-c_p$,

and the limit set Λ of Γ is a subset of $\mathbb{R}$.



Fundamental domain ω for Γ = exterior of 6 circles.

Fundamental domain $\tilde{\omega}$ for $G = \omega \cap$ (upper half-plane).

Let $E(z) = -z$. Then $E_j = E \circ A_j = A_j^{-1} \circ E$ because the maps on both side of the equality agree at $a_j \pm r_j$ and $a_j + ir_j$. Furthermore, E_j fixes $a_j \pm ir_j$. It follows that G, the group generated by $E_0, \ldots, E_p$, represents a sphere with $2p + 2$ ramification points of order 2. The involution $J(z) = \bar{z}$, $z \in \hat{\mathbb{C}}$, commutes with each

element of Γ and induces an involution on Ω/Γ that permutes the Weierstrass points. The algebraic curve represented by Γ is symmetric in the sense that the branch values e_j, $j = 1,\ldots,2p+2$, consist of $0,\infty$ and p pairs of conjugate complex numbers.

Another interesting example is obtained by choosing $A_j \in PSL(2,\mathbb{R})$ to satisfy

$$A_j(a_j \pm r_j) = -a_j \mp r_j,\ A_j(a_j + ir_j) = -a_j + ir_j.$$

Now Γ is (also) a Fuchsian group of the second kind. The Weierstrass points are fixed by the anti-conformal involution and the branch values are all real.

§11. Geometric interpretations.

All of the examples of cusp forms in this paper were for $q=2$. This case is easier than the general case because it is geometrically significant. Let

$$\Phi : \Delta \to \mathrm{Hom}(\Gamma, PSL(2,\mathbb{C}))$$

be a holomorphic family of homomorphisms of Γ into $PSL(2,\mathbb{C})$ defined on the open unit disk Δ. Assume that $\Phi(0)$ is the identity isomorphism. We can associate to Φ a cohomology class $\chi \in H^1(\Gamma, \Pi_2)$ as follows: For $\gamma \in \Gamma$,

$$\chi(\gamma)(z) = \lim_{t\to 0} \frac{(\gamma^{-1}\circ\Phi(t)(\gamma)(z)) - z}{t}.$$

The cohomology class χ is trivial whenever there exists a holomorphic function A from an ε-neighborhood of 0 to $PSL(2,\mathbb{C})$ such that

$$\chi(t)(\gamma) = A(t)\circ\gamma\circ A(t)^{-1}, \ |t| < \varepsilon.$$

Further, $\chi \in PH^1(\Gamma,\Pi_2)$ whenever $\Phi(t)$ sends parabolic elements to parabolic elements or the identity for all sufficiently small $|t|$. (For details, see [8].)

A given element $A \in \Gamma$ defines, in general, two holomorphic functions in a neighborhood of zero: the multiplier function K (defined only if A is <u>not</u> parabolic) and the trace function T (always defined). The two functions are related by the formula $\pm T = K^{1/2} + K^{-1/2}$. We set

$$K(t) = \text{multiplier of } \Phi(t)(A),$$
$$T(t) = \text{trace of } \Phi(t)(A).$$

To specify choices, if A is loxodromic choose $|K(0)| < 1$, and if A is parabolic choose $T(0) = 2$.

It follows that

$$\ell_A(\chi) = \frac{K'(0)}{K(0)} \quad \text{for loxodromic } A,$$

and

$$\ell_A(\chi) = T'(0) \quad \text{for parabolic } A.$$

(The formulae have trivial generalizations to the elliptic case.)

If Γ is Fuchsian operating on U, the upper half plane, and if $\Phi(t)$ is also Fuchsian for all sufficiently small $|t|$, and if $A \in \Gamma$ is hyperbolic, then $-\log(K(t))$ represents the length of the geodesic on $U/\Gamma(t)$ obtained by projecting the axis of $A(t)$. Thus $-\ell_A(\chi)$ represents the variation of the length of this geodesic. (Above, $\Gamma(t) = \Phi(t)(\Gamma)$.)

The case $q = 2$ is also significant because $\mathbb{A}_2(\Gamma)$ can be naturally identified with the cotangent space to the Teichmüller or deformation space $T(\Gamma)$ at the origin (see [16] and [17]) and fixed points of elements of Γ yield global coordinates for $T(\Gamma)$. Further, the traces of elements of Γ yield local coordinates on $T(\Gamma)$.

§12. Concluding remarks.

The determination of the algebraic curves represented by a Kleinian group requires a transcendental intermediate step. Poincaré and relative Poincaré series provide this step. Much is known about this tool. However, to solve the general problem mentioned in the introduction and in §1 seems at this time to be a distant dream. It is slightly surprising that in the hyperelliptic case, our tools are sufficient to recover the curve from the group.

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