ON AUGMENTED SCHOTTKY SPACES AND INTERCHANGE OPERATORS

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§0. Introduction.

Problem 1. Let $S$ be a compact Riemann surface with nodes. Does there exist a point in an augmented Schottky space representing the surface $S$?

Problem 2. We give a point $\tau$ in an augmented Schottky space $\hat{G}_g(\tilde{\Sigma}_0)$ associated with a basic system of Jordan curves $\tilde{\Sigma}_0$, which represents a compact Riemann surface $S$ with nodes. Then for any sequence of points $\{\tau_n\}$ in the Schottky space $G_g(\tilde{\Sigma}_0)$ tending to the point $\tau$, does the Riemann surface $S(\tau_n)$ represented by $\tau_n$ converge to $S$ as marked surfaces as $n \to \infty$?

The answer to Problem 1 is affirmative:

THEOREM 1. There exists a point in an augmented Schottky space which represents a given Riemann surface with nodes.

The answer to Problem 2 is negative in the general case, namely in the case where $\tilde{\Sigma}_0$ is a basic system of Jordan curves. However the answer is affirmative in a special case, namely in the case where $\tilde{\Sigma}_0$ is a standard system of Jordan curves. Now the following question arises: To what Riemann surface does the
sequence of Riemann surfaces \( \{S(\tau_n)\} \) converge as marked surface as \( n \to \infty \) in the general case?

**Theorem 2.** Given a point \( \tau \in \widehat{G}_g(\Sigma_0) \). Then there exists a sequence of points \( \{\tau_n\} \subset \widehat{G}_g(\Sigma_0) \) tending to \( \tau \) such that \( S(\tau_n) \) converges to \( S(\tau) \) as marked surfaces.

**Theorem 3.** Let \( <G_0> \) and \( \tilde{\Sigma}_0 \) be a fixed marked Schottky group and a fixed basic system of Jordan curves for \( <G_0> \), respectively. Given a point \( \tau \in \delta^{I, J}\widehat{G}_g(\Sigma_0) \), where \( I \supset I(J) \neq \emptyset \). Let \( \Sigma_0^*, I^* \), and \( J^* \) be a basic system of loops, a subset of \( I \), and a subset of \( J \), respectively, obtained from \( \tilde{\Sigma}_0 \), \( I \) and \( J \) by applying certain interchange operators. Let \( \tau^* \in \delta^{I^*, J^*}\widehat{G}_g(\Sigma_0^*) \) be a point representing a compact Riemann surface with \( |I^*| + |J^*| \) nodes. Then there exists the following sequence of points \( \{\tau_n\} \subset \widehat{G}_g(\Sigma_0) \):

\[
\tau_n \to \tau \quad \text{and} \quad S(\tau_n) \to S(\tau^*) \quad \text{as} \quad n \to \infty ,
\]
as marked surfaces.

§ 1. Definitions.

**Definition 1.** Let \( C_1, C_{g+1}, C_2, C_{g+2}, \ldots, C_g, C_{2g} \) be a set of \( 2g \) mutually disjoint Jordan curves on the Riemann sphere \( \hat{c} \) which comprise the boundary of a \( 2g \)-ply connected region \( \omega \). Suppose there are \( g \) Möbius transformations \( A_1, \ldots, A_g \) which have the property that \( A_j \) maps \( C_j \) onto \( C_{g+j} \) and \( A_j(\omega) \cap \omega = \emptyset \) \( (1 \leq j \leq g) \). Then \( A_j \) \( (j=1, 2, \ldots, g) \) generates a marked Schottky
group \( <G> = \langle A_1, A_2, \cdots, A_g \rangle \). \( C_1, \cdots, C_{2g} \) are called defining curves of \( <G> \).

We say two marked Schottky groups \( <G> = \langle A_1, \cdots, A_g \rangle \) and \( \hat{G} = \langle \hat{A}_1, \cdots, \hat{A}_g \rangle \) being equivalent if there exists a Möbius transformation \( T \) such that \( \hat{A}_j = TA_jT^{-1} \) \((j=1,2,\cdots,g)\), and we denote it by \( <G> \sim \hat{G} \).

**DEFINITION 2.** The Schottky space of genus \( g \), denoted by \( \mathbb{C}_g \), is the set of all equivalent classes of Schottky groups of genus \( g \geq 1 \).

**DEFINITION 3.** Let \( C_1, \cdots, C_{2g} \) be defining curves of \( <G> = \langle A_1, \cdots, A_g \rangle \). If mutually disjoint Jordan curves \( C_1, \cdots, C_{2g}, C_{2g+1}, \cdots, C_{4g-3} \) on \( \hat{C} \) have the following properties (i) and (ii), then we call \( \hat{E} = \{ C_1, \cdots, C_{2g}, C_{2g+1}, \cdots, C_{4g-3} \} \) a basic system of Jordan curves (B.S.J.C.) for \( <G> \): (i) \( C_{2g+j} \) \((j=1, \cdots, 2g-3)\) lie in \( \omega \). (ii) Each component of \( \hat{C} \setminus \bigcup_{j=1}^{2g-3} C_{2g+j} \) is a triply connected domain. In particular, if a B.S.J.C. \( \hat{E} \) has the following property (iii), we call \( \hat{E} \) a standard system of Jordan curves (S.S.J.C.) for \( <G> \): (iii) For each \( i=1,2,\cdots,g \) and \( j=1,2,\cdots,2g-3 \), \( C_i \) and \( C_{g+j} \) lie on the same side of \( C_{2g+j} \). See Examples 1 and 2 on p.13.

**DEFINITION 4.** Let \( S \) be a compact Riemann surface. We call the set \( E = \{ a_1, \cdots, a_g, \gamma_1, \cdots, \gamma_{2g-3} \} \) of loops on \( S \) having the following property a basic system of loops (B.S.L.): Each component of \( S \setminus \bigcup_{i=1}^{g} a_i \setminus \bigcup_{j=1}^{2g-3} \gamma_j \) is a planar and triply connected region. If, in particular, the number of nondividing loops is equal
to \( g \), we call a B.S.L. \( \Sigma \) a standard system of loops (S.S.L.).

Let \( \Omega(G) \) be the region of discontinuity of \( \langle G \rangle \). Let \( \Pi: \Omega(G)/\langle G \rangle = S \) be the natural projection. If \( \tilde{\Sigma} = \{C_1, \ldots, C_{2g}; C_{2g+1}, \ldots, C_{4g-3} \} \) is a B.S.J.C. (resp. S.S.J.C.), then the projection \( \Sigma = \Pi(\tilde{\Sigma}) = \{a_1, \ldots, a_g; \gamma_1, \ldots, \gamma_{2g-3} \} \), \( a_i = \Pi(C_i) \) and \( \gamma_j = \Pi(C_{2g+j}) \), is a B.S.L. (resp. S.S.L.). We call \( \Sigma \) the projection of \( \tilde{\Sigma} \). See Examples 1 and 2 on p.13.

§2. Introduction of new coordinates to \( \mathcal{C}_g \).

We fix a marked Schottky group \( \langle G_0 \rangle = \langle A_1, 0, \ldots, A_g, 0 \rangle \). Let \( \tilde{\Sigma}_0 = \{C_1, 0, \ldots, C_{2g}, 0; C_{2g+1}, 0, \ldots, C_{4g-3}, 0 \} \) be a fixed B.S.J.C. for \( \langle G_0 \rangle \). Let \( \langle G \rangle = \langle A_1, \ldots, A_g \rangle \) be a marked Schottky group. Let \( \lambda_j (|\lambda_j| > 1) \), \( p_j \) and \( p_{g+j} \) be the multiplier, the repelling and the attracting fixed points of \( A_j \), respectively. We normalize \( \langle G \rangle \) by setting \( p_1 = 0 \), \( p_{g+1} = \infty \) and \( p_2 = 1 \). Then a point in the Schottky space \( \mathcal{C}_g \) is identified with

\[
\tilde{\tau} = (\lambda_1, \ldots, \lambda_g, p_{g+2}, p_3, p_{g+3}, \ldots, p_g, p_{2g}) \in \mathcal{C}^{3g-3}.
\]

Now we will introduce new coordinates with respect to \( \tilde{\Sigma}_0 \):

\[
\tau = (t_1, t_2, \ldots, t_g, \rho_1, \ldots, \rho_{2g-3}) \in \mathcal{C}^{3g-3}.
\]

First define \( t_i \) by setting \( t_i = 1/\lambda_i \) (i=1,\ldots,g). Thus \( t_i \in \mathbb{D}^* = \{z|0 < |z| < 1\} \). Next in order to define \( \rho_j \) associated with \( C_{2g+j} = C(i_0, i_1, \ldots, i_{2g-3}) \in \tilde{\Sigma}_0 \) (j=1,2,\ldots,2g-3), we determine integers \( k(j), \ell(j), m(j) \) and \( n(j) \) which are \( \geq 1 \) and \( \leq 2g \) as
follows, where $C(i_0, i_1, \ldots, i_\mu)$ is the multi-suffix of $C_{2g+j}$ (see [4] for the definition): $k(j) = 1$, $C_k(j) = C(i_0, i_1, \ldots, i_{\mu-1}, 1-i_\mu, 0, \ldots, 0)$, $C_m(j) = C(i_0, i_1, \ldots, i_\mu, 0, \ldots, 0)$ and $C_n(j) = C(i_0, i_1, \ldots, i_\mu, 0, \ldots, 0)$. The coordinate $\rho_j$ is now defined as follows:

$$(P_k(j), P_l(j), P_m(j), P_n(j)) = (0, 1, \infty, \rho_j),$$

where $(a, b, c, d)$ means the cross ratio of $a, b, c,$ and $d$.

We define a mapping $\phi$ by $\phi(<G>) = \tau$. We note that if $<G> \sim <\tilde{G}>$, then $\phi(<G>) = \phi(<\tilde{G}>).$ We denote by $G_g(\tilde{\Sigma}_0)$ the set

$$G_g(\tilde{\Sigma}_0) = \{ \tau = \phi(<G>) | <G> \in G_g \}.$$

Then $G_g(\tilde{\Sigma}_0) \cong G_g$ and $G_g(\tilde{\Sigma}_0) \subset D^{*g} \times (C \{ 0, 1 \})^{2g-3}$. We call $G_g(\tilde{\Sigma}_0)$ the Schottky space associated with $\tilde{\Sigma}_0$.

§3. **Augmented Schottky spaces.**

Let $<G_0>$ and $\tilde{\Sigma}_0$ be a fixed Schottky group and a fixed B.S.J.C. as in §2.

**DEFINITION 5.** We say $C_{2g+j} = C(i_1, \ldots, i_\mu)$ (resp. $C_i = C(j_1, \ldots, j_\mu)$) is behind $C_{2j+j} = C(i'_1, \ldots, i'_\nu)$ if $\nu < \mu$ and $i_k = i'_k$ ($k=1, 2, \ldots, \nu$) (resp. $\nu < \sigma$ and $j_k = i'_k$ ($k=1, 2, \ldots, \nu$)), and denote the fact $C_{2g+l} < C_{2g+j}$ (resp. $C_{2g+l} < C_i$). Otherwise, we say that $C_{2g+j}$ (resp. $C_i$) is not behind $C_{2g+l}$ and we denote the fact by $C_{2g+l} \not< C_{2g+j}$ (resp. $C_{2g+l} \not< C_i$).

We define the ordered cycle corresponding to $a_i$ as follows.
We denote the shortest path from $C_i$ to $C_{g+i}$ on the tree of $\widetilde{\Sigma}_0$ by

$$(1) \quad C_i, C_{2g+1}(1), C_{2g+1}(2), \ldots, C_{2g+1}(k), C_{g+i}$$

(see [4] and Fig. 1 on p.13 in this paper for trees.) Here $\delta(\ell)$ $(\ell=1,2,\ldots,k)$ are determined by $\delta(\ell) = +1$ or $\delta(\ell) = -1$ according as $C_{2g+\ell} < C_{g+i}$ or $C_{2g+\ell} < C_i$.

**DEFINITION 6.** The projection

$$(\alpha_i; \gamma_{i}^{\delta(1)}, \ldots, \gamma_{i}^{\delta(k)})$$

of (1) onto $S_0 = \Omega(G_0)/\langle G_0 \rangle$ is called the *ordered cycle* corresponding to $\alpha_i$, and is denoted by $L_{0,i}$.

Let $I$ be a subset of $\{1,2,\ldots,g\}$ and $J$ a subset of $\{1,2,\ldots,2g-3\}$. We denote by $|I|$ and $|J|$ the cardinality of $I$ and $J$, respectively. Let $L_{0,j}(1), L_{0,j}(2), \ldots, L_{0,j}(t)$ be the complete list of cycles containing $\gamma_j^{\delta}$, and let $\alpha_{0,k}$ be the "$\alpha$-loops" contained in $L_{0,k}$ ($1 \leq k \leq t$), where $t = t(j)$ depends on $j$. We define the subset $I(J)$ of $\{1,2,\ldots,g\}$ by

$I(J) = \{i \in \{1,2,\ldots,g\} | \alpha_{0,i}$ is contained in $L_{0,j}(k)$ for some $k (1 \leq k \leq t(j))$ and for some $j \in J\}$.

**Remark.** If $\widetilde{\Sigma}_0$ is a S.S.J.C. , then $I(J) = \emptyset$.

We define the following sets $X = \delta^I,J \Omega_g(\widetilde{\Sigma}_0)$ with $I \supset I(J)$:

(i) When $I = J = \emptyset$, we define $X$ as $\Omega_g(\widetilde{\Sigma}_0)$, the Schottky space associated with $\widetilde{\Sigma}_0$.  

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(ii) When $I \neq \emptyset$, $j = \emptyset$, we define $X$ as follows:

$$\tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \ldots, t_g, \rho_1, \ldots, \rho_{2g-3}) | t_i = 0 \ (i \in I), \ t_i \neq 0 \ (i \notin I), \ \rho_j \neq 1 \ (j = 1, \ldots, 2g-3), \ \text{and} \ \tau \ \text{represents a Riemann surface with nodes such that only} \ \alpha_i \ (i \in I) \ \text{are nodes}. \}$$

(iii) When $I = \emptyset$, $J \neq \emptyset$, we define $X$ as follows:

$$\tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \ldots, t_g, \rho_1, \ldots, \rho_{2g-3}) | t_i \neq 0 \ (i = 1, \ldots, g), \ 
\rho_j = 1 \ (j \in J), \ \rho_j \neq 1 \ (j \notin J) \ \text{and} \ \tau \ \text{represents a Riemann surface with nodes such that only} \ \gamma_j \ (j \in J) \ \text{are nodes}. \}$$

(iv) When $I \supset I(J) \neq \emptyset$, $X$ is defined as follows:

$$\tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \ldots, t_g, \rho_1, \ldots, \rho_{2g-3}) | t_i = 0 \ (i \in I), \ t_i \neq 0 \ (i \notin I), \ \rho_j = 1 \ (j \in J), \ \rho_j \neq 1 \ (j \notin J) \ \text{and} \ \rho \ \text{represents a compact Riemann surface such that only} \ \alpha_i \ (i \in I) \ \text{and} \ \gamma_j \ (j \in J) \ \text{are nodes}. \}$$

**DEFINITION 7.**

$$\widehat{\mathcal{G}}_g^*(\tilde{\Sigma}_0) = \bigcup \{ \tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) | I \subset \{1, 2, \ldots, g\}, \ J \subset \{1, 2, \ldots, 3g-3\} \ \text{with} \ I \supset I(J) \}$$

is called the augmented Schottky space associated with $\tilde{\Sigma}_0$.

**Remark.** Let $S(\tau)$ be the Riemann surface represented by $\tau$. 
$\{ S(\tau) | \tau \in \widehat{\mathcal{G}}_g^*(\tilde{\Sigma}_0) \}$ is the sets of all Riemann surfaces in Fig.2 and Fig.3 in the cases of Example 1 and Example 2, respectively.
§ 4. Interchange operators.

For simplicity, we will only consider interchange operators in the case of Example 1 (see Fig. 4). For detail, see Sato [5]. Choose \( j \) with \( I([j]) \neq \emptyset \). Let \( i \in I([j]) \). For these \( i \) and \( j \), we introduce the interchange operators \( I_g(i,j) \).

Remark. Since \( I(J) \) is always empty in the case where \( \Sigma \) is a S.S.J.C., we can not define an interchange operator in this case.

For simplicity, we only consider \( I_g(1,2) \), which is defined as follows (see Fig. 4 on p. 15): For a B.S.J.C. \( \Sigma \),

\[
I_g(1,2)(\Sigma) = \Sigma^* = \{C_1^*, C_2^*, \ldots, C_6^*, C_7^*, C_8^*, C_9^*\},
\]

where \( C_1^* = A_1^{-1}(C_8) \), \( C_2^* = A_1^{-1}(C_2) \), \( C_3^* = C_3 \), \( C_4^* = C_8 \), \( C_5^* = C_5 \), \( C_6^* = C_6 \), \( C_7^* = C_7 \), \( C_8^* = C_1 \), and \( C_9^* = C_9 \).

For a B.S.L. \( \Sigma = \{a_1, a_2, a_3; \gamma_1, \gamma_2, \gamma_3\} \), \( I_g(1,2)(\Sigma) = \{a_1^*, a_2^*, a_3^*, \gamma_1^*, \gamma_2^*, \gamma_3^*\} \), where \( a_1^* = \gamma_2 \), \( a_2^* = a_2 \), \( a_3^* = a_3 \), \( \gamma_1^* = \gamma_1 \), \( \gamma_2^* = \gamma_2 \), \( a_1, \gamma_3^* = \gamma_3 \).

For ordered cycles \( L_1, L_2 \) and \( L_3, L_1^* = I_g(1,2)(L_1) = (a_1^*, \gamma_2^*, \gamma_1^*), L_2^* = I_g(1,2)(L_2) = (a_2^*, \gamma_2^*, \gamma_1^*, \gamma_3^*) \) and \( L_3^* = I_g(1,2)(L_3) = (a_3^*, \gamma_3^*-1, \gamma_1^*-1) \), where we write \( \gamma_j^* \) for \( \gamma_j^{*+1} \) for simplicity.

For a marked Schottky group \( \langle G \rangle = \langle A_1, A_2, A_3 \rangle \), \( \langle G^* \rangle = I_g(1,2)(\langle G \rangle) = \langle A_1^*, A_2^*, A_3^* \rangle \), where \( A_1^* = A_1 \), \( A_2^* = A_2A_1 \), \( A_3^* = A_3 \).

We obtain Theorem 1 by using interchange operators. See Sato [5] for details.
§ 5. Relations between limits of Schottky groups and limits of Riemann surfaces.

Here we will consider Problem 2. Let $S$ be a compact Riemann surface of genus $g$ with or without nodes. We denote by $N(S)$ the set of all nodes on $S$. We assume that each component of $S \setminus N(S)$ has the Poincaré metric. The Poincaré metric $\lambda(z)dz$ on $S$ is defined as the Poincaré metric on each component of $S \setminus N(S)$.

**DEFINITION 8.** If the following conditions are satisfied, a sequence of Riemann surfaces $\{S_n\}$ converges to a surface $S$ as marked surfaces: There exists a locally quasiconformal mapping $\phi_n : S \setminus N(S) \to S_n \setminus P(S_n)$ such that (i) $\lambda_n(\phi_n(z))|d\phi_n(z)|$ uniformly converges to $\lambda(z)|dz|$ on every compact subset of $S \setminus N(S)$, where $\lambda_n(z)|dz|$ and $\lambda(z)|dz|$ are the Poincaré metrics on $S_n$ and $S$, respectively, (ii) $\phi_n$ maps a deleted neighborhood $N(a_i) \setminus \{a_i\}$ (resp. $N(\gamma_j) \setminus \{\gamma_j\}$) of $a_i$ (resp. $\gamma_j$) to a deleted neighborhood $N(a_i, n) \setminus \{a_i, n\}$ (resp. $N(\gamma_j, n) \setminus \{\gamma_j, n\}$) of $a_i, n$ (resp. $\gamma_j, n$) if $a_i \in N(S)$ (resp. $\gamma_j \in N(S)$), and (iii) $\phi_n$ maps a neighborhood $N(a_i)$ (resp. $N(\gamma_j)$) of $a_i$ (resp. $\gamma_j$) to a neighborhood $N(a_i, n)$ (resp. $N(\gamma_j, n)$) of $a_i, n$ (resp. $\gamma_j, n$) if $a_i \in N(S)$ (resp. $\gamma_j \in N(S)$), where $P(S_n) = f_n^{-1}(N(S))$ and $f_n : S_n \to S$ is a deformation.

By constructing locally quasiconformal mappings, we have Theorem 2. See Sato [6] for details.
Let $<G_0>$ and $\tilde{\Sigma}_0$ be a fixed marked Schottky group and a fixed B.S.J.C. for $<G_0>$, respectively. Set $S_0 = \Omega(G_0)/\langle G_0 \rangle$. Given a point $\tau \in \delta I, J \subseteq \tilde{\Sigma}_0$, where $I \supset I(J) \neq \emptyset$. Then $S(\tau)$ is a compact Riemann surface with $|I| + |J|$ nodes of genus $g$. We define the following sets: $J_1 = \{ j \in J \ | \ \gamma_j \text{ is a dividing loop on } S_0 \}$, $J_2 = \text{any subset of } J \setminus J_1$, $\tilde{\Sigma}_1 = \tilde{I}_g(i_k(1), j_k(1))(\tilde{\Sigma}_0)$ with $i_k(1) \in I(\{j_k(1)\})$, $j_k(1) \in J_2$ and $J_{21} = J_2 \setminus \{ j_k(1) \}$. Choose $j_k(2) \in J_{21}$ such that $I_1(\{j_k(2)\}) \cap (I(J_2) \setminus \{ i_k(1) \}) \neq \emptyset$. Set $\tilde{\Sigma}_2 = \tilde{I}_g(i_k(2), j_k(2))(\tilde{\Sigma}_1)$ with $i_k(2) \in I_1(\{j_k(2)\})$, $j_k(2) \neq i_k(1)$. We set $J_{22} = J_2 \setminus \{ j_k(2) \} = J_2 \setminus \{ j_k(1), j_k(2) \}$. By the same way, we determined the following: $j_k(3)$, $i_k(3)$, $J_{23}$, $\tilde{\Sigma}_3$, $I_3(J_{23})$; \ldots; $j_k(s)$, $i_k(s)$, $J_{2s}$, $\tilde{\Sigma}_s$. Here $s$ is the integer satisfying the following (i) and (ii): (i) $I_{s-1}(\{j_k(s)\}) \cap I(J_2) \setminus \{ i_k(1), i_k(2), \ldots, i_k(s-1) \} \neq \emptyset$, (ii) $I_s(\{ j \}) \subseteq \{ i_k(1), \ldots, i_k(s) \}$ for any $j \in J_2 \setminus \{ j_k(1), j_k(2), \ldots, j_k(s) \}$.

We set $J_3 = J \setminus (J_1 \cup J_2)$, $J_4 = \{ j_k(1), j_k(2), \ldots, j_k(s) \}$, $J_5 = J_2 \setminus J_4$, $I_1 = I \setminus I(J)$, $I_4 = \{ i_k(1), i_k(2), \ldots, i_k(s) \}$, $I_3 = I_s(J_3)$, $I_5 = I \setminus (I_1 \cup I_3 \cup I_4)$, $I_6$ is a subset of $I_5$, $I_7 = I_5 \setminus I_6$, $I^* = I \setminus I_7$ and $J^* = J \setminus J_4$. Then we have Theorem 3. See Sato [6] for the proof.

**Corollary.** Given $\tau \in \delta I, J \subseteq \tilde{\Sigma}_0$, where $I \supset I(J) \neq \emptyset$. Then there exists a sequence of points $\{ \tau_n \} \subseteq \tilde{\Sigma}_0$ such that (i) $\tau_n \to \tau$ as $n \to \infty$ and (ii) $S(\tau_n)$ does not converge to $S(\tau)$ as marked surfaces.

**Remark.** By a similar method to the proof of Theorem 2, we
have the following. If \( \tilde{\mathcal{L}}_0 \) is a S.S.J.C., then \( S(\tau_n) \) converges to \( S(\tau) \) as marked surfaces for any point \( \tau \in \mathcal{G}^*(\tilde{\mathcal{L}}_0) \) and for any sequence of points \( \{\tau_n\} \subset \mathcal{G}(\tilde{\mathcal{L}}_0) \) with \( \tau_n \to \tau \).

§6. Appendices.

We will consider the following in the forthcoming papers [7,8].

1. Properties of interchange operators. There are five kinds of interchange operators as follows: (1) \( I_g(a_i, a_i^{-1}) = I_g(C_i, C_{g+i}) \), (2) \( I_g(a_i, a_j) = I_g(C_i, C_j) \), (3) \( I_g(\gamma_j, \gamma_j^{-1}) = I_g(C_{2g+j}, C_{2g+j}) \), (4) \( I_g(\gamma_i, \gamma_j) = I_g(C_{2g+i}, C_{2g+j}) \) and (5) \( I_g(a_i, \gamma_j) = I_g(C_i, C_{2g+j}) \).

Here we only considered and used interchanged operators in case (5).

2. Relations between Nielsen isomorphisms and interchange operators. Here Nielsen isomorphisms are

\[
N_1(A_1, A_1) : <A_1, A_2, \cdots, A_1, \cdots, A_g> \to <A_1, A_2, \cdots, A_1, \cdots, A_g>.
\]

\[
N_2(A_1, A_1^{-1}) : <A_1, A_2, \cdots, A_g> \to <A_1^{-1}, A_2, \cdots, A_g>.
\]

\[
N_3(A_1, A_2) : <A_1, A_2, A_3, \cdots, A_g> \to <A_1, A_1A_2, A_3, \cdots, A_g>.
\]

3. Boundary behavior of the space of marked Schottky groups of real type of genus 2. We say \( <G> = <A_1, A_2> \) a Schottky group of real type if \( A_1, A_2 \in \text{SL}(2, \mathbb{R}) \).

References

[1] W. Abikoff, Degenerating families of Riemann surfaces, Ann. of


Example 1.

\begin{align*}
\langle G \rangle &= \langle A_1, A_2, A_3 \rangle \\
E &= \{ C_1, \ldots, C_6; C_7, C_8, C_9 \}
\end{align*}

Example 2.

\begin{align*}
C_4 &= C(0) \\
C_2 &= C(1, 0, 0) \\
C_5 &= C(1, 0, 1) \\
C_3 &= C(1, 1, 0) \\
C_6 &= C(1, 1, 1)
\end{align*}

Fig. 1
Example 1.

Fig. 2.

Example 2.

Fig. 3.
\[ \bar{\mathcal{E}}^* = I_{g(1,2)}(\bar{\Sigma}) = \{c_1^*, \ldots, c_6^*, c_7^*, c_8^*, c_9^*\} \]

\[ \mathcal{E}^* = I_{g(1,2)}(\Sigma) = \{a_1^*, a_2^*, a_3^*, \gamma_1^*, \gamma_2^*, \gamma_3^*\} \]

Fig. 4.