ON AUGMENTED SCHOTTKY SPACES AND INTERCHANGE OPERATORS

Hiroki Sato 佐藤 準樹
Department of Mathematics, Shizuoka University

§0. Introduction.

Problem 1. Let \( S \) be a compact Riemann surface with nodes. Does there exist a point in an augmented Schottky space representing the surface \( S \)?

Problem 2. We give a point \( \tau \) in an augmented Schottky space \( \mathcal{G}_g(\Sigma_0) \) associated with a basic system of Jordan curves \( \Sigma_0 \), which represents a compact Riemann surface \( S \) with nodes. Then for any sequence of points \( \{\tau_n\} \) in the Schottky space \( \mathcal{G}_g(\Sigma_0) \) tending to the point \( \tau \), does the Riemann surface \( S(\tau_n) \) represented by \( \tau_n \) converge to \( S \) as marked surfaces as \( n \to \infty \)?

The answer to Problem 1 is affirmative:

THEOREM 1. There exists a point in an augmented Schottky space which represents a given Riemann surface with nodes.

The answer to Problem 2 is negative in the general case, namely in the case where \( \Sigma_0 \) is a basic system of Jordan curves. However the answer is affirmative in a special case, namely in the case where \( \Sigma_0 \) is a standard system of Jordan curves. Now the following question arises: To what Riemann surface does the
sequence of Riemann surfaces \( \{S(\tau_n)\} \) converge as marked surface as \( n \to \infty \) in the general case?

**Theorem 2.** Given a point \( \tau \in \hat{G}_g(\Sigma_0) \). Then there exists a sequence of points \( \{\tau_n\} \subset \hat{G}_g(\Sigma_0) \) tending to \( \tau \) such that \( S(\tau_n) \) converges to \( S(\tau) \) as marked surfaces.

**Theorem 3.** Let \( <G_0> \) and \( \Sigma_0 \) be a fixed marked Schottky group and a fixed basic system of Jordan curves for \( <G_0> \), respectively. Given a point \( \tau \in \delta^I,J \hat{G}_g(\Sigma_0) \), where \( I \supset I(J) \neq \emptyset \). Let \( \Sigma_0^* \), \( I^* \), and \( J^* \) be a basic system of loops, a subset of \( I \), and a subset of \( J \), respectively, obtained from \( \Sigma_0 \), \( I \) and \( J \) by applying certain interchange operators. Let \( \tau^* \in \delta^{I^*},J^* \hat{G}_g(\Sigma_0^*) \) be a point representing a compact Riemann surface with \( |I^*| + |J^*| \) nodes. Then there exists the following sequence of points \( \{\tau_n\} \subset \hat{G}_g(\Sigma_0) \):

\[ \tau_n \to \tau \text{ and } S(\tau_n) \to S(\tau^*) \text{ as } n \to \infty, \]

as marked surfaces.

§1. Definitions.

**Definition 1.** Let \( C_1, C_{g+1}, C_2, C_{g+2}, \ldots, C_g, C_{2g} \) be a set of \( 2g \) mutually disjoint Jordan curves on the Riemann sphere \( \hat{C} \) which comprise the boundary of a \( 2g \)-ply connected region \( \omega \). Suppose there are \( g \) Möbius transformations \( A_1, \ldots, A_g \) which have the property that \( A_j \) maps \( C_j \) onto \( C_{g+j} \) and \( A_j(\omega) \cap \omega = \emptyset \) \((1 \leq j \leq g)\). Then \( A_j \) \((j=1,2,\ldots,g)\) generates a marked Schottky
group $\langle G \rangle = \langle A_1, A_2, \ldots, A_g \rangle$. $C_1, \ldots, C_{2g}$ are called defining curves of $\langle G \rangle$.

We say two marked Schottky groups $\langle G \rangle = \langle A_1, \ldots, A_g \rangle$ and $\bar{G} = \langle \hat{A}_1, \ldots, \hat{A}_g \rangle$ being equivalent if there exists a Möbius transformation $T$ such that $\hat{A}_j = TA_jT^{-1}$ ($j=1,2,\ldots,g$), and we denote it by $\langle G \rangle \sim \langle \bar{G} \rangle$.

**Definition 2.** The Schottky space of genus $g$, denoted by $\mathbb{C}_g$, is the set of all equivalent classes of Schottky groups of genus $g \geq 1$.

**Definition 3.** Let $C_1, \ldots, C_{2g}$ be defining curves of $\langle G \rangle = \langle A_1, \ldots, A_g \rangle$. If mutually disjoint Jordan curves $C_1, \ldots, C_{2g}, C_{2g+1}, \ldots, C_{4g-3}$ on $\hat{\mathbb{C}}$ have the following properties (i) and (ii), then we call $\mathcal{E} = \{C_1, \ldots, C_{2g}, C_{2g+1}, \ldots, C_{4g-3}\}$ a basic system of Jordan curves (B.S.J.C.) for $\langle G \rangle$: (i) $C_{2g+j}$ ($j=1, \ldots, 2g-3$) lie in $\omega$. (ii) Each component of $\hat{\mathbb{C}} \setminus \bigcup_{j=1}^{2g-3} C_{2g+j}$ is a triply connected domain. In particular, if a B.S.J.C. $\mathcal{E}$ has the following property (iii), we call $\mathcal{E}$ a standard system of Jordan curves (S.S.J.C.) for $\langle G \rangle$: (iii) For each $i=1,2,\ldots,g$ and $j=1,2,\ldots,2g-3$, $C_i$ and $C_{g+i}$ lie on the same side of $C_{2g+j}$. See Examples 1 and 2 on p.13.

**Definition 4.** Let $S$ be a compact Riemann surface. We call the set $\mathcal{E} = \{\alpha_1, \ldots, \alpha_g, \gamma_1, \ldots, \gamma_{2g-3}\}$ of loops on $S$ having the following property a basic system of loops (B.S.L.): Each component of $S \setminus \bigcup_{i=1}^{g} \alpha_i \setminus \bigcup_{j=1}^{2g-3} \gamma_j$ is a planar and triply connected region. If, in particular, the number of nondividing loops is equal
to \( g \), we call a B.S.L. \( \Sigma \) a standard system of loops (S.S.L.).

Let \( \Omega(G) \) be the region of discontinuity of \( \langle G \rangle \). Let \( \Pi: \Omega(G) \to \Omega(G)/\langle G \rangle = S \) be the natural projection. If \( \tilde{\Sigma} = \{C_{1}, \cdots, C_{2g}; C_{2g+1}, \cdots, C_{4g-3}\} \) is a B.S.J.C. (resp. S.S.J.C.), then the projection \( \Sigma = \Pi(\tilde{\Sigma}) = \{a_{1}, \cdots, a_{g}; \gamma_{1}, \cdots, \gamma_{2g-3}\} \) and \( \gamma_{j} = \Pi(C_{2g+j}) \), is a B.S.L. (resp. S.S.L.). We call \( \Sigma \) the projection of \( \tilde{\Sigma} \). See Examples 1 and 2 on p.13.

§2. Introduction of new coordinates to \( \mathcal{G}_{g} \).

We fix a marked Schottky group \( \langle G_{0} \rangle = \langle A_{1}, 0; \cdots, A_{g}, 0 \rangle \). Let \( \tilde{\Sigma}_{0} = \{C_{1}, 0; \cdots, C_{2g}, 0; C_{2g+1}, 0; \cdots, C_{4g-3}, 0\} \) be a fixed B.S.J.C. for \( \langle G_{0} \rangle \). Let \( \langle G \rangle = \langle A_{1}, \cdots, A_{g} \rangle \) be a marked Schottky group. Let \( \lambda_{j} (|\lambda_{j}| > 1), \; p_{j} \) and \( p_{g+j} \) be the multiplier, the repelling and the attracting fixed points of \( A_{j} \), respectively. We normalize \( \langle G \rangle \) by setting \( p_{1} = 0, \; p_{g+1} = \infty \) and \( p_{2} = 1 \). Then a point in the Schottky space \( \mathcal{G}_{g} \) is identified with

\[
\tilde{\tau} = (\lambda_{1}, \cdots, \lambda_{g}, p_{g+2}, p_{3}, p_{g+3}, \cdots, p_{g}, p_{2g}) \in \mathbb{C}^{3g-3}.
\]

Now we will introduce new coordinates with respect to \( \tilde{\Sigma}_{0} \):

\[
\tau = (t_{1}, t_{2}, \cdots, t_{g}, \rho_{1}, \cdots, \rho_{2g-3}) \in \mathbb{C}^{3g-3}.
\]

First define \( t_{i} \) by setting \( t_{i} = 1/\lambda_{i} \) \((i=1, \cdots, g)\). Thus \( t_{i} \in D^{*} = \{z|0 < |z| < 1\} \). Next in order to define \( \rho_{j} \) associated with \( C_{2g+j} = C(i_{0}, i_{1}, \cdots, i_{y}) \in \tilde{\Sigma}_{0} \) \((j=1, 2, \cdots, 2g-3)\), we determine integers \( k(j), \ell(j), m(j) \) and \( n(j) \) which are \( \geq 1 \) and \( \leq 2g \) as
follows, where \( C(i_0, i_1, \ldots, i_\mu) \) is the multi-suffix of \( C_{2g+j} \) (see [4] for the definition): 
\[ k(j) = 1, \quad C_k(j) = C(i_0, i_1, \ldots, i_\mu, i_{\mu-1}, \ldots, i_1, i_0, 0, \ldots, 0), \quad C_m(j) = C(i_0, i_1, \ldots, i_\mu, 0, \ldots, 0) \quad \text{and} \quad C_n(j) = C(i_0, i_1, \ldots, i_\mu, 0, \ldots, 0). \] 
The coordinate \( \rho_j \) is now defined as follows:

\[
(P_k(j), P_l(j), P_m(j), P_n(j)) = (0, 1, \infty, \rho_j),
\]

where \((a, b, c, d)\) means the cross ratio of \(a, b, c,\) and \(d.\)

We define a mapping \( \phi \) by \( \phi(\langle G \rangle) = \tau. \) We note that if \( \langle G \rangle \sim \langle \hat{G} \rangle, \) then \( \phi(\langle G \rangle) = \phi(\langle \hat{G} \rangle). \) We denote by \( \mathcal{G}_g(\hat{\Sigma}_0) \) the set

\[
\mathcal{G}_g(\hat{\Sigma}_0) = \{ \tau = \phi(\langle G \rangle) | \langle G \rangle \in \mathcal{G}_g \}.
\]

Then \( \mathcal{G}_g(\hat{\Sigma}_0) \cong \mathcal{G}_g \) and \( \mathcal{G}_g(\hat{\Sigma}_0) \subset D^g \times (\mathcal{C}\backslash \{0, 1\})^{2g-3}. \) We call \( \mathcal{G}_g(\hat{\Sigma}_0) \) the **Schottky space** associated with \( \hat{\Sigma}_0. \)

§3. **Augmented Schottky spaces.**

Let \( \langle G_0 \rangle \) and \( \hat{\Sigma}_0 \) be a fixed Schottky group and a fixed B.S.J.C. as in §2.

**DELI\(C\)NTION 5.** We say \( C_{2g+j} = C(i_1, \ldots, i_\mu) \) (resp. \( C_i = C(j_1, \ldots, j_\nu) \)) is **behind** \( C_{2j+l} = C(i_1', \ldots, i_\nu') \) (resp. \( C_i' = C(j_1, \ldots, j_\nu') \)) if \( \nu < \mu \) and \( i_k = i_k' \) \((k=1, 2, \ldots, \nu)\) (resp. \( \nu < \sigma \) and \( j_k = j_k' \) \((k=1, 2, \ldots, \nu)\)), and denote the fact \( C_{2g+l} \subset C_{2g+j} \) (resp. \( C_{2g+l} \subset C_i \)). Otherwise, we say that \( C_{2g+j} \) (resp. \( C_i \)) is not **behind** \( C_{2g+l} \) and we denote the fact by \( C_{2g+l} \not\subset C_{2g+j} \) (resp. \( C_{2g+l} \not\subset C_i \)).

We define the ordered cycle corresponding to \( a_i \) as follows.
We denote the shortest path from \( C_i \) to \( C_{g+i} \) on the tree of \( \widetilde{\Sigma}_0 \) by

\[
(1) \quad C_i, C_{2g+i}(1), C_{2g+i}(2), \ldots, C_{2g+i}(k), C_{g+i}
\]

(see [4] and Fig. 1 on p.13 in this paper for trees.) Here \( \delta(l) \) \( (l=1,2,\ldots,k) \) are determined by \( \delta(l) = +1 \) or \( \delta(l) = -1 \) according as \( C_{2g+l} < C_{g+i} \) or \( C_{2g+l} < C_i \).

**DEFINITION 6.** The projection

\[
(\alpha_i^{\delta(1)}, \ldots, \alpha_i^{\delta(k)})
\]

of (1) onto \( S_0 = \Omega(G_0)/<G_0> \) is called the ordered cycle corresponding to \( \alpha_i \), and is denoted by \( L_{0,i} \).

Let \( I \) be a subset of \( \{1,2,\ldots,g\} \) and \( J \) a subset of \( \{1,2,\ldots,2g-3\} \). We denote by \( |I| \) and \( |J| \) the cardinality of \( I \) and \( J \), respectively. Let \( L_{0,j}(1), L_{0,j}(2), \ldots, L_{0,j}(t) \) be the complete list of cycles containing \( \gamma_j^{\delta} \), and let \( \alpha_0,k \) be the "\( \alpha \)-loops" contained in \( L_{0,k} \) \( (1 \leq k \leq t) \), where \( t = t(j) \) depends on \( j \). We define the subset \( I(J) \) of \( \{1,2,\ldots,g\} \) by

\[
I(J) = \{ i \in \{1,2,\ldots,g\} | \alpha_0,i \text{ is contained in } L_{0,j}(k) \text{ for some } k (1 \leq k \leq t(j)) \text{ and for some } j \in J \}.
\]

**Remark.** If \( \widetilde{\Sigma}_0 \) is a S.S.J.C., then \( I(J) = \emptyset \).

We define the following sets \( X = \delta I,J \otimes_g (\widetilde{\Sigma}_0) \) with \( I \supset I(J) \):

(i) When \( I = J = \emptyset \), we define \( X \) as \( \otimes_g (\widetilde{\Sigma}_0) \), the Schottky space associated with \( \widetilde{\Sigma}_0 \).
(ii) When $I \neq \emptyset$, $J = \emptyset$, we define $X$ as follows:

$$
\hat{\mathcal{S}}^I \emptyset \mathcal{G}_g(\hat{\mathcal{S}}_0) = \{ \tau = (t_1, \ldots , t_g, \rho_1, \ldots , \rho_{2g-3}) | t_i = 0 \ (i \in I), \ t_i \neq 0 \ (i \notin I), \ \rho_j = 1 \ (j \notin J), \ \rho_j \neq 1 \ (j \in J) \} \quad \text{and } \tau \ \text{represents a Riemann surface with nodes such that only } \gamma_i \ (i \in I) \ \text{are nodes}.
$$

(iii) When $I = \emptyset$, $J \neq \emptyset$, we define $X$ as follows:

$$
\hat{\mathcal{S}}^J \emptyset \mathcal{G}_g(\hat{\mathcal{S}}_0) = \{ \tau = (t_1, \ldots , t_g, \rho_1, \ldots , \rho_{2g-3}) | t_i \neq 0 \ (i = 1, \ldots , g), \ \rho_j = 1 \ (j \in J), \ \rho_j \neq 1 \ (j \notin J) \} \quad \text{and } \tau \ \text{represents a Riemann surface with nodes such that only } \gamma_j \ (j \in J) \ \text{are nodes}.
$$

(iv) When $I \supseteq I(J) \neq \emptyset$, $X$ is defined as follows:

$$
\hat{\mathcal{S}}^I_J \mathcal{G}_g(\hat{\mathcal{S}}_0) = \{ \tau = (t_1, \ldots , t_g, \rho_1, \ldots , \rho_{2g-3}) | t_i = 0 \ (i \in I), \ t_i \neq 0 \ (i \notin I), \ \rho_j = 1 \ (j \in J), \ \rho_j \neq 1 \ (j \notin J) \} \quad \text{and } \tau \ \text{represents a compact Riemann surface such that only } \gamma_i \ (i \in I) \ \text{and } \gamma_j \ (j \in J) \ \text{are nodes}.
$$

**DEFINITION 7.**

$$
\hat{\mathcal{G}}^\ast_g(\hat{\mathcal{S}}_0) = \bigcup \{ \hat{\mathcal{S}}^I_J \mathcal{G}_g(\hat{\mathcal{S}}_0) | I \subseteq \{1,2,\ldots ,g\}, \ J \subseteq \{1,2,\ldots ,3g-3\} \}
$$

with $I \supseteq I(J)$

is called the **augmented Schottky space** associated with $\hat{\mathcal{S}}_0$.

**Remark.** Let $S(\tau)$ be the Riemann surface represented by $\tau$. 

$$
\{ S(\tau) | \tau \in \hat{\mathcal{G}}^\ast_3(\hat{\mathcal{S}}_0) \}
$$

is the sets of all Riemann surfaces in Fig.2 and Fig.3 in the cases of Example 1 and Example 2, respectively.
§ 4. Interchange operators.

For simplicity, we will only consider interchange operators in the case of Example 1 (see Fig. 4). For detail, see Sato [5]. Choose \( j \) with \( I([j]) \neq \emptyset \). Let \( i \in I([j]) \). For these \( i \) and \( j \), we introduce the interchange operators \( I_\Sigma(i,j) \).

Remark. Since \( I(J) \) is always empty in the case where \( \Sigma \) is a S.S.J.C., we can not define an interchange operator in this case.

For simplicity, we only consider \( I_\Sigma(1,2) \), which is defined as follows (see Fig. 4 on p. 15): For a B.S.J.C. \( \Sigma \),

\[
I_\Sigma(1,2)(\Sigma) = \Sigma^* = \{ C_1^*, C_2^*, \ldots, C_6^*, C_7^*, C_8^*, C_9^* \},
\]

where \( C_1^* = A_1^{-1}(C_8) \), \( C_2^* = A_1^{-1}(C_2) \), \( C_3^* = C_3 \), \( C_4^* = C_8 \), \( C_5^* = C_5 \), \( C_6^* = C_6 \), \( C_7^* = C_7 \), \( C_8^* = C_1 \), and \( C_9^* = C_9 \).

For a B.S.L. \( \Sigma = \{ a_1^*, a_2^*, a_3^*; \gamma_1, \gamma_2, \gamma_3 \} \), \( I_\Sigma(1,2)(\Sigma) = \{ a_1^*, a_2^*, a_3^*; \gamma_1^*, \gamma_2^*, \gamma_3^* \} \), where \( a_1^* = \gamma_2 \), \( a_2^* = a_2 \), \( a_3^* = a_3 \), \( \gamma_1^* = \gamma_1 \), \( \gamma_2^* = \gamma_2 \), \( \gamma_3^* = \gamma_3 \).

For ordered cycles \( L_1 \), \( L_2 \) and \( L_3 \), \( I_\Sigma(L_1) = \{ a_1^*; \gamma_2^*, \gamma_1^* \} \), \( I_\Sigma(L_2) = \{ a_2^*; \gamma_2^*, \gamma_1^*, \gamma_3^* \} \) and \( I_\Sigma(L_3) = \{ a_3^*; \gamma_3^*-1, \gamma_1^*-1 \} \), where we write \( \gamma_j^* \) for \( \gamma_j^{*+1} \) for simplicity.

For a marked Schottky group \( <G> = \langle A_1, A_2, A_3 \rangle \), \( <G^*> = I_\Sigma(1,2)(<G>) = \langle A_1^*, A_2^*, A_3^* \rangle \), where \( A_1^* = A_1 \), \( A_2^* = A_2 A_1 \), \( A_3^* = A_3 \).

We obtain Theorem 1 by using interchange operators. See Sato [5] for details.
§ 5. Relations between limits of Schottky groups and limits of Riemann surfaces.

Here we will consider Problem 2. Let $S$ be a compact Riemann surface of genus $g$ with or without nodes. We denote by $N(S)$ the set of all nodes on $S$. We assume that each component of $S \setminus N(S)$ has the Poincaré metric. The Poincaré metric $\lambda(z)|dz|$ on $S$ is defined as the Poincaré metric on each component of $S \setminus N(S)$.

**Definition 8.** If the following conditions are satisfied, a sequence of Riemann surfaces $\{S_n\}$ converges to a surface $S$ as marked surfaces: There exists a locally quasiconformal mapping $\phi_n : S \setminus N(S) \rightarrow S_n \setminus P(S_n)$ such that (i) $\lambda_n(\phi_n(z))|d\phi_n(z)|$ uniformly converges to $\lambda(z)|dz|$ on every compact subset of $S \setminus N(S)$, where $\lambda_n(z)|dz|$ and $\lambda(z)|dz|$ are the Poincaré metrics on $S_n$ and $S$, respectively, (ii) $\phi_n$ maps a deleted neighborhood $N(a_i) \setminus \{a_i\}$ (resp. $N(\gamma_j) \setminus \{\gamma_j\}$) of $a_i$ (resp. $\gamma_j$) to a deleted neighborhood $N(a_i) \setminus \{a_i\}$ (resp. $N(\gamma_j) \setminus \{\gamma_j\}$) of $a_i$ (resp. $\gamma_j$) if $a_i \in N(S)$ (resp. $\gamma_j \in N(S)$), and (iii) $\phi_n$ maps a neighborhood $N(a_i)$ (resp. $N(\gamma_j)$) of $a_i$ (resp. $\gamma_j$) to a neighborhood $N(a_i)$ (resp. $N(\gamma_j)$) of $a_i$ (resp. $\gamma_j$) if $a_i \in N(S)$ (resp. $\gamma_j \in N(S)$), where $P(S_n) = f_n^{-1}(N(S))$ and $f_n : S_n \rightarrow S$ is a deformation.

By constructing locally quasiconformal mappings, we have Theorem 2. See Sato [6] for details.
Let \(<G_0>\) and \(\tilde{\Sigma}_0\) be a fixed marked Schottky group and a fixed B.S.J.C. for \(<G_0>\), respectively. Set \(S_0 = \Omega(G_0)/<G_0>\).

Given a point \(\tau \in \delta \mathcal{I}, J, \mathcal{G}_{g}(\tilde{\Sigma}_0)\), where \(I \supset I(\mathcal{J}) \neq \emptyset\). Then \(S(\tau)\) is a compact Riemann surface with \(|I| + |J|\) nodes of genus \(g\). We define the following sets: \(J_1 = \{j \in J| \gamma_j\) is a dividing loop on \(S_0\}\), \(J_2 = \) any subset of \(J \setminus J_1\), \(\tilde{\Sigma}_1 = I_g(i_{k(1)}, j_{l(1)})(\tilde{\Sigma}_0)\) with \(i_{k(1)} \in I_1\{j_{l(1)}\}\), \(j_{l(1)} \in J_2\) and \(J_{21} = J_2 \setminus \{j_{l(1)}\}\). Choose \(j_{l(2)} \in J_{21}\) such that \(I_1\{j_{l(2)}\}\) \(\cap (I(\mathcal{J}_2) \setminus \{i_{k(1)}\}) \neq \emptyset\). Set \(\tilde{\Sigma}_2 = I_g(i_{k(2)}, j_{l(2)})(\tilde{\Sigma}_1)\) with \(i_{k(2)} \in I_1\{j_{l(2)}\}\), \(i_{k(2)} \neq i_{k(1)}\).

We set \(J_{22} = J_{21} \setminus \{j_{l(2)}\} = J_2 \setminus \{j_{l(1)}, j_{l(2)}\}\). By the same way, we determined the following: \(j_{l(3)}, i_{k(3)}, J_{23}, \tilde{\Sigma}_3, I_3(J_{23}), \ldots, j_{l(s)}, i_{k(s)}, J_{2s}, \tilde{\Sigma}_s\). Here \(s\) is the integer satisfying the following (i) and (ii): (i) \(I_{s-1}\{j_{l(s)}\}\) \(\cap I(\mathcal{J}_2) \setminus \{i_{k(1)}, i_{k(2)}, \ldots, i_{k(s-1)}\} \neq \emptyset\), (ii) \(I_s\{j\} \subseteq \{i_{k(1)}, \ldots, i_{k(s)}\}\) for any \(j \in J_2 \setminus \{j_{l(1)}, j_{l(2)}, \ldots, j_{l(s)}\}\).

We set \(J_3 = J \setminus (J_1 \cup J_2), J_4 = \{j_{l(1)}, j_{l(2)}, \ldots, j_{l(s)}\}, J_5 = J_2 \setminus J_4, I_1 = I \setminus I(\mathcal{J}), I_4 = \{i_{k(1)}, i_{k(2)}, \ldots, i_{k(s)}\}, I_3 = I_s(J_3), I_5 = I \setminus (I_1 \cup I_3 \cup I_4), I_6 = \) a subset of \(I_5\), \(I_7 = I_5 \setminus I_6, I^* = I \setminus I_7\) and \(J^* = J \setminus J_4\). Then we have Theorem 3. See Sato [6] for the proof.

**COROLLARY.** Given \(\tau \in \delta \mathcal{I}, J, \mathcal{G}_{g}(\tilde{\Sigma}_0)\), where \(I \supset I(\mathcal{J}) \neq \emptyset\). Then there exists a sequence of points \(\{\tau_n\} \subset \mathcal{G}_{g}(\tilde{\Sigma}_0)\) such that (i) \(\tau_n \to \tau\) as \(n \to \infty\) and (ii) \(S(\tau_n)\) does not converge to \(S(\tau)\) as marked surfaces.

**Remark.** By a similar method to the proof of Theorem 2, we
have the following. If \( \tilde{\Sigma}_0 \) is a S.S.J.C., then \( S(\tau_n) \) converges to \( S(\tau) \) as marked surfaces for any point \( \tau \in \hat{\mathbb{G}}_g(\tilde{\Sigma}_0) \) and for any sequence of points \( \{\tau_n\} \subset \mathbb{G}_g(\tilde{\Sigma}_0) \) with \( \tau_n \to \tau \).

§6. Appendices.

We will consider the following in the forthcoming papers [7,8].

1. Properties of interchange operators. There are five kind of interchange operators as follows: (1) \( I_g(a_i,a_i^{-1}) = I_g(C_i,C_{g+i}) \), (2) \( I_g(a_i,a_j) = I_g(C_i,C_j) \), (3) \( I_g(\gamma_j,\gamma_j^{-1}) = I_g(C_{2g+j},C_{2g+j}) \), (4) \( I_g(\gamma_i,\gamma_j) = I_g(C_{2g+i},C_{2g+j}) \) and (5) \( I_g(a_i,\gamma_j) = I_g(C_i,C_{2g+j}) \). Here we only considered and used interchanged operators in case (5).

2. Relations between Nielsen isomorphisms and interchange operators. Here Nielsen isomorphisms are

\[
N_1(A_1,A_i) : \langle A_1,A_2,\cdots,A_i,\cdots,A_g \rangle \to \langle A_1,A_2,\cdots,A_i,\cdots,A_g \rangle .
\]

\[
N_2(A_1,A_i^{-1}) : \langle A_1,A_2,\cdots,A_g \rangle \to \langle A_i^{-1},A_2,\cdots,A_g \rangle .
\]

\[
N_3(A_1,A_2) : \langle A_1,A_2,A_3,\cdots,A_g \rangle \to \langle A_1,A_1A_2,A_3,\cdots,A_g \rangle .
\]

3. Boundary behavior of the space of marked Schottky groups of real type of genus 2. We say \( \langle G \rangle = \langle A_1,A_2 \rangle \) a schottky group of real type if \( A_1, A_2 \in \text{SL}(2,\mathbb{R}) \).

References

[1] W. Abikoff, Degenerating families of Riemann surfaces, Ann. of
Math. 105 (1977), 29-44.


Example 1.

\[ \langle G \rangle = \langle A_1, A_2, A_3 \rangle \]

B.S.J.C.  S.S.J.C.
\[ \Sigma = \{ C_1, \ldots, C_6; C_7, C_8, C_9 \} \]

Example 2.

\[ C_4 = C(0) \]
\[ C_2 = C(1, 0, 0) \]
\[ C_1 \]
\[ C_7 \]
\[ C_9 \]
\[ C_5 = C(1, 0, 1) \]
\[ C_3 = C(1, 1, 0) \]
\[ C_6 = C(1, 1, 1) \]

Fig. 1
Example 1.

\[(\emptyset, \emptyset), (\{1\}, \emptyset), (\{2\}, \emptyset), (\{3\}, \emptyset), (\{1, 2\}, \emptyset), (\{1, 3\}, \emptyset), (\{2, 3\}, \emptyset), (\{1, 2, 3\}, \emptyset)\]
\[(\{1, 2, 3\}, \emptyset), (\{1, 2, 3\}, \{1\}), (\{1, 2, 3\}, \{2\}), (\{1, 2, 3\}, \{3\}), (\{1, 2, 3\}, \{1, 2\}), (\{1, 2, 3\}, \{1, 3\}), (\{1, 2, 3\}, \{2, 3\}), (\{1, 2, 3\}, \{1, 2, 3\})\]

*Fig. 2.*

Example 2.

\[(\emptyset, \emptyset), (\{1\}, \emptyset), (\{2\}, \emptyset), (\{3\}, \emptyset), (\{1, 2\}, \emptyset), (\{1, 3\}, \emptyset), (\{2, 3\}, \emptyset), (\{1, 2, 3\}, \emptyset)\]
\[(\emptyset, \{1\}), (\{1\}, \{1\}), (\{2\}, \{1\}), (\{3\}, \{1\}), (\{1, 2\}, \{1\}), (\{1, 3\}, \{1\}), (\{2, 3\}, \{1\}), (\{1, 2, 3\}, \{1\})\]
\[(\emptyset, \{2\}), (\{1\}, \{2\}), (\{2\}, \{2\}), (\{3\}, \{2\}), (\{1, 2\}, \{2\}), (\{1, 3\}, \{2\}), (\{2, 3\}, \{2\}), (\{1, 2, 3\}, \{2\})\]
\[(\emptyset, \{3\}), (\{1\}, \{3\}), (\{2\}, \{3\}), (\{3\}, \{3\}), (\{1, 2\}, \{3\}), (\{1, 3\}, \{3\}), (\{2, 3\}, \{3\}), (\{1, 2, 3\}, \{3\})\]
\[(\emptyset, \{1, 2\}), (\{1\}, \{1, 2\}), (\{2\}, \{1, 2\}), (\{3\}, \{1, 2\}), (\{1, 2\}, \{1, 2\}), (\{1, 3\}, \{1, 2\}), (\{2, 3\}, \{1, 2\}), (\{1, 2, 3\}, \{1, 2\})\]

*Fig. 3.*
$\Sigma^* = I_g(1,2)(\Sigma) = \{c_1^*, \ldots, c_6^*, c_7^*, c_8^*, c_9^*\}$

$E^* = I_g(1,2)(\Sigma) = \{a_1^*, a_2^*, a_3^*, \gamma_1^*, \gamma_2^*, \gamma_3^*\}$

Fig. 4.