ON AUGMENTED SCHOTTKY SPACES AND INTERCHANGE OPERATORS

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§0. Introduction.

Problem 1. Let $S$ be a compact Riemann surface with nodes. Does there exist a point in an augmented Schottky space representing the surface $S$?

Problem 2. We give a point $\tau$ in an augmented Schottky space $\mathcal{G}_g(\Sigma_0)$ associated with a basic system of Jordan curves $\Sigma_0$, which represents a compact Riemann surface $S$ with nodes. Then for any sequence of points $\{\tau_n\}$ in the Schottky space $\mathcal{G}_g(\Sigma_0)$ tending to the point $\tau$, does the Riemann surface $S(\tau_n)$ represented by $\tau_n$ converge to $S$ as marked surfaces as $n \to \infty$?

The answer to Problem 1 is affirmative:

THEOREM 1. There exists a point in an augmented Schottky space which represents a given Riemann surface with nodes.

The answer to Problem 2 is negative in the general case, namely in the case where $\Sigma_0$ is a basic system of Jordan curves. However the answer is affirmative in a special case, namely in the case where $\Sigma_0$ is a standard system of Jordan curves. Now the following question arises: To what Riemann surface does the
sequence of Riemann surfaces \( \{S(\tau_n)\} \) converge as marked surface as \( n \to \infty \) in the general case?

**Theorem 2.** Given a point \( \tau \in \tilde{G}_g(\tilde{\Sigma}_0) \). Then there exists a sequence of points \( \{\tau_n\} \subset G_g(\tilde{\Sigma}_0) \) tending to \( \tau \) such that \( S(\tau_n) \) converges to \( S(\tau) \) as marked surfaces.

**Theorem 3.** Let \( <G_0> \) and \( \tilde{\Sigma}_0 \) be a fixed marked Schottky group and a fixed basic system of Jordan curves for \( <G_0> \), respectively. Given a point \( \tau \in \delta I, J G_g(\tilde{\Sigma}_0) \), where \( I \supset I(J) \neq \emptyset \). Let \( \tilde{\Sigma}_0^*, I^*, \) and \( J^* \) be a basic system of loops, a subset of \( I \), and a subset of \( J \), respectively, obtained from \( \tilde{\Sigma}_0, I \) and \( J \) by applying certain interchange operators. Let \( \tau^* \in \delta I^*, J^* G_g(\tilde{\Sigma}_0^*) \) be a point representing a compact Riemann surface with \( |I^*| + |J^*| \) nodes. Then there exists the following sequence of points \( \{\tau_n\} \subset G_g(\tilde{\Sigma}_0) \):

\[
\tau_n \to \tau \text{ and } S(\tau_n) \to S(\tau^*) \text{ as } n \to \infty,
\]
as marked surfaces.

§1. Definitions.

**Definition 1.** Let \( C_1, C_{g+1}; C_2, C_{g+2}; \ldots; C_g, C_{2g} \) be a set of \( 2g \) mutually disjoint Jordan curves on the Riemann sphere \( \hat{\mathbb{C}} \) which comprise the boundary of a \( 2g \)-ply connected region \( \omega \). Suppose there are \( g \) Möbius transformations \( A_1, \ldots, A_g \) which have the property that \( A_j \) maps \( C_j \) onto \( C_{g+j} \) and \( A_j(\omega) \cap \omega = \emptyset \) (\( 1 \leq j \leq g \)). Then \( A_j \) (\( j = 1, 2, \ldots, g \)) generates a marked Schottky.
group \( \langle G \rangle = \langle A_1, A_2, \cdots, A_g \rangle \). \( C_1, \cdots, C_{2g} \) are called defining curves of \( \langle G \rangle \).

We say two marked Schottky groups \( \langle G \rangle = \langle A_1, \cdots, A_g \rangle \) and \( \langle \hat{G} \rangle = \langle \hat{A}_1, \cdots, \hat{A}_g \rangle \) being equivalent if there exists a Möbius transformation \( T \) such that \( \hat{A}_j = TA_jT^{-1} \) \( (j=1,2,\cdots,g) \), and we denote it by \( \langle G \rangle \sim \langle \hat{G} \rangle \).

**DEFINITION 2.** The Schottky space of genus \( g \), denoted by \( \mathcal{G}_g \), is the set of all equivalent classes of Schottky groups of genus \( g \geq 1 \).

**DEFINITION 3.** Let \( C_1, \cdots, C_{2g} \) be defining curves of \( \langle G \rangle = \langle A_1, \cdots, A_g \rangle \). If mutually disjoint Jordan curves \( C_1, \cdots, C_{2g}; C_{2g+1}, \cdots, C_{4g-3} \) on \( \hat{\mathcal{C}} \) have the following properties (i) and (ii), then we call \( \mathcal{E} = \{ C_1, \cdots, C_{2g}; C_{2g+1}, \cdots, C_{4g-3} \} \) a basic system of Jordan curves (B.S.J.C.) for \( \langle G \rangle \): (i) \( C_{2g+j} \) \( (j=1, \cdots, 2g-3) \) lie in \( \omega \). (ii) Each component of \( \hat{\mathcal{C}} \setminus \bigcup_{j=1}^{2g-3} C_{2g+j} \) is a triply connected domain. In particular, if a B.S.J.C. \( \mathcal{E} \) has the following property (iii), we call \( \mathcal{E} \) a standard system of Jordan curves (S.S.J.C.) for \( \langle G \rangle \): (iii) For each \( i=1,2,\cdots,g \) and \( j=1,2,\cdots,2g-3 \), \( C_i \) and \( C_{g+i} \) lie on the same side of \( C_{2g+j} \). See Examples 1 and 2 on p.13.

**DEFINITION 4.** Let \( S \) be a compact Riemann surface. We call the set \( \mathcal{E} = \{ a_1, \cdots, a_g; \gamma_1, \cdots, \gamma_{2g-3} \} \) of loops on \( S \) having the following property a basic system of loops (B.S.L.): Each component of \( S \setminus \bigcup_{i=1}^{g} a_i \setminus \bigcup_{j=1}^{2g-3} \gamma_j \) is a planar and triply connected region. If, in particular, the number of nondividing loops is equal
to \( g \), we call a B.S.L. \( \Sigma \) a **standard system of loops** (S.S.L.).

Let \( \Omega(G) \) be the region of discontinuity of \( \langle G \rangle \). Let \( \Pi : \Omega(G) \to \Omega(G)/\langle G \rangle = S \) be the natural projection. If \( \tilde{\Sigma} = \{ C_1, \cdots, C_{2g}; C_{2g+1}, \cdots, C_{4g-3} \} \) is a B.S.J.C. (resp. S.S.J.C.), then the projection \( \Sigma = \Pi(\tilde{\Sigma}) = \{ \alpha_1, \cdots, \alpha_g; \gamma_1, \cdots, \gamma_{2g-3} \} \), \( \alpha_i = \Pi(C_i) \) and \( \gamma_j = \Pi(C_{2g+j}) \), is a B.S.L. (resp. S.S.L.). We call \( \Sigma \) the projection of \( \tilde{\Sigma} \). See Examples 1 and 2 on p.13.

§2. **Introduction of new coordinates to \( \mathcal{G}_g \).**

We fix a marked Schottky group \( \langle G_0 \rangle = \langle A_1, 0, \cdots, A_g, 0 \rangle \). Let \( \tilde{\Sigma}_0 = \{ C_1, 0, \cdots, C_{2g}, 0; C_{2g+1}, 0, \cdots, C_{4g-3}, 0 \} \) be a fixed B.S.J.C. for \( \langle G_0 \rangle \). Let \( \langle G \rangle = \langle A_1, \cdots, A_g \rangle \) be a marked Schottky group. Let \( \lambda_j (|\lambda_j| > 1) \), \( p_j \) and \( p_{g+j} \) be the multiplier, the repelling and the attracting fixed points of \( A_j \), respectively. We normalize \( \langle G \rangle \) by setting \( p_1 = 0 \), \( p_{g+1} = \infty \) and \( p_2 = 1 \). Then a point in the Schottky space \( \mathcal{G}_g \) is identified with

\[
\tilde{\tau} = (\lambda_1, \cdots, \lambda_g, p_{g+2}, p_3, p_{g+3}, \cdots, p_g, p_{2g}) \in \mathbb{C}^{3g-3}.
\]

Now we will introduce new coordinates with respect to \( \tilde{\Sigma}_0 \):

\[
\tau = (t_1, t_2, \cdots, t_g, \rho_1, \cdots, \rho_{2g-3}) \in \mathbb{C}^{3g-3}.
\]

First define \( t_i \) by setting \( t_i = 1/\lambda_i \) (\( i=1, \cdots, g \)). Thus \( t_i \in D^* = \{ z | 0 < |z| < 1 \} \). Next in order to define \( \rho_j \) associated with \( C_{2g+j} = C(i_0, i_1, \cdots, i_{\mu}) \in \tilde{\Sigma}_0 \) (\( j=1, 2, \cdots, 2g-3 \)), we determine integers \( k(j), \ell(j), m(j) \) and \( n(j) \) which are \( \geq 1 \) and \( \leq 2g \) as
follows, where \( C(i_0, i_1, \ldots, i_\mu) \) is the multi-suffix of \( C_{2g+j} \) (see [4] for the definition): \( k(j) = 1, \quad C(1) = C(i_0, i_1, \ldots, i_{\mu-1}, 0, \ldots, 0), \quad C_m(j) = C(i_0, i_1, \ldots, i_\mu, 0, \ldots, 0) \) and \( C_n(j) = C(i_0, i_1, \ldots, i_\mu, 0, \ldots, 0) \). The coordinate \( \rho_j \) is now defined as follows:

\[
(P_k(j), P_L(j), P_m(j), P_n(j)) = (0, 1, \infty, \rho_j),
\]

where \((a, b, c, d)\) means the cross ratio of \(a, b, c,\) and \(d\).

We define a mapping \( \phi \) by \( \phi(<G>) = \tau \). We note that if \( <G> \sim <\hat{G}> \), then \( \phi(<G>) = \phi(<\hat{G}>) \). We denote by \( \mathcal{G}_g(\tilde{\Sigma}_0) \) the set

\[
\mathcal{G}_g(\tilde{\Sigma}_0) = \{ \tau = \phi(<G>) | <G> \in \mathcal{G}_g \}.
\]

Then \( \mathcal{G}_g(\tilde{\Sigma}_0) \cong \mathcal{G}_g \) and \( \mathcal{G}_g(\tilde{\Sigma}_0) \subset D^g \times (C \backslash \{0, 1\})^{2g-3} \). We call \( \mathcal{G}_g(\tilde{\Sigma}_0) \) the Schottky space associated with \( \tilde{\Sigma}_0 \).

§3. Augmented Schottky spaces.

Let \( <G_0> \) and \( \tilde{\Sigma}_0 \) be a fixed Schottky group and a fixed B.S.J.C. as in §2.

**DEFINITION 5.** We say \( C_{2g+j} = C(i_1, \ldots, i_\mu) \) (resp. \( C_i = C(j_1, \ldots, j_\nu) \)) is **behind** \( C_{2j+\ell} = C(i_1', \ldots, i_\nu) \) if \( \nu < \mu \) and \( i_k = i_k' \) (\( k=1, 2, \ldots, \nu \)) (resp. \( \nu < \sigma \) and \( j_k = j_k' \) (\( k=1, 2, \ldots, \nu \))), and denote the fact \( C_{2g+\ell} < C_{2g+j} \) (resp. \( C_{2g+\ell} < C_i \)). Otherwise, we say that \( C_{2g+j} \) (resp. \( C_i \)) is **not behind** \( C_{2g+\ell} \) and we denote the fact by \( C_{2g+j} \not\leq C_{2g+j} \) (resp. \( C_{2g+\ell} \not\leq C_i \)).

We define the ordered cycle corresponding to \( a_i \) as follows.
We denote the shortest path from $C_i$ to $C_{g+i}$ on the tree of $\tilde{\Sigma}_0$ by

$$C_i, C_{2g+1}^\delta(1), C_{2g+1}^\delta(2), \cdots, C_{2g+1}^\delta(k), C_{g+i}$$

(see [4] and Fig. 1 on p.13 in this paper for trees.) Here $\delta(\lambda)$ ($\lambda=1, 2, \cdots, k$) are determined by $\delta(\lambda) = +1$ or $\delta(\lambda) = -1$ according as $C_{2g+\lambda} < C_{g+i}$ or $C_{2g+\lambda} < C_i$.

**DEFINITION 6.** The projection

$$(a_i^\alpha; \gamma_i^\delta(1), \cdots, \gamma_i^\delta(k))$$

of (1) onto $S_0 = \Omega(G_0)/<G_0>$ is called the ordered cycle corresponding to $a_i^\alpha$, and is denoted by $L_{0,i}$.

Let $I$ be a subset of $\{1, 2, \cdots, g\}$ and $J$ a subset of $\{1, 2, \cdots, 2g-3\}$. We denote by $|I|$ and $|J|$ the cardinality of $I$ and $J$, respectively. Let $L_{0,j}(1), L_{0,i}(2), \cdots, L_{0,i}(t)$ be the complete list of cycles containing $\gamma_j^\delta$, and let $a_{0,k}$ be the "$a$-loops" contained in $L_{0,k}$ ($1 \leq k \leq t$), where $t = t(j)$ depends on $j$. We define the subset $I(J)$ of $\{1, 2, \cdots, g\}$ by

$$I(J) = \{i \in \{1, 2, \cdots, g\} | a_{0,i} \text{ is contained in } L_{0,j}(k) \text{ for some } k (1 \leq k \leq t(j)) \text{ and for some } j \in J\}.$$

**Remark.** If $\tilde{\Sigma}_0$ is a S.S.J.C., then $I(J) = \emptyset$.

We define the following sets $X = \delta_I, J \mathcal{G}_g(\tilde{\Sigma}_0)$ with $I \supset I(J)$:

(i) When $I = J = \emptyset$, we define $X$ as $\mathcal{G}_g(\tilde{\Sigma}_0)$, the Schottky space associated with $\tilde{\Sigma}_0$. 
(ii) When $I \neq \emptyset, j = \emptyset$, we define $X$ as follows:
\[
\tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \cdots, t_g, \rho_1, \cdots, \rho_{2g-3}) \mid t_1 = 0 \ (i \in I), \ t_i \neq 0 \ (i \notin I), \ \rho_j \neq 1 \ (j = 1, \cdots, 2g-3), \ \text{and } \tau \ \text{represents a Riemann surface with nodes such that only } \alpha_i \ (i \in I) \ \text{are nodes} \}.
\]

(iii) When $I = \emptyset, J \neq \emptyset$, we define $X$ as follows:
\[
\tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \cdots, t_g, \rho_1, \cdots, \rho_{2g-3}) \mid t_i \neq 0 \ (i = 1, \cdots, g), \ \rho_j = 1 \ (j \in J), \ \rho_j \neq 1 \ (j \notin J) \ \text{and } \tau \ \text{represents a Riemann surface with nodes such that only } \gamma_j \ (j \in J) \ \text{are nodes} \}.
\]

(iv) When $I \supset I(J) \neq \emptyset$, $X$ is defined as follows:
\[
\tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) = \{ \tau = (t_1, \cdots, t_g, \rho_1, \cdots, \rho_{2g-3}) \mid t_1 = 0 \ (i \in I), \ t_i \neq 0 \ (i \notin I), \ \rho_j = 1 \ (j \in J), \ \rho_j \neq 1 \ (j \notin J) \ \text{and } \rho \ \text{represents a compact Riemann surface such that only } \alpha_i \ (i \in I) \ \text{and } \gamma_j \ (j \in J) \ \text{are nodes} \}.
\]

**DEFINITION 7.**
\[
\hat{\mathcal{G}}_g(\tilde{\Sigma}_0) = \bigcup \{ \tilde{\mathcal{G}}_g(\tilde{\Sigma}_0) \mid I \subset \{1, 2, \cdots, g\}, J \subset \{1, 2, \cdots, 3g-3\} \}
\]
with $I \supset I(J)$

is called the augmented Schottky space associated with $\tilde{\Sigma}_0$.

**Remark.** Let $S(\tau)$ be the Riemann surface represented by $\tau$.
\[
\{ S(\tau) \mid \tau \in \hat{\mathcal{G}}_g(\tilde{\Sigma}_0) \}
\]
is the sets of all Riemann surfaces in Fig.2 and Fig.3 in the cases of Example 1 and Example 2, respectively.
§ 4. Interchange operators.

For simplicity, we will only consider interchange operators in the case of Example 1 (see Fig.4). For detail, see Sato [5]. Choose j with \( I(\{j\}) \neq \emptyset \). Let \( i \in I(\{j\}) \). For these \( i \) and \( j \), we introduce the interchange operators \( I_g(i,j) \).

Remark. Since \( I(J) \) is always empty in the case where \( \Sigma \) is a S.S.J.C., we can not define an interchange operator in this case.

For simplicity, we only consider \( I_g(1,2) \), which is defined as follows (see Fig.4 on p.15): For a B.S.J.C. \( \Sigma \),

\[
I_g(1,2)(\Sigma) = \Sigma^* = \{ C_1^*, C_2^*, \ldots, C_6^*, C_7^*, C_8^*, C_9^* \},
\]

where \( C_1^* = A_1^{-1}(C_9) \), \( C_2^* = A_1^{-1}(C_2) \), \( C_3^* = C_3 \), \( C_4^* = C_8 \), \( C_5^* = C_5 \), \( C_6^* = C_6 \), \( C_7^* = C_7 \), \( C_8^* = C_1 \), and \( C_9^* = C_9 \).

For a B.S.L. \( \Sigma = \{ a_1, a_2, a_3; \gamma_1, \gamma_2, \gamma_3 \} \), \( I_g(1,2)(\Sigma) = \{ a_1^*, a_2^*, a_3^*, \gamma_1^*, \gamma_2^*, \gamma_3^* \} \), where \( a_1^* = \gamma_2 \), \( a_2^* = a_2 \), \( a_3^* = a_3 \), \( \gamma_1^* = \gamma_1 \), \( \gamma_2^* = \gamma_2 \), \( \gamma_3^* = \gamma_3 \).

For ordered cycles \( L_1, L_2 \) and \( L_3 \), \( L_1^* = I_g(1,2)(L_1) = (a_1^*, \gamma_2^*, \gamma_1^*) \), \( L_2^* = I_g(1,2)(L_2) = (a_3^*, \gamma_2^*, \gamma_1^*, \gamma_3^*) \) and \( L_3^* = I_g(1,2)(L_3) = (a_3^*, \gamma_3^*, \gamma_1^*, \gamma_3^*) \), where we write \( \gamma_j^* \) for \( \gamma_j^{**} \) for simplicity.

For a marked Schottky group \( \langle G \rangle = \langle A_1, A_2, A_3 \rangle \), \( \langle G^* \rangle = I_g(1,2)(\langle G \rangle) = \langle A_1^*, A_2^*, A_3^* \rangle \), where \( A_1^* = A_1 \), \( A_2^* = A_2 A_1 \), \( A_3^* = A_3 \).

We obtain Theorem 1 by using interchange operators. See Sato [5] for details.
§ 5. Relations between limits of Schottky groups and limits of Riemann surfaces.

Here we will consider Problem 2. Let $S$ be a compact Riemann surface of genus $g$ with or without nodes. We denote by $N(S)$ the set of all nodes on $S$. We assume that each component of $S \setminus N(S)$ has the Poincaré metric. The Poincaré metric $\lambda(z)|dz|$ on $S$ is defined as the Poincaré metric on each component of $S \setminus N(S)$.

**Definition 8.** If the following conditions are satisfied, a sequence of Riemann surfaces $\{S_n\}$ converges to a surface $S$ as marked surfaces: There exists a locally quasiconformal mapping $\phi_n: S \setminus N(S) \to S_n \setminus P(S_n)$ such that (i) $\lambda_n(\phi_n(z))|d\phi_n(z)|$ uniformly converges to $\phi(z)|dz|$ on every compact subset of $S \setminus N(S)$, where $\lambda_n(z)|dz|$ and $\lambda(z)|dz|$ are the Poincaré metrics on $S_n$ and $S$, respectively, (ii) $\phi_n$ maps a deleted neighborhood $N(a_i) \setminus \{a_i\}$ (resp. $N(\gamma_j) \setminus \{\gamma_j\}$) of $a_i$ (resp. $\gamma_j$) to a deleted neighborhood $N(a_i, n) \setminus \{a_i, n\}$ (resp. $N(\gamma_j, n) \setminus \{\gamma_j, n\}$) of $a_i, n$ (resp. $\gamma_j, n$) if $a_i \in N(S)$ (resp. $\gamma_j \in N(S)$), and (iii) $\phi_n$ maps a neighborhood $N(a_i)$ (resp. $N(\gamma_j)$) of $a_i$ (resp. $\gamma_j$) to a neighborhood $N(a_i, n)$ (resp. $N(\gamma_j, n)$) of $a_i, n$ (resp. $\gamma_j, n$) if $a_i \notin N(S)$ (resp. $\gamma_j \notin N(S)$), where $P(S_n) = f_n^{-1}(N(S))$ and $f_n: S_n \to S$ is a deformation.

By constructing locally quasiconformal mappings, we have Theorem 2. See Sato [6] for details.
Let $<G_0>$ and $\tilde{\Sigma}_0$ be a fixed marked Schottky group and a fixed B.S.J.C. for $<G_0>$, respectively. Set $S_0 = \Omega(G_0)/<G_0>$. Given a point $\tau \in \delta I, J G_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Then $S(\tau)$ is a compact Riemann surface with $|I| + |J|$ nodes of genus $g$. We define the following sets: $J_1 = \{j \in J \mid \gamma_j$ is a dividing loop on $S_0\}$, $J_2 = \text{any subset of } J \setminus J_1$, $\tilde{\Sigma}_1 = I_g(i_k(1), j_k(1))(\tilde{\Sigma}_0)$ with $i_k(1) \in I(J_1)$, $j_k(1) \in J_2$ and $J_21 = J_2 \setminus \{j_k(1)\}$. Choose $j_k(2) \in J_21$ such that $I_1(\{j_k(2)\}) \cap (I(J_2) \setminus \{i_k(1)\}) \neq \emptyset$. Set $\tilde{\Sigma}_2 = I_g(i_k(2), j_k(2))(\tilde{\Sigma}_1)$ with $i_k(2) \in I_1(\{j_k(2)\})$, $i_k(2) \neq i_k(1)$. We set $J_22 = J_21 \setminus \{j_k(2)\} = J_2 \setminus \{j_k(1), j_k(2)\}$. By the same way, we determined the following: $j_k(3), i_k(3), J_23, \tilde{\Sigma}_3, I_3(J_23); \ldots$ $j_k(s), i_k(s), J_2s, \tilde{\Sigma}_s$: Here $s$ is the integer satisfying the following (i) and (ii): (i) $I_{s-1}(\{j_k(s)\}) \cap I(J_2) \setminus \{i_k(1), i_k(2), \ldots, i_k(s-1)\} \neq \emptyset$, (ii) $I_s(\{j\}) \subseteq \{i_k(1), \ldots, i_k(s)\}$ for any $j \in J_2 \setminus \{j_k(1), j_k(2), \ldots, j_k(s)\}$.

We set $J_3 = J \setminus (J_1 U J_2)$, $J_4 = \{j_k(1), j_k(2), \ldots, j_k(s)\}$, $J_5 = J_2 \setminus J_4$, $I_1 = I \setminus I(J)$, $I_4 = \{i_k(1), i_k(2), \ldots, i_k(s)\}$, $I_3 = I_s(\tilde{\Sigma}_3)$, $I_5 = I \setminus (I_1 U I_3 U I_4)$, $I_6$ is a subset of $I_5$, $I_7 = I_5 \setminus I_6$, $I^* = I \setminus I_7$ and $J^* = J \setminus J_4$. Then we have Theorem 3. See Sato [6] for the proof.

**COROLLARY.** Given $\tau \in \delta I, J G_g(\tilde{\Sigma}_0)$, where $I \supset I(J) \neq \emptyset$. Then there exists a sequence of points $\{\tau_n\} \subset G_g(\tilde{\Sigma}_0)$ such that (i) $\tau_n \rightarrow \tau$ as $n \rightarrow \infty$ and (ii) $S(\tau_n)$ does not converge to $S(\tau)$ as marked surfaces.

**Remark.** By a similar method to the proof of Theorem 2, we
have the following. If $\tilde{\tau}_0$ is a S.S.J.C., then $S(\tau_n)$ converges
to $S(\tau)$ as marked surfaces for any point $\tau \in \hat{G}_g(\tilde{\tau}_0)$ and for
any sequence of points $\{\tau_n\} \subset \hat{G}_g(\tilde{\tau}_0)$ with $\tau_n \to \tau$.

§6. Appendices.

We will consider the following in the forthcoming papers [7,8].

1. Properties of interchange operators. There are five kind
of interchange operators as follows: (1) $I_g(\alpha_i, \alpha_i^{-1}) = I_g(C_i, C_{g+i})$,
(2) $I_g(\alpha_i, \alpha_j) = I_g(C_i, C_j)$, (3) $I_g(\gamma_j, \gamma_j^{-1}) = I_g(C_{2g+j}, C_{2g+j})$,
(4) $I_g(\gamma_i, \gamma_j) = I_g(C_{2g+i}, C_{2g+j})$ and (5) $I_g(\alpha_i, \gamma_j) = I_g(C_i, C_{2g+j})$.
Here we only considered and used interchanged operators in case (5).

2. Relations between Nielsen isomorphisms and interchange
operators. Here Nielsen isomorphisms are

$N_1(A_1, A_i) : \langle A_1, A_2, \ldots, A_i, \ldots, A_g \rangle \to \langle A_1, A_2, \ldots, A_i, \ldots, A_g \rangle$.

$N_2(A_1, A_i^{-1}) : \langle A_1, A_2, \ldots, A_g \rangle \to \langle A_i^{-1}, A_2, \ldots, A_g \rangle$.

$N_3(A_1, A_2) : \langle A_1, A_2, A_3, \ldots, A_g \rangle \to \langle A_1, A_1A_2, A_3, \ldots, A_g \rangle$.

3. Boundary behavior of the space of marked Schottky groups
of real type of genus 2. We say $\langle G \rangle = \langle A_1, A_2 \rangle$ a schottky group
of real type if $A_1, A_2 \in SL(2, \mathbb{R})$.

References

[1] W. Abikoff, Degenerating families of Riemann surfaces, Ann. of
Math. 105 (1977), 29-44.


Example 1.

\[ \langle G \rangle = \langle A_1, A_2, A_3 \rangle \]

\[ \Sigma = C_1, \ldots, C_6; C_7, C_8, C_9 \]

Example 2.

\[ \text{tree} \]

\[ \alpha_1, \alpha_2, \alpha_3 \]

Fig. 1
Example 1.

Example 2.

Fig. 2.

Fig. 3.
\[ \hat{\Sigma} = I_g(1,2)(\Sigma) = \{c_1^*, \ldots, c_6^*, c_7, c_8, c_9\} \]
\[ \Sigma^* = I_g(1,2)(\Sigma) = \{a_1^*, a_2^*, a_3^*, \gamma_1^*, \gamma_2^*, \gamma_3^*\} \]

Fig. 4.