数理解析研究所講究録
571巻 1985年 11-15

Brownian Motions on Riemann Surfaces of Inverse Functions

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§1. Introduction.

Let $B = (B_t, t \geq 0)$ be a complex Brownian motion starting at 0 defined on a probability space $(\Omega, \mathbf{F}, P)$ and f be a non-constant analytic function in the unit disc Δ. Define φ_t and $\mathbf{W}$ by

$$\varphi_t = \int_0^t |f'(B_s)|^2 ds,$$

up to the first exit time σ of B from Δ and

$$\mathbf{W} = (W_t) = (f(B_{\varphi_t^{-1}})).$$

Then the process $\mathbf{W}$ is also a Brownian motion up to the time φ_σ. It is known that $E[\varphi(\sigma)^{p/2}] \approx \|f\|_p$ for $0 < p < \infty$ (Burkholder, Gundy and Silverstein [2]). In 1979 Davis [3] noted that φ_σ is the first exit time of the Euclidean Brownian motion $\mathbf{W}$ from $f(\Delta)$. Precisely let S be the Riemann surface of f^{-1} such that S is a covering surface of $f(\Delta)$ with the natural projection p and that there exists a one-to-one onto mapping f^{-1} with $f^{-1} \circ f = p$. Such a surface is called the Riemann surface of inverse function. The

Brownian motion $\mathbf{W} = (W_t)$ can be lifted continuously on S. Let $\mathbf{W}^*$ $= (W_t^*)$ be the lifted Brownian motion on S. Since the generator of $\mathbf{W}^*$ is 1/2 times the Laplace-Beltrami operator corresponding to the pull-backed metric on S from the Euclidean metric on $f(\Delta)$, $\mathbf{W}^*$ is a Brownian motion corresponding to this metric and φ_σ is the first exit time of $\mathbf{W}^*$ from S.

In the present paper we shall study analogously spherical Brownian motions on Riemann surfaces of inverse functions.

§1. Result.

Let $w = f(z)$ be a non-constant meromorphic function in the z-plane to the w-sphere. We may regard f and its restriction $f|_{\{|z|<r\}}$ as one-to-one onto mappings from the complex plane C and $\{|z|<r\}$ onto Riemann surfaces of inverse functions S and S_r respectively. We may assume $S_r \subset S$. Now we can define a spherical metric on S by

$$\rho(w^*)dw^*d\bar{w}^* = \frac{dwd\bar{w}}{(1+|w|^2)^2},$$

for each local coordinate w^* with $w = p(w^*)$. Let A denote the spherical area on S, then

$$A(r,f) \equiv A(S_r) = \int_{|z|\leq r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\, dxdy.$$

Define the Ahlfors-Shimizu characteristic $T(r,f)$ by

$$T(r,f) = \int_0^r \frac{A(x,f)}{x}\, dx.$$

Then it is well-known that

$$T(r,f) = \frac{1}{\pi}\int_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2}\, g(z)\, dxdy,$$

where g is the Green's function of $\{|z|<r\}$ with a pole at 0 and $z = x+iy$.

Let $w_0^* = f(0) \in S_r$. The spherical metric ρ does not only define $A(r,f)$ and $T(r,f)$ but also generates a Brownian motion $W^* = (W_t)$ starting at w_0^* on S defined on some probability space (Ω^*, F^*, P^*) such that

$$\lim_{t\downarrow 0} \frac{1}{t} E^*[\ u(W_t^*) - u(w_o^*)\ ] = \frac{1}{2}(L_\rho u)(w_0^*), \qquad (2.1)$$

for each C^2-bounded function u on S where E^* denotes the mathematical expectation with respect to P^* and L_ρ is the Laplace-Beltrami operator corresponding to ρ. Let σ_r^* be the first exit time of W^* from S_r. Then we have,

Theorem. For each r, r > 0, it holds

$$E^*[\ \sigma_r^*\ ] = T(r,\ f).$$

§3. Proof. We can construct W^* by the standard time change-argument (Blumenthal and Getoor [1] p.212). Define φ_t by

$$\varphi_t = \int_0^t \frac{|f'(B_s)|^2}{(1+|f(B_s)|^2)^2}\, ds,$$

and put $\psi_t = \varphi_t^{-1}$. Then $W = (W_t) \equiv (f(B_{\psi_t}))$ is a spherical Brownian motion on the w-sphere. Let $W^* = (W_t^*)$ be a lifted process of W such that W^* has continuous paths a.s. with $p(W_t^*) = W_t$ and $W_0^* = w_0^*$. Without loss of generality we assume $f'(0) \neq 0$. Then a simple application of Itô's formula (Ikeda and Watanabe [4] p.66) shows (2.1). Since σ_r^* is the first exit time of W^* from S_r, we have

$$\sigma_r^* = \inf\{\ t\ ;\ W_t^* \in S_r\ \}$$

$$= \inf \{ t ; f^{-1}(W_t^*) \in f^{-1}(S_r) \}$$

$$= \inf \{ t ; |B_{\psi_t}| \geq r \}$$

$$= \inf \{ \varphi_t ; |B_t| \geq r \}$$

$$= \varphi_{\sigma_r},$$

where σ_r is the first exit time of B from $\{|z|<r\}$. Hence we have

$$E[\sigma_r^*] = E[\varphi_{\sigma_r}]$$

$$=E[\int_0^{\sigma_r} \frac{|f'(B_s)|^2}{(1+|f(B_s)|^2)^2} ds].$$

Let $p(s, z) = P(s < \sigma_r, B_s \in dxdy)$ is the density function of the random variable $B_{s\wedge\sigma_r}$ with respect to the Euclidean area element. Then it is well-known (Itô-McKean [5] p.237) that

$$\int_0^\infty p(s,z) ds = \frac{1}{\pi} g(z).$$

This shows

$$E[\sigma_r^*] = \frac{1}{\pi} \int_{\{|z|<r\}} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} g(z) dxdy$$

$$= T(r,f).$$

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