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Brownian Motions on Riemann Surfaces of Inverse Functions

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§1. Introduction.

Let $B = (B_t, t \geq 0)$ be a complex Brownian motion starting at 0 defined on a probability space $(\Omega, F, P)$ and $f$ be a non-constant analytic function in the unit disc $\Delta$. Define $\varphi_t$ and $W$ by

$$\varphi_t = \int_0^t |f'(B_s)|^2 ds,$$

up to the first exit time $\sigma$ of $B$ from $\Delta$ and

$$W = (W_t) = (f(B_{\varphi_t})).$$

Then the process $W$ is also a Brownian motion up to the time $\varphi_\sigma$. It is known that $E[\varphi(\sigma)^p] \approx \|f\|_p$ for $0 < p < \infty$ (Burkholder, Gundy and Silverstein [2]). In 1979 Davis [3] noted that $\varphi_\sigma$ is the first exit time of the Euclidean Brownian motion $W$ from $f(\Delta)$. Precisely let $S$ be the Riemann surface of $f^{-1}$ such that $S$ is a covering surface of $f(\Delta)$ with the natural projection $p$ and that there exists a one-to-one onto mapping $f^{-1}$ with $f^{-1} \cdot f = p$. Such a surface is called the Riemann surface of inverse function. The
Brownian motion $W = (W_t^*)$ can be lifted continuously on $S$. Let $\tilde{W}^* = (\tilde{W}_t^*)$ be the lifted Brownian motion on $S$. Since the generator of $\tilde{W}^*$ is $\frac{1}{2}$ times the Laplace-Beltrami operator corresponding to the pull-backed metric on $S$ from the Euclidean metric on $f(A)$, $\tilde{W}^*$ is a Brownian motion corresponding to this metric and $\varphi_\sigma$ is the first exit time of $\tilde{W}^*$ from $S$.

In the present paper we shall study analogously spherical Brownian motions on Riemann surfaces of inverse functions.

§1. Result.

Let $w = f(z)$ be a non-constant meromorphic function in the $z$-plane to the $w$-sphere. We may regard $f$ and its restriction $f|\{|z|<r\}$ as one-to-one onto mappings from the complex plane $\mathbb{C}$ and $\{|z|<r\}$ onto Riemann surfaces of inverse functions $S$ and $S_r$ respectively. We may assume $S_r \subset S$. Now we can define a spherical metric on $S$ by

$$\rho(\tilde{w}^*) dw^* dw^* = \frac{dw dz}{(1+|w|^2)^2},$$

for each local coordinate $\tilde{w}^*$ with $w = p(\tilde{w}^*)$. Let $A$ denote the spherical area on $S$, then

$$A(r,f) = A(S_r) = \int_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} \ dx dy.$$ 

Define the Ahlfors-Shimizu characteristic $T(r,f)$ by

$$T(r,f) = \int_0^F \frac{A(x,f)}{x} \ dx.$$ 

Then it is well-known that

$$T(r,f) = \frac{1}{2} \int_{|z|<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} g(z) \ dx dy,$$
where \( g \) is the Green's function of \(|z|<r\) with a pole at 0 and \( z = x + iy \).

Let \( \omega^*_0 = f(0) \in S_r \). The spherical metric \( \rho \) does not only define \( A(r,f) \) and \( T(r,f) \) but also generates a Brownian motion \( \omega^* = (\omega^*_t) \) starting at \( \omega^*_0 \) on \( S \) defined on some probability space \((\Omega^*, F^*, P^*)\) such that

\[
\lim_{t \to 0} \frac{1}{t} E^*[u(\omega^*_t) - u(\omega^*_0)] = \frac{1}{2} (L^*_\rho u)(\omega^*_0),
\]

for each \( C^2 \)-bounded function \( u \) on \( S \) where \( E^* \) denotes the mathematical expectation with respect to \( P^* \) and \( L^*_\rho \) is the Laplace-Beltrami operator corresponding to \( \rho \). Let \( \sigma^*_r \) be the first exit time of \( \omega^* \) from \( S_r \). Then we have,

**Theorem.** For each \( r, r > 0 \), it holds

\[
E^*[\sigma^*_r] = T(r, f).
\]

§3. Proof. We can construct \( \omega^* \) by the standard time change-argument (Blumenthal and Getoor [1] p.212). Define \( \varphi_t \) by

\[
\varphi_t = \int_0^t \frac{|f'(B^*_s)|^2}{(1+|f(B^*_s)|^2)^2} \, ds,
\]

and put \( \psi_t = \varphi_t^{-1} \). Then \( \omega = (\omega^*_t) = (f(B^*_t)) \) is a spherical Brownian motion on the \( w \)-sphere. Let \( \omega^* = (\omega^*_t) \) be a lifted process of \( \omega \) such that \( \omega^* \) has continuous paths a.s. with \( p(\omega^*_t) = \omega^*_t \) and \( \omega^*_0 = \omega^*_0 \). Without loss of generality we assume \( f'(0) \neq 0 \). Then a simple application of Itô's formula (Ikeda and Watanabe [4] p.66) shows \((2.1)\). Since \( \sigma^*_r \) is the first exit time of \( \omega^* \) from \( S_r \), we have

\[
\sigma^*_r = \inf \{ t : \omega^*_t \in S_r \}
\]
\[ = \inf \{ t : \phi_t^{-1}(W^*_t) \in f^{-1}(S_r) \} \]
\[ = \inf \{ t : |B_{\psi_t}| \geq r \} \]
\[ = \inf \{ \sigma_t : |B_t| \geq r \} \]
\[ = \sigma_r' \]

where \( \sigma_r' \) is the first exit time of \( B \) from \((|z|<r)\). Hence we have

\[ E[\sigma_r'^*] = E[\sigma_r'] \]

\[ = E[\int_0^{\sigma_r} \frac{|f'(B_s)|^2}{(1+|f(B_s)|^2)^2} ds] \]

Let \( p(s, z) = P(s < \sigma_r, B_S \in dx dy) \) is the density function of the random variable \( B_{S \wedge \sigma_r} \) with respect to the Euclidean area element. Then it is well-known (Itô-McKean [5] p.237) that

\[ \int_0^{\infty} p(s, z) ds = \frac{1}{\pi} g(z). \]

This shows

\[ E[\sigma_r'^*] = \frac{1}{\pi} \int_{||z||<r} \frac{|f'(z)|^2}{(1+|f(z)|^2)^2} g(z) \, dx \, dy \]

\[ = T(r, f). \]

REFERENCES


