数理解析研究所講究録
572巻 1985年 109-129

ON DEDEKIND SUMS II

REMARKS ON HIGHER DIMENSIONAL DEDEKIND SUMS

EIKO HIROOKA(広岡栄子)
Department of Mathematics,
Kobe University

§1. Classical Dedekind sums

Dedekind uses the symbol $s(a,c)$ for classical Dedekind sums. He originally introduced these sums in connection with the transformation properties, under the modular group, of his η-function

$$\eta(z)=e^{\pi iz/12}\prod_{n=1}^{\infty}(1-e^{2\pi inz}) \quad (\mathrm{Im}z>0),$$

namely, for $c>0$,

$$\log\eta\left(\frac{az+b}{cz+d}\right)=\log\eta(z)+\frac{1}{2}\log\frac{cz+d}{i}+\pi i\left(\frac{a+d}{12c}-s(d,c)\right),$$

where the principal branch is taken for the logarithm.

Let a, c be integers such that $(a,c)=1$, $c\geq 1$. Then $s(a,c)$ was evaluated by Dedekind and he proved the relation

$$s(a,c)=\sum_{k=1}^{c-1}\left(\left(\frac{k}{c}\right)\right)\left(\left(\frac{ak}{c}\right)\right).$$

Here the symbol $((x))$ is defined by

$$((x))=\begin{cases} x-[x]-1/2 & \text{if } x\notin Z, \\ 0 & \text{if } x\in Z, \end{cases}$$

where $[x]$ denotes the greatest integer function. And these sums are represented by cotangents,

namely

$$s(a,c)=\frac{1}{4c}\sum_{k=1}^{c-1}\cot\frac{\pi k}{c}\cot\frac{\pi ka}{c}.$$

We forcus our attention now on its properties and its connection with other mathematical topics([6]).

Proposition 1. (Reciprocity Low)

Let a, c be pogitive integers, $(a,c)=1$. Then,

$$s(a,c)+s(c,a)=\frac{1}{12ac}(a^2+c^2+1-3ac).$$

The number of lattice points in a tetrahedron has been related to Dedekind sums by L.J.Mordell([4]). He proved the following result:

Proposition 2. (Lattice Points)

Let a, b and c be pairwise coprime, positive integers and let $N_3(a,b,c)$ be the number of lattice points in the tetrahedron

$$0\leqq x<a,\ 0\leqq y<b,\ 0\leqq z<c,\ 0<\frac{x}{a}+\frac{y}{b}+\frac{z}{c}<1;$$

then

$$N_3(a,b,c)=-\{s(bc,a)+s(ca,b)+s(ab,c)\}+\frac{1}{6}abc+\frac{1}{4}(bc+ca+ab)$$
$$+\frac{1}{4}(a+b+c)+\frac{1}{12}\left(\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c}\right)+\frac{1}{12abc}-2.$$

Since the Dedekind sums are rational, we can evaluate its denominators:

Proposition 3. (Evaluation of the Denominators)

$2c(3,c)s(a,c)$ is an integer.

The range of values of $s(a,c)$ is not fully known. Salié([5]) proved that $s(a,c)$ always satisfies one of the following five congruences: $6cs(a,c)\equiv 0,\ \pm 1,\ \pm 3 \pmod 9$. So the problem of the

"Missing Values" ([1]) has arisen.

Dedekind sums are connected with the Jacobi symbol:

Proposition 4. (Connection with the Jacobi Symbol)

If c is odd, then

$$6cs(a,c) \equiv \frac{c+1}{2} - \left(\frac{a}{c}\right) \pmod 4.$$

This means that the Dedekind sums become the Jacobi symbol if we have them modulo 4. Hence we can consider that the Dedekind sums explain the Jacobi symbol in detail. We have some other properties, for example, the range of s(a,c) is dense on the real axis and connected with class numbers of quadratic fields etc. Now we explain about the classical Dedekind sums no more and we come to the higher dimensional Dedekind sums that are one of the extented Dedekind sums.

§2. Higher Dimensional Dedekind Sums

2.1 Let p be a positive integer and let $a_1,\ldots,a_n$ be integers prime to p, and n is even. Then we define

$$d(p:a_1,\ldots,a_n) = (-1)^{n/2}\sum_{k=1}^{p-1} \cot ka_1/p \ldots \cot ka_n/p .$$

Remark.

If n is odd, the sum is clearly equal to zero.

Zagier called these sums the higher dimensional Dedekind sums for the first time. Why he was interested in sums of this type ? There are two reasons. Firstly the case n=2 is, up to a factor, the classical Dedekind sums:

$$d(p:a_1,a_2) = -4ps(a_1a_2^{-1},p),$$

where a_2^{-1} is an inverse of $a_2 (\mathrm{mod}\ p)$. And as I mentioned at §1, the Dedekind sums are connection with other mathematical topics. So the same things may hold about higher dimensional analogues. The second ground for studying this sum is that it arises in topological situations. The first appearance of trigonometric functions in this context was the index theorem of Hirzebruch, which evaluates the signature (a certain invariant of the homology of a differentiable manifold) by means of a formula involving cotangents([3]). In that case, if the dimension of the manifold is four, classical Dedekind sums appear and generally if the dimension of the manifold is 2n, $d(p:a_1,\ldots,a_n)$ appears. After that we shall pay more attention to the number theorical part of higher dimensional case.

2.2 Some Properties of the Higher Dimensional Dedekind Sums

<u>Proposition 5.</u> (Rational Number ([7]))

$d(p:a_1,\ldots,a_n)$ is rational:

$$d(p:a_1,\ldots,a_n) = 2^n p \sum_{\substack{0<k_i<p \\ p \mid k_1 a_1+\ldots+k_n a_n}} ((k_1/p))\ldots((k_n/p)).$$

<u>Proposition 6.</u> (Reciprocity Law ([7]))

Let $a_0,\ldots,a_n$ be pairwise coprime positive integers.

Then $$\sum_{j=0}^{n} \frac{1}{a_j} d(a_j:a_0,\ldots,\hat{a}_j,\ldots,a_n) = 1 - 1_n(a_0,\ldots,a_n)/a_0\ldots a_n,$$

where $l_n(a_0,\ldots,a_n)$ is a polynomial defined by the next formula:

$$\sum_{n=0}^{\infty} l_n(a_0,\ldots,a_n)t^n = \prod_{j=0}^{n} a_j t/\tanh a_j t$$
$$= \prod_{j=0}^{n} (1 + a_j^2 t^2/3 - a_j^4 t^4/45 + 2a_j^6 t^6/945 - \ldots)$$

Remark.

$l_n(a_0,\ldots,a_n)$ is nearly equal to the Hizebruch L-polynomial.

In Proposition 5, $d(p:a_1,\ldots,a_n)$ is rational ; so we can evaluate the denominator:

<u>Proposition 7.</u> (Evaluation of the Denominators ([7]))

Let

$$c(n,p) = \prod_{\substack{m:\text{odd prime} \\ m \nmid p}} m^{[n/(m-1)]}, \text{ then}$$

$c(n,p)d(p:a_1,\ldots,a_n)$ is an integer.

And for n=2,4 the denominator of $d(p:a_1,\ldots,a_n)$ equals to $c(n,p)$.

We want to add much more about the evaluation of the denominators, but before that let's consider the generalization of the lattice points in the Proposition 2:

<u>Proposition 8.</u> (Generalization of the Lattice Points ([8]))

Let $a_1,\ldots,a_n$ (n:odd) be positive integers and N a common multiple of the a_i's. Let

$$t(a_1,\ldots,a_n) = \mathrm{Card}\{(x_1,\ldots,x_n) \mid 0<x_i\le a_i,\ 0 < \sum_{i=1}^{n} x_i/a_i < 1 \pmod 2\}$$
$$-\mathrm{Card}\{(x_1,\ldots,x_n) \mid 0<x_i<a_i,\ 1 < \sum_{i=1}^{n} x_i/a_i < 2 \pmod 2\}.$$

Then

$$t(a_1,\ldots,a_n) = \frac{(-1)^{(n-1)/2}}{N} \sum_{k=1,k:\text{odd}}^{2N-1} \cot\frac{\pi k}{2N} \cot\frac{\pi k}{2a_1} \cdots \cot\frac{\pi k}{2a_n}$$

This is a kind of the higher dimensional Dedekind sums, too. And the formula by which we define $t(a_1,\ldots,a_n)$ is the value found by Brieskorn([2]) for the signature of the (2n-3)-dimensional manifold defined by

$$V = \{(z_1,\ldots,z_n)\in C^n \mid z_1^{a_1} + \ldots + z_n^{a_n} = 0,\ |z_1|^2+\ldots+|z_n|^2 = 1\}.$$

2.3 From now on we shall give some new results for the higher dimensional Dedekind sums.

The first is a result about the evaluation of the denominators. In §1 we saw the congruence which $s(a,c)$ satisfies. And for $n=4$ we have the next congruences:

<u>Theorem 1.</u>

Let

$$d(p,4) = d(p:a_1,a_2,a_3,a_4).$$

Then

1) $d(p,4) \equiv 0 \pmod 2$ if $3,5 \mid p$,
2) $9d(p,4) \equiv \pm 2 \pmod{18}$ if $3 \mid p$, $5 \nmid p$,
3) $5d(p,4) \equiv 0 \pmod 2$, $\neq 0$ if $5 \mid p$, $3 \nmid p$,
4) $45d(p,4) \equiv \pm 10 \pmod{18}$ if $15 \mid p$.

From this the problem of the "Missing Values" arises. But we can't find the algorithm of the $d(p:a_1,a_2,a_3,a_4)$ yet.

To prove this theorem we need some lemmas.

<u>Lemma 1.</u>

For the a_i's we can choose a_i''s as $d(p:a_1,a_2,a_3,a_4) = d(p:a_1',a_2',a_3',1)$ (p,a_1',a_2',a_3' are pairwise coprime.)

Proof. Since $(p,a_4)=1$, a_4' exists such that $a_4a_4'\equiv 1 \pmod p$. Hence by the definition $d(p:a_1,a_2,a_3,a_4)=d(p:a_1a_4',a_2a_4',a_3a_4',1)$. And by Dirichlet's theorem on primes in arithmetic progressions, a_ia_4' can be replaced by prime number a_i' congruent to a_ia_4' modulo p. Since the value of $d(p:a_1,\ldots,a_n)$ only depends on the values of the $a_i \pmod p$, $d(p:a_1,a_2,a_3,a_4)=d(p:a_1',a_2',a_3',1)$. But they can be chosen as arbitrarily large primes,so in particular they can be made prime to one another and then to p. Therefore we have the result.

Lemma 2.

$$45d(p:a_1,a_2,a_3,a_4) \equiv 0 \pmod 2$$

Proof. It suffices to prove this lemma for $d(p:a_1,a_2,a_3,1)$ (p,a_1,a_2,a_3 are pairwise coprime.) by lemma 1. We first assume that p is odd. By proposition 5,

$$d(p:a_1,a_2,a_3,1) = 2^4p \sum_{\substack{0\leqq k_i<p \\ p\mid a_1k_1+a_2k_2+a_3k_3+k_4}} \left(\left(\frac{k_1}{p}\right)\right)\left(\left(\frac{k_2}{p}\right)\right)\left(\left(\frac{k_3}{p}\right)\right)\left(\left(\frac{k_4}{p}\right)\right).$$

We fix one of the (k_1,k_2,k_3) $(0\leqq k_i<p)$. For this (k_1,k_2,k_3) only one k_4 $(0\leqq k_4<p)$ exists such that $p\mid a_1k_1+a_2k_2+a_3k_3+k_4$. Therefore

$$d(p:a_1,a_2,a_3,1) = 2^4p \sum_{\substack{0\leqq k_i<p \\ (i=1,2,3)}} \left(\left(\frac{k_1}{p}\right)\right)\left(\left(\frac{k_2}{p}\right)\right)\left(\left(\frac{k_3}{p}\right)\right)\left(\left(\frac{Np-(a_1k_1+a_2k_2+a_3k_3)}{p}\right)\right) \quad \text{(N:integer)}$$

$$= -2^4p \sum_{0\leqq k_i<p} \left(\left(\frac{k_1}{p}\right)\right)\left(\left(\frac{k_2}{p}\right)\right)\left(\left(\frac{k_3}{p}\right)\right)\left(\left(\frac{a_1k_1+a_2k_2+a_3k_3}{p}\right)\right)$$

$$= -2^4 p \sum_{0\leq k_i<p} ((\frac{k_1}{p}))((\frac{k_2}{p}))((\frac{k_3}{p}))\{\frac{a_1k_1+a_2k_2+a_3k_3}{p} - [\frac{a_1k_1+a_2k_2+a_3k_3}{p}] - \frac{1}{2}\}$$

$$-2^3 p \sum_{\substack{0\leq k_i<p \\ p\mid a_1k_1+a_2k_2+a_3k_3}} ((\frac{k_1}{p}))((\frac{k_2}{p}))((\frac{k_3}{p})).$$

Here we pay attention to

$$\sum_{0\leq k_i<p} ((\frac{k_i}{p})) = 0:$$

for example

$$\sum_{0\leq k_i<p} ((\frac{k_1}{p}))((\frac{k_2}{p}))((\frac{k_3}{p}))k_1 = \sum_{k_1} k_1((\frac{k_1}{p}))\sum_{k_2}((\frac{k_2}{p}))\sum_{k_3}((\frac{k_3}{p}))$$

$$= 0 .$$

Therefore

$$d(p:a_1,a_2,a_3,1) = 2^4 p \sum_{0\leq k_i<p} ((\frac{k_1}{p}))((\frac{k_2}{p}))((\frac{k_3}{p}))[\frac{a_1k_1+a_2k_2+a_3k_3}{p}]$$

$$-d(p:a_1,a_2,a_3)$$

$$= 2^4 p \sum_{0\leq k_i<p} ((\frac{k_1}{p}))((\frac{k_2}{p}))((\frac{k_3}{p}))[\frac{a_1k_1+a_2k_2+a_3k_3}{p}]$$

$$= 16p \sum_{k_i=1}^{p-1} [\frac{a_1k_1+a_2k_2+a_3k_3}{p}]((\frac{k_1}{p}))((\frac{k_2}{p}))((\frac{k_3}{p}))$$

$$= 16p \sum_{k_i=1}^{p-1} [\frac{a_1k_1+a_2k_2+a_3k_3}{p}]\{\frac{k_1k_2k_3}{p^3} - \frac{1}{2p^2}(k_1k_2+k_2k_3+k_3k_1)$$

$$+ \frac{1}{4p}(k_1+k_2+k_3) - \frac{1}{8}\} .$$

$45d(p:a_1,a_2,a_3,1)$ is an integer and since p is odd,

$$45d(p:a_1,a_2,a_3,1) \equiv 45p^2 d(p:a_1,a_2,a_3,1)$$

$$\equiv 0 \pmod 2 .$$

If p is even, we use the reciprocity low and we find

$$45a_1a_2a_3d(p:1,a_1,a_2,a_3) + 45pa_2a_3d(a_1:p,1,a_2,a_3)$$

$$+ 45pa_1a_3d(a_2:p,1,a_1,a_3) + 45pa_1a_2d(a_3:p,1,a_1,a_2)$$

$$= 45pa_1a_2a_3 + p^4 + 1 + a_1^4 + a_2^4 + a_3^4 - 5(p^2 + p^2a_1^2 + \ldots + a_2^2a_3^2)$$

modulo 2. Since a_i's are odd,

$$45d(p:1,a_1,a_2,a_3) \equiv 4-5\cdot 6 \equiv 0 \pmod 2 .$$

<u>Proof of Theorem 1.</u>

1) Let's recall that

$$\prod_{\substack{m:\text{odd prime}\\ m|p}} m^{[4/(m-1)]}d(p:a_1,a_2,a_3,a_4) \in Z .$$

Now $3,5 \nmid p$ and $m|p$, so $m \geq 7$. Hence $[4/(m-1)]=0$ and $\prod_{\substack{m:\text{odd prime}\\ m|p}} m^{[4/(m-1)]}=1$.

Therefore $d(p:a_1,a_2,a_3,a_4)$ is an integer. We combine this fact with lemma 2, and we have the result.

About 2),3),4) it suffices to prove for $d(p:a_1,a_2,a_3,1)$ (p,a_1,a_2,a_3 are pairwise coprime.). Therefore we use the reciprocity law:

$$(*) \quad 45a_1a_2a_3d(p:1,a_1,a_2,a_3) + 45pa_2a_3d(a_1:p,1,a_2,a_3)$$

$$+45pa_1a_3d(a_2:p,1,a_1,a_3) + 45pa_1a_2d(a_3:p,1,a_1,a_2)$$

$$= 45pa_1a_2a_3 + p^4 + 1 + a_1^4 + a_2^4 + a_3^4 - 5(p^2 + p^2a_1^2 + \ldots + a_2^2a_3^2) .$$

2) We first assume that $5 \nmid a_i$ for all a_i's. Since $3 \nmid a_i$ for all

a_i's, $d(a_i:p,1,\ldots)\in Z$. Hence we consider (*) modulo 45 and we have

$$45a_1a_2a_3d(p:1,a_1,a_2,a_3) \equiv p^4 + 1 + a_1^4 + a_2^4 + a_3^4 - 5(a_1^2 + a_2^2 + a_3^2 + a_1^2a_2^2 + a_1^2a_3^2 + a_2^2a_3^2)$$

$$\equiv 5(a_1 \pm a_2a_3)^2 + 5(a_2 \pm a_1a_3)^2 - 5(a_3 \pm a_1a_2)^2 + p^4 + 1 + a_1^4 + a_2^4 + a_3^4$$

$$\mp 10a_1a_2a_3 - 10(a_1^2 + a_2^2 + a_2^2a_3^2 + a_1^2a_3^2) \pmod{45}.$$

Since either $a_i \pm a_{i+1}a_{i+2}$ (i mod 3) is a multiple of 3 at the same time,

$$5(a_i \pm a_{i+1}a_{i+2})^2 \equiv 0 \pmod{45}.$$

Therefore

$$45a_1a_2a_3d(p:1,a_1,a_2,a_3)$$

$$\equiv p^4 + 1 + a_1^4 + a_2^4 + a_3^4 - 10(a_1^2 + a_2^2 + a_2^2a_3^2 + a_1^2a_3^2) \mp 10a_1a_2a_3$$

$$\equiv p^4 + 1 + a_1^4 + a_2^4 + a_3^4 + 5(a_1^2-1)^2 + 5(a_2^2-1)^2 + 5(a_2^2-a_3^2)^2$$

$$+5(a_1^2-a_3^2)^2 \mp 10a_1a_2a_3 - 10(1 + a_1^4 + a_2^4 + a_3^4)$$

$$\equiv p^4 - 9(1 + a_1^4 + a_2^4 + a_3^4) \mp 10a_1a_2a_3 \pmod{45}.$$

The last congruence is ground in the fact that either $a_i \pm 1$ is a multiple of 3. Put

$$X = p^4 - 9(1+a_1^4+a_2^4+a_3^4):$$

then by $3|p$ we find $9|X$. Since $p\not|5$ and $a_i\not|5$, $p^4, a_i^4 \equiv 1 \pmod 5$. Hence $X \equiv 0 \pmod 5$ and $x \equiv 0 \pmod{45}$. Therefore

$$45a_1a_2a_3d(p:1,a_1,a_2,a_3) \equiv \mp 10a_1a_2a_3 \pmod{45}.$$

By $(a_1a_2a_3,45)=1$ we have

$$45d(p:1,a_1,a_2,a_3)\equiv \mp 10 \pmod{45}.$$

We combine this with lemma 2 and we have

$$9d(p:1,a_1,a_2,a_3) \equiv \pm 2 \pmod{18}.$$

If $5|a_i$, we may assume $5|a_1$. And we consider (*) modulo 45 so

$$45a_1a_2a_3d(p:1,a_1,a_2,a_3) + 45pa_2a_3d(a_1;p,1,a_2,a_3)$$
$$\equiv p^4+1+a_1^4+a_2^4+a_3^4-5(a_1^2+a_2^2+\ldots+a_2^2a_3^2) \pmod{45}.$$

We multiply 5 in the both sides and we notice that

$$225pa_2a_3d(a_1:p,1,a_2,a_3) \equiv 0 \pmod{45}.$$

Then

$$225a_1a_2a_3d(p:1,a_1,a_2,a_3)$$
$$\equiv 5(p^4+1+a_1^4+a_2^4+a_3^4) - 25(a_1^2+\ldots+a_2^2a_3^2) \pmod{45}.$$

By using the same way we have

$$225a_1a_2a_3d(p:1,a_1,a_2,a_3)$$
$$\equiv 25(a_1\pm a_2a_3)^2+25(a_2\pm a_1a_3)^2-25(a_3\pm a_1a_2)^2+5(1+a_1^4+a_2^4+a_3^4)$$
$$\mp 50a_1a_2a_3-50(a_1^2+a_2^2+a_2^2a_3^2+a_1^2a_3^2)$$
$$\equiv 5(1+a_1^4+a_2^4+a_3^4) \mp 50a_1a_2a_3+25(a_1^2-1)^2+25(a_2^2-1)^2$$
$$+25(a_2^2-a_3^2)^2+25(a_1^2-a_3^2)^2-50(1+a_1^4+a_2^4+a_3^4)$$
$$\equiv -45(1+a_1^4+a_2^4+a_3^4) \mp 50a_1a_2a_3$$
$$\equiv \mp 50a_1a_2a_3 \pmod{45}.$$

And by $(a_2a_3,45)=1$

$$225a_1d(p:1,a_1,a_2,a_3) \equiv \mp 50a_1 \pmod{45}.$$

Hence

$$45a_1 d(p:1,a_1,a_2,a_3) \equiv \mp 10a_1 \pmod 9$$

and by $(5a_1,9)=1$, we have

$$9d(p:1,a_1,a_2,a_3) \equiv \mp 2 \pmod 9.$$

By lemma 2 we have also

$$9d(p:1,a_1,a_2,a_3) \equiv \pm 2 \pmod{18}.$$

3) Since $3 \nmid p$ and $5 \mid p$, $5d(p:1,a_1,a_2,a_3)$ is an integer. By using the lemma 2 we have

$$5d(p:1,a_1,a_2,a_3) \equiv 0 \pmod 2.$$

4) Since $3, 5 \nmid a_i$ for all a_i's, $d(a_i:p,1,\cdots) \in Z$.

Hence

$$45a_1a_2a_3 d(p:1,a_1,a_2,a_3) \equiv 1+a_1^4+a_2^4+a_3^4-5(a_1^2+\ldots a_2^2a_3^2) \pmod 9.$$

By using the same way of 1), we have

$$45a_1a_2a_3 d(p:1,a_1,a_2,a_3) \equiv -9(1+a_1^4+a_2^2+a_3^4) \mp 10a_1a_2a_3$$
$$\equiv \mp a_1a_2a_3 \pmod 9.$$

And by $(a_1a_2a_3,9)=1$

$$45d(p:1,a_1,a_2,a_3) \equiv \mp 1 \pmod 9.$$

By lemma 2 we have

$$45d(p:1,a_1,a_2,a_3) \equiv \pm 10 \pmod{18}.$$

Q.E.D.

Higher dimensional Dedekind sums are connected with n's Jacobi symbols:

Theorem 2.

Put

$$\mu_{n/2} = \prod_{m:\text{odd prime}} m^{[n/(m-1)]} .$$

If p is odd, then

$$\mu_{n/2} d(p:a_1,\ldots,a_n) \equiv \mu_{n/2}(p-1) - 2n + 2\{(\frac{a_1}{p})+\ldots+(\frac{a_n}{p})\} \pmod 8 .$$

Proof. We first show the theorem for the type $d(p:b_1,\ldots,b_{n-1},1)$. By the definition

$$d(p:b_1,\ldots,b_{n-1},1) = 2^n \sum_{\substack{0\leq k_i<p \\ p\mid b_1k_1+\ldots+b_{n-1}k_{n-1}+k_n}} ((\frac{k_1}{p}))\ldots((\frac{k_n}{p})).$$

And we use the same way in the proof of lemma 2, then

$$d(p:b_1,\ldots,b_{n-1},1) = 2^n p \sum_{\substack{0\leq k_i<p \\ (i=1,2,\ldots,n-1)}} [\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}]((\frac{k_1}{p}))\ldots((\frac{k_{n-1}}{p}))$$

$$= A + 4\sum_{k_i=1}^{p-1} (k_1+\ldots+k_{n-1})[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}] - 2p\sum [\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}],$$

where A is an integer and

$$\mu_{n/2}A \equiv 0 \pmod 8.$$

Therefore

$$\mu_{n/2} d(p:b_1,\ldots,b_{n-1},1) \equiv 4\sum_{k_i=1}^{p-1} (k_1+\ldots+k_{n-1})[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}]$$

$$-2p\mu_{n/2} [\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}] \pmod 8.$$

First we calculate $\sum k_1[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}]$ modulo 2.

$$\sum_{k_{n-1}=1}^{p-1} \left[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}\right] =$$

$$\sum_{k_{n-1}=0}^{p-1} \left(\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p} - \left\{\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}\right\}\right) - \left[\frac{b_1k_1+\ldots+b_{n-2}k_{n-2}}{p}\right],$$

where { } denotes the fractional part and

$$\sum_{k_{n-1}=0}^{p-1} \frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p} = \frac{p-1}{2}.$$

Hence

$$\sum_{k_{n-1}=1}^{p-1} \left[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}\right] = b_1k_1+\ldots+b_{n-2}k_{n-2} + \frac{(p-1)(b_{n-1}-1)}{2}$$

$$- \left[\frac{b_1k_1+\ldots+b_{n-2}k_{n-2}}{p}\right].$$

And by noticing that p is odd,

$$\sum_{k_{n-2}=1}^{p-1} \left(\sum_{k_{n-1}} \left[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}\right]\right) \equiv \frac{b_{n-2}p(p-1)}{2}$$

$$- \sum_{k_{n-2}=1}^{p-1} \left[\frac{b_1k_1+\ldots+b_{n-2}k_{n-2}}{p}\right] -$$

$$\equiv \frac{b_{n-2}p(p-1)}{2} - (b_1k_1+\ldots+b_{n-3}k_{n-3}) - \frac{(p-1)(b_{n-2}-1)}{2}$$

$$+ \left[\frac{b_1k_1+\ldots b_{n-3}k_{n-3}}{p}\right] \pmod 2$$

$$\sum_{k_{n-3}} \left(\sum_{k_{n-2},k_{n-1}} \left[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}\right]\right) \equiv$$

$$- \frac{b_{n-3}p(p-1)}{2} + \sum_{k_{n-3}=1}^{n-1} \left[\frac{b_1k_1+\ldots+b_{n-3}k_{n-3}}{p}\right] \pmod 2.$$

We repeat this operation and we notice that n is even. So

$$\sum_{k_2,\ldots,k_{n-1}} [\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}] \equiv$$

$$\frac{b_2p(p-1)}{2} - b_1k_1 - \frac{(p-1)(b_2-1)}{2} + [\frac{b_1k_1}{p}]$$

$$\equiv - b_1k_1 + \frac{p-1}{2} + [\frac{b_1k_1}{2}] \pmod 2.$$

$$\sum_{k_1,\ldots,k_{n-1}} k_1[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}] \equiv$$

$$- \frac{b_1p(p-1)(2p-1)}{6} + \frac{p(p-1)^2}{4} + \sum_{k_1} k_1[\frac{b_1k_1}{p}] \pmod 2.$$

Here

$$\sum_{k_1=1}^{n-1} k_1[\frac{b_1k_1}{p}] \equiv \sum_{\substack{k_1=1\\ k_1:\text{odd}}}^{n-1} [\frac{b_1k_1}{p}] \equiv \frac{(p-1)(b_1-1)}{2} + \frac{1}{2}\{(\frac{b_1}{p})-1\} \pmod 2,$$

hence

$$4 \sum_{k_1,\ldots,k_{n-1}} k_1[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}] \equiv$$

$$- \frac{2b_1p(p-1)(2p-1)}{3} + p(p-1)^2 + 4(p-1)(b_1-1) + 4\{(\frac{b_1}{p})-1\} \pmod 8.$$

Since in the first term p or p-1 or $2p-1 \equiv 0 \pmod 3$ and $(3,4)=1$,

$$\frac{p(p-1)(2p-1)}{3} \equiv n-1 \pmod 4.$$

Thus

$$4\sum k_1[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}] \equiv$$

$$-2b_1(p-1) + 2(p-1) + 2(b_1-1)(p-1) + 2\{(\frac{b_1}{p})-1\}$$

$$\equiv 2(p-1)(-b_1+1+b_1-1) + 2\{(\frac{b_1}{p})-1\}$$

$$\equiv 2\{(\frac{b_1}{p}) - 1\} \pmod 8.$$

Therefore

$$4\sum(k_1+\cdots+k_{n-1})\left[\frac{b_1k_1+\cdots+b_{n-1}k_{n-1}}{p}\right]$$
$$\equiv 2\{(\frac{b_1}{p})+\cdots+(\frac{b_{n-1}}{p})-(n-1)\} \pmod 8.$$

Next we calculate $\sum_{k_i=1}^{p-1}\left[\frac{b_1k_1+\cdots+b_{n-1}k_{n-1}}{p}\right]$ modulo 4.

$$\sum_{k_{n-1}=1}^{p-1}\left[\frac{b_1k_1+\cdots+b_{n-1}k_{n-1}}{p}\right] = b_1k_1+\cdots+b_{n-2}k_{n-2}+\frac{(p-1)(b_{n-1}-1)}{2}$$
$$-\left[\frac{b_1k_1+\cdots+b_{n-2}k_{n-2}}{p}\right]$$

$$\sum_{k_{n-2},k_{n-1}}\left[\frac{b_1k_1+\cdots b_{n-1}k_{n-1}}{p}\right] = (p-1)\{b_1k_1+\cdots+b_{n-3}k_{n-3}$$
$$+\frac{(p-1)(b_{n-1}-1)}{2}\} + \frac{b_{n-2}p(p-1)}{2} - \sum_{k_{n-2}}\left[\frac{b_1k_1+\cdots+b_{n-2}k_{n-2}}{p}\right].$$

$$\sum_{k_{n-3},k_{n-2},k_{n-1}}\left[\frac{b_1k_1+\cdots+b_{n-1}k_{n-1}}{p}\right] \equiv \frac{b_{n-3}p(p-1)^2}{2} + \frac{b_{n-2}p(p-1)^2}{2}$$
$$-(p-1)\{b_1k_1+\cdots+b_{n-4}k_{n-4}+\frac{(p-1)(b_{n-2}-1)}{2}\} - \frac{b_{n-3}p(p-1)}{2}$$
$$+\sum_{k_{n-3}}\left[\frac{b_1k_1+\cdots+b_{n-3}k_{n-3}}{p}\right] \pmod 4.$$

We repeat this operation and we obtain

$$\sum_{k_i}\left[\frac{b_1k_1+\cdots+b_{n-1}k_{n-1}}{p}\right] \equiv \frac{b_1p(p-1)^2}{2} + \frac{b_2p(p-1)^2}{2} - \frac{(p-1)^2(b_2-1)}{2}$$
$$-\frac{b_1p(p-1)}{2} + \sum_{k_1}\left[\frac{b_1k_1}{p}\right]$$
$$\equiv \frac{b_1p(p-1)(p-2)}{2} + \frac{(p-1)^2\{b_2(p-1)+1\}}{2} + \sum_{k_1}\left[\frac{b_1k_1}{p}\right]$$
$$\equiv \frac{(p-1)^2}{2} + \frac{b_1p(p-1)(p-2)}{2} + \frac{(p-1)(b_1-1)}{2} \pmod 4.$$

$$2p\mu_{n/2}\Sigma\left[\frac{b_1k_1+\ldots+b_{n-1}k_{n-1}}{p}\right] \equiv p\mu_{n/2}\{(p-1)^2+b_1p(p-1)(p-2)$$
$$+(p-1)(b_1-1)\}$$
$$\equiv p\mu_{n/2}\{2(1-p)+3b_1(p-1)+(p-1)(b_1-1)\}$$
$$\equiv p\mu_{n/2}(p-1)(-2+3b_1+b_1-1)$$
$$\equiv p\mu_{n/2}(p-1)(4b_1-3)$$
$$\equiv -3\mu_{n/2}p(p-1)$$
$$\equiv 3\mu_{n/2}(p-1) \pmod 8.$$

Therefore

$$\mu_{n/2}d(p:b_1,\ldots,b_{n-1},1) \equiv 2\{(\frac{b_1}{p})+\ldots+(\frac{b_{n-1}}{p})-(n-1)\}+5\mu_{n/2}(p-1)$$
$$\equiv \mu_{n/2}(p-1)+2\{(\frac{b_1}{p})+\ldots+(\frac{b_{n-1}}{p})-(n-1)\} \pmod 8.$$

By the way

$$d(p:a_1,a_2,\ldots,a_n) = d(p:a_1x,\ldots,a_{n-1}x,1) \quad (a_nx\equiv 1 \pmod p).$$

Hence

$$\mu_{n/2}d(p:a_1,\ldots,a_n) = \mu_{n/2}d(p:a_1x,\ldots,a_{n-1}x,1)$$
$$\equiv \mu_{n/2}(p-1)+2\{(\frac{a_1x}{p})+\ldots+(\frac{a_{n-1}x}{p})\}-2(n-1)$$
$$\equiv \mu_{n/2}(p-1)+2(\frac{x}{p})\{(\frac{a_1}{p})+\ldots+(\frac{a_{n-1}}{p})\}-2(n-1) \pmod 8.$$

And since $a_nx\equiv 1 \pmod p$, $(\frac{x}{p}) = (\frac{a_n}{p})$.

By using this fact we obtain that

$$\mu_{n/2}d(p:a_1,\ldots,a_n) \equiv \mu_{n/2}(p-1)+2(\frac{a_n}{p})\{(\frac{a_1}{p})+\ldots+(\frac{a_{n-1}}{p})\}-2(n-1)$$
$$\equiv \mu_{n/2}(p-1)+2(\frac{a_n}{p})\{(\frac{a_1}{p})+\ldots+(\frac{a_n}{p})\}-2n$$
$$\equiv \mu_{n/2}(p-1)+2\{(\frac{a_1}{p})+\ldots+(\frac{a_n}{p})\}-2n \pmod 8.$$

We have the last congruence by noticing that

$$(\frac{a_1}{p})+\ldots+(\frac{a_n}{p}) \text{ is always even.}$$

Q.E.D.

We use this theorem in the reciprocity law and we obtain the next corollarys:

Corollary 1.

Let $a_0,\ldots,a_n$ be odd and pairwise coprime.

Then

$$2\sum_{\substack{i,j=0\\ i<j}}^{n}\{a_i(\frac{a_j}{a_i}) + a_j(\frac{a_i}{a_j})\}$$

$$\equiv -n\mu_{n/2} + (\mu_{n/2}+2n)(a_0+\ldots+a_n)-\mu_{n/2}a_0\cdots a_n \pmod 8.$$

This formula is very beautiful but I don't know what it means. To prove this we need the next lemma:

Lemma 3.

J_n is defined as the coefficient of t^n in the power series expansion of $(\frac{t}{\tanh t})^{n+1}$, where n is even. Then

$$J_n = 1.$$

This lemma has been shown ([3]).

Proof of the Corollary 1.

Since a_i's are odd, $a_i^2 \equiv 1 \pmod 8$. Hence

$$\mu_{n/2}l_n(a_0,\ldots,a_n) \equiv \mu_{n/2}J_n \equiv \mu_{n/2} \pmod 8.$$

We multiply $a_0^2\cdots a_n^2$ ($\equiv 1 \pmod 8$) in the both sides of the reciprocity law:

$$\sum_{j=0}^{n} a_j d(a_j : a_0, \ldots, \hat{a}_j, \ldots, a_n) \equiv 1 - a_0 \cdots a_n 1_n(a_0, \ldots, a_n) \pmod 8.$$

$$\mu_{n/2} \sum_{j=0}^{n} a_j d(a_j : a_0, \ldots, \hat{a}_j, \ldots, a_n) \equiv \mu_{n/2} - a_0 \cdots a_n \mu_{n/2} 1_n(a_0, \ldots, a_n)$$

$$(**) \quad \equiv \mu_{n/2} - \mu_{n/2} a_0 \cdots a_n \pmod 8.$$

Here we use the theorem 2:

$$\text{the left side} \equiv \sum_{j=0}^{n} [a_j \mu_{n/2}(a_j - 1) - 2a_j n + 2a_j \{(\frac{a_0}{a_j}) + \overset{j}{\hat{\ldots}} + (\frac{a_n}{a_j})\}]$$

$$(n+1)\mu_{n/2} - (\mu_{n/2} + 2n)(a_0 + \ldots + a_n) + 2 \sum_{\substack{i,j=0 \\ i<j}}^{n} \{a_i(\frac{a_j}{a_i}) + a_j(\frac{a_i}{a_j})\} \pmod 8.$$

We substitute this for (**) and then

$$2 \sum_{\substack{i,j=0 \\ i<j}}^{n} \{a_i(\frac{a_j}{a_i}) + a_j(\frac{a_i}{a_j})\} \equiv -n\mu_{n/2} + (\mu_{n/2} + 2n)(a_0 + \ldots + a_n)$$

$$- \mu_{n/2} a_0 \cdots a_n \pmod 8.$$

Q.E.D.

<u>Corollary 2.</u>

Let a,b be odd and (a,b)=1. Then

$$a(\frac{b}{a}) + b(\frac{a}{b}) \equiv \frac{3}{2}(a+1)(b+1) \pmod 4.$$

This is the quadratic recirpocity law. First we assume that $a, b \equiv -1 \pmod 4$. Then the right side of the corollary is zero modulo 4,

namely

$$a(\frac{b}{a}) + b(\frac{a}{b}) \equiv 0 \pmod 4.$$

$$(\frac{b}{a}) + (\frac{a}{b}) \equiv 0 \pmod 4.$$

Since $(\frac{b}{a}) = \pm 1$,

$$(\frac{b}{a}) + (\frac{a}{b}) = 0.$$

We use the same way in the remainder part and then we obtain the

result.

<u>Proof of the Corollary 2.</u>

The case n=2 we use the theorem 2. Since $\mu_1 = \prod_{m:\text{odd prime}} m^{[2/(m-1)]} = 3$,

$$3d(a:b,1) \equiv 3(a-1)-4+2\{(\frac{b}{a}) + (\frac{1}{a})\}$$

$$\equiv 3(a+1)+2(\frac{b}{a}) \pmod 8.$$

Similarly

$$3d(b:a,1) \equiv 3(b+1)+2(\frac{a}{b}) \pmod 8.$$

We substitute these formulas for the new reciprocity law ((**) in the proof of the cor.1)and then

$$a\{3(a+1)+2(\frac{b}{a})\}+b\{3(b+1)+2(\frac{a}{b})\} \equiv 3-3ab \pmod 8.$$

Since a and b are odd, $a^2 \equiv b^2 \equiv 1 \pmod 8$.

Hence

$$3(a+1)+2a(\frac{b}{a})+3(b+1)+2b(\frac{a}{b}) \equiv 3-3ab \pmod 8.$$

$$2\{a(\frac{b}{a}) + b(\frac{a}{b})\} \equiv -3(a+b+1+ab)$$

$$\equiv -3(a+1)(b+1)$$

$$\equiv 3(a+1)(b+1) \pmod 8.$$

$$a(\frac{b}{a}) + b(\frac{a}{b}) \equiv \frac{3}{2}(a+1)(b+1) \pmod 4.$$

Q.E.D.

As previously stated we can't find the algorithm yet. After this to do first of all is to find the algorithm.

References

[1] T.Asai : Some arithmetic on Dedekind sums. To appear in Proceedings of the Conference on Number Theory. Okayama, Japan: World Scientific, (1984).

[2] E.Brieskorn : Beispiele zur Differentialtopologie von Singularitäten. Invent.Math.2, (1966) 1-14.

[3] F.Hirzebruch : Topological methods in algebraic geometry. Third Enlarged Edition. Berlin-Heidelberg-New York: Springer 1966.

[4] L.J.Modell : Lattice points in a tetrahedron and generalized Dedekind sums. J.Indian Math.Soc.15,(1951) 41-46.

[5] H.Salié : Zum Wertevorrat der Dedekindschen Summen. Math.Z.72, (1959) 61-75.

[6] H.Rademacher and E.Grosswald : Dedekind Sums. Carus Mathematical. Monographs Nr,16, MAA (1972).

[7] D.Zagier : Higher Dimensional Dedekind Sums. Math.Ann. 202,(1973) 149-172.

[8] D.Zagier and F.Hirzebruch : The Atiyah-Singer Theorem and Elementary Number Theory. Math.Lecture Series 3, Publish or Perich,Inc.,Boston, (1974).