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<td>Author(s)</td>
<td>Kano, T.</td>
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<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1985), 572: 1-7</td>
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<tr>
<td>Issue Date</td>
<td>1985-11</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/99194">http://hdl.handle.net/2433/99194</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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Kyoto University
ON THE SIZE OF SOME TRIGONOMETRIC POLYNOMIALS

T. Kano
Okayama University
Okayama 700, Japan

Let $E_n(f)$ denote the best approximation of $f \in C[0,2\pi]$, i.e.,

$$E_n(f) = \inf \| f - T_n \|,$$

where $\| \cdot \|$ is the supremum norm and $T_n(x)$ are trigonometric polynomials
of degree $\leq n$. It was Bernstein [3] who first showed the close connection
between absolute convergence of the Fourier series of $f(x) \in \text{Lip } \alpha$
and $E_n(f)$. In particular, he obtained the following theorem which shows
his result being best possible.

Theorem A. For any given sequence $\varepsilon_n \downarrow 0$ such that

$$\sum_{n=1}^{\infty} \frac{\varepsilon_n}{\sqrt{n}} = \infty,$$

we can find an $f \in C[0,2\pi]$ whose Fourier series is not absolutely con-
vergent at any point at all and yet satisfies the inequality $E_n(f) \leq \varepsilon_n$.

To prove this theorem, he invented the following lemma which may well
deserve an independent interest.

Lemma A. For any given natural number $N$, we can find a trigonometric
polynomial of the form

$$T_N(x) = \sum_{\substack{n=1 \leq n \leq N \leq N/2}} \cos(nx + \rho_n)$$

such that uniformly in $x$,

$$T_N(x) \ll \sqrt{N}.$$
Bernstein's original proof of this lemma is due to the theory of characters, while Bari [1] applied Kuzmin's lemma instead and Kahane[4] used Rudin-Shapiro theorem. Actually Bernstein considered the sum

\[ S = \sum_{A \leq n \leq B} e\left(\frac{n^2}{N} + xn\right), \quad (e(u) = \exp(2\pi iu)) \]

where \( a > 0, 0 \leq x \leq 1 \) and \( A, B, N \in \mathbb{N} \) are such that \( 1 \leq A < B \leq N \).

First we remark that if we apply Salem's lemma (Lemma 2 below) to \( S \), then we obtain

\[ S \ll (\sqrt{a} + 1/\sqrt{a})\sqrt{N}, \]

which holds uniformly in \( x, A, B \).

On the one hand, from a different standpoint, there is a problem of finding the polynomials

\[ P_N(z) = \sum_{n \leq N} c_{n,N} z^n, \]

with \(|c_{n,N}| = 1\) and \(|z| = 1\) such that

\[ \sqrt{N} \ll P_N(z) \ll \sqrt{N}, \]

for all \( z \). Parseval's formula shows

\[ \max_{|z|=1} |P_N(z)| \geq \sqrt{N}. \]

See e.g. Kahane[5] for recent results. The next example of (3) in the literature seems to be the following one due to Hardy and Littlewood[cf. 7, p.199]:

\[ P_N(z) = \sum_{n \leq N} e(cn \log n + xn), \quad z = e(x) \]

which satisfies \( P_N(z) \ll \sqrt{N} \) uniformly in \( x \). However, as far as I know, it seems open whether it satisfies \( P_N(z) \gg \sqrt{N} \) for all \( x \). We notice that in their example the coefficients \( c_n = e(cn \log n) \) are independent of \( N \).

The main purpose of this note is to show that the size of (3) may be sometimes smaller than \( \sqrt{N} \). We shall show it effectively by constructing examples.
First we prove

**Theorem 1.** For any given \( N > 1 \), we can find a sequence \( c_{n,N} \in \mathbb{C} \) with 
\[
|c_{n,N}| = 1 \quad (1 \leq n \leq N)
\]
such that

\[
\sum_{n \leq N} c_{n,N} z^n \ll N^{1/4},
\]
for all \( z \in \mathbb{C} \) with \( |z| = 1 \), where \( \ll \) depends on \( z \).

**Proof.** Consider the sum

\[
S_N = \sum_{n \leq N} e(xn - 2s\sqrt{n}),
\]
where \( 0 < x \leq 1 \) and \( s \geq 1 \) will be suitably chosen (as a function of \( N \)) later.

If we put \( f(t) = xt - 2s\sqrt{t} \) (\( 1 \leq t \leq N \)), then

\[
-s/\sqrt{t} < f'(t) \leq 1 - s/\sqrt{t} \leq 1 - s/\sqrt{N}.
\]

Therefore, if \( 4s^2 \leq t \leq N \), then

\[
|f'(t)| \leq 1 - s/\sqrt{N},
\]
because then \( -(1 - s/\sqrt{N}) \leq -s/\sqrt{t} \).

Now we shall apply the following known lemma due to van der Corput [cf. 2].

**Lemma 1.** If \( f(t) \) is monotone and satisfies

\[
|f'(t)| \leq 1 - \varepsilon, \quad (0 < \varepsilon < 1)
\]
throughout \((a, b)\), then

\[
\sum_{a \leq n \leq b} e(f(n)) = \int_a^b e(f(t))dt + O(1/\varepsilon),
\]
where the constant implied by \( O \) is absolute.

If we insert \( \varepsilon = s/\sqrt{N} \) in the above lemma, then we obtain
\[ 4s^2 \sum_{4s^2 \leq n \leq N} e(xn - 2s\sqrt{n}) = \int_{4s^2}^{N} e(xt - 2s\sqrt{t})dt + O(\sqrt{N}/s). \]

Thus we have

\[ (9) \quad S_N = \sum_{1 \leq n \leq 4s^2} e(xn - 2s\sqrt{n}) + \int_{4s^2}^{N} e(xt-2s\sqrt{t})dt + O(\sqrt{N}/s). \]

We appeal to the known lemma below to estimate the first sum in (9).

Lemma 2 (Salem[cf. 7, p.226]). If \( f''(t) > 0 \) is monotone, then

\[ \sum_{a \leq n \leq b} e(f(n)) = O(\max_{a \leq t \leq b} \frac{1}{\sqrt{f''(t)}}) + O(\int_{a}^{b} (\sqrt{f''(t)} + f''(t))dt), \]

where the implied constants by 0's are absolute.

Now for \( f(t) = xt - 2s\sqrt{t} \) we have \( f''(t) = \frac{s}{2} t^{-3/2} \). Hence by Lemma 2 we have

\[ (10) \quad \sum_{1 \leq n \leq 4s^2} e(f(n)) = O(\max_{1 \leq t \leq 4s^2} \frac{1}{\sqrt{5}} t^{3/4}) + O(\int_{1}^{4s^2} \sqrt{5} t^{-3/4}dt + \int_{1}^{4s^2} st^{-3/2}dt) = O(s) + O(s) + O(s) = O(s). \]

Next we shall estimate the integral

\[ I_N = \int_{4s^2}^{N} e(xt - 2s\sqrt{t})dt. \]

If we put \( t = u^2 \), then

\[ I_N = 2 \int_{2s}^{\sqrt{N}} u e(\sqrt{xu^2} - 2s)du \]

\[ = \frac{1}{2\sqrt{x}} \int_{2s}^{\sqrt{N}} (e(\sqrt{xu^2} - 2su))'du + \frac{2s}{x} \int_{2s}^{\sqrt{N}} e(\sqrt{xu^2} - 2su)du \]
\[ \frac{2s}{x} \int_{2s}^{\sqrt{N}} e(xu^2 - 2su)du = O(1/x). \]

Lemma 3 [cf. 6 & 7]. If \( f''(t) \geq r > 0 \) throughout \((a, b)\), then

\[ \int_a^b e(f(t))dt \ll 1/\sqrt{r}, \]

where \( \ll \) is absolute.

From this lemma we have

\[ \int_{2s}^{\sqrt{N}} e(xu^2 - 2su)du = O(1/\sqrt{x}). \]  

(11)

Thus we obtain from (9)-(11)

\[ S_N = O(s) + O(sx^{-3/2}) + O(\sqrt{N}/s) + O(1/x). \]

Finally, by choosing \( s = \frac{1}{2} N^{1/4} \), we get

\[ S_N = O(N^{1/4}), \]

where the implied constant by \( O \) depends on \( x \). \( \square \)

If the coefficients \( c_n, N \) are independent of \( N \), then the situation in general becomes more difficult and we then have the following result.

Theorem 2. For any given \( \varepsilon > 0 \), there exist a natural number \( N_0 = N_0(\varepsilon) \) and a sequence \( c_n = c_n(\varepsilon) \in \mathbb{C} \) with \( |c_n| = 1 \) \((1 \leq n \leq N)\) such that for all \( N \geq N_0 \) and \( z \) with \( |z| = 1 \),

\[ \sum_{n \leq N} c_nz^n \ll \varepsilon^2 N^{2/5} + \varepsilon. \]

Proof. We only indicate the outline of the proof since it is similar to that of Theorem 1. In this case we consider the sum
\[ S_N = \sum_{n \leq N} e(xn - n^c/c), \]

where \(0 < x \leq 1\) and \(0 < c < 1\). If we put \(f(t) = xt - t^c/c\) \((2 \leq t \leq N)\), then we have by Lemma 1

\[ \sum_{2 \leq n \leq N} e(f(n)) = \int_{2}^{N} e(f(t))dt + O(N^{1-c}). \]

Next we apply a known lemma [7, p.62] in order to estimate the above integral, say \(I(N)\). Then after simple calculation, we have for

\[ N \geq 2(2/x)^{1/(1-c)} \]

\[ I(N) - I(N/2) = O(N^{1-3c/5}), \]

where \(O\) depends on \(c\) and \(x\). Hence substituting in \(N\) successively

\[ N/2, N/2^2, \ldots \]

and adding them all, we get \(I(N) = O(N^{1-3c/5})\).

Therefore we finally obtain

\[ S_N = O(N^{1-3c/5}) + O(N^{1-c}) = O(N^{1-3c/5}) \]

\[ = O(N^{2/5} + \varepsilon), \quad (c = 1 - 5\varepsilon/3). \]

References
