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Periodic Orbits of some kinds of Periodic Systems

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ABSTRACT

We apply a modified version of the method of Sinai and others by means of computer to the existence of a closed orbit which appears in the periodic system of special type.

1. INTRODUCTION

Ja. Sinai studied the ordinary differential equations having the form $\dot{x}_i = f_i(x_1, \ldots, x_d)$, i=1, ..., d, where f_i are polynomials of degree not more than two. He applied his criterion to the Lorentz model and showed rigorously the existence of a closed orbit.

In this paper we consider a time dependent system

$$\dot{x}_i = f_i(t,X), \quad X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad i=1,2$$

where $f_i(t,X)$ are some special types described below. We use Sinai's criterion and estimate each constant with the use of a computer.

2. A CRITERION FOR THE EXISTENCE OF A CLOSED ORBIT

We consider a differential equation in R^2

$$\dot{X} = F(t,X)$$

where $F(t,X) = \begin{pmatrix} f_1(t,x) \\ f_2(t,X) \end{pmatrix}$ is defined as follows:

$$f_1(t,X) = f_1(X)$$

$$f_2(t,X) = f_2(X) + f(t).$$

$f_i(x)$ (i=1,2) are polynomials of degree not more than two and $f(t)$ is defined as follows:

$$f(t) = bf_a(t), \quad f_a(t) = \begin{cases} 1 - \frac{2}{a}t & \text{for } 0 \leq t \leq a, \\ -3 + \frac{2}{a}t & \text{for } a \leq t \leq 2a \end{cases}$$

where a, b are some real constants. We put

$$\overline{F}(X) = \begin{pmatrix} f_1(X) \\ f_2(X) \end{pmatrix}.$$

Then $F(t,X)$ can be rewritten as follows:

$$F(t,X) = \overline{F}(X) + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

This system is periodic in t of period $T = 2a$. Let $S_t X$ be the solution of (2.1) with initial condition X. Let P be the Poincaré map induced by the solution S_t. We put

$$Y = X - X^0, \quad Q(Y) = P(X) - X^0, \quad X \in R^2$$

and write Q in the form

$$Q(Y) = Y^0 + LY + K(Y).$$

Here $Y^0 = P(X^0) - X^0$, L is the matrix of linear part of Q at the point $Y = 0$ and K is a nonlinear term. The mapping K satisfies the following condition; there exist positive constants ρ_0, K_0 such that $|K(Y') - K(Y^2)| \leq K_0 \rho |Y^1 - Y^2|$ for an arbitrary $\rho \leq \rho_0$ and arbitrary Y^1, Y^2, $\| Y^1 \| \leq \rho$, $\| Y^2 \| \leq \rho$. This inequality expresses the quadratic character of K. We put

$$\varepsilon = |P(X^0) - X^0|.$$

Criterion. (J. Sinai[1]) For some $\overline{\rho}_0 < \rho_0$ $(L-E)^{-1} (\varepsilon/\overline{\rho}_0 + K_0 \overline{\rho}_0) \leq 1$.

It is easily proved under this criterion, in the $\overline{\rho}_0$ neighbourhood of X^0 there exists one and only one fixed point of Q. Now, we estimate L, ε, K_0 with the use of a computer.

3. PSEUDO ORBITS

We introduce the pseudo orbits which play the fundamental role in our problem. We define $(X,Y) = x_1y_1 + x_2y_2$ where $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$. We write $f_i(X)$ in the form

$$f_i(X) = (\ell_i, X) + (B^i X, X)$$

where (ℓ_i, X) is linear form and $(B^i X, X)$ is quadratic form. In vector notation we denote $\overline{F}(X) = (\ell, X) + (\overline{B}X, X)$ where $(\ell, X) = \begin{pmatrix} (\ell_1, X) \\ (\ell_2, X) \end{pmatrix}$ and $(\overline{B}X, X) = \begin{pmatrix} (B^1 X, X) \\ (B^2 X, X) \end{pmatrix}$. We fix a constant Δ called the step and we write

$$f(t) = At + B$$

Suppose a pseudo orbit X_i is given, we define a pseudo orbit X_{i+1} in the following. Put $t_i = i\Delta$. We take the zeroth approximation $X_0(t) \equiv X_i$; then the first approximation is given by

$$X_1(t) = X_i + \int_{t_i}^{t} F(s, X_0(s))ds.$$

Consider the second approximation

$$X_2(t_{i+1}) = X_i + \int_{t_i}^{t_{i+1}} F(s, X_1(s))ds.$$

We define the function R as follows

$$R(x_i) \equiv X_i + \Delta[(\ell, X_i) + (\overline{B}X_i, X_i) + \begin{pmatrix} 0 \\ At_i + B \end{pmatrix}] + \Delta^2 M,$$

$$M = \frac{1}{2}(\ell, \overline{F}(X_i)) + (\overline{B}X_i, \overline{F}(X_i)) + \frac{1}{2}(At_i + B)\begin{pmatrix} \ell_{12} \\ \ell_{22} \end{pmatrix} + (At_i + B)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \begin{pmatrix} 0 \\ A/2 \end{pmatrix},$$

where

$$X_i = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \ell_j = \begin{pmatrix} \ell_{j1} \\ \ell_{j2} \end{pmatrix}, \quad B^j = (B^j_{mn}), \quad \alpha_j = B^j_{12}x_1 + B^j_{22}x_2, \quad j = 1, 2.$$

R is the main part of the second approximation. This implies that we ignore the following terms of degree V than or equal to three with respect to Δ, i.e.

$$\Delta^3[\frac{1}{3}(\overline{B}\,\overline{F}(X_i),\ F(X_i) + \frac{A}{6}\begin{pmatrix}\ell_{12}\\ \ell_{22}\end{pmatrix} + \frac{2}{3}(At_i+B)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \frac{1}{3}(At_i+B)^2\begin{bmatrix}B^1_{22}\\ B^2_{22}\end{bmatrix}],$$

$$\Delta^4[\frac{A}{4}\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + \frac{A}{4})At_i+B)\begin{bmatrix}B^1_{22}\\ B^2_{22}\end{bmatrix}],\quad \text{and}$$

$$\Delta^5[\frac{A^2}{20}\begin{bmatrix}B^1_{22}\\ B^2_{22}\end{bmatrix}]$$

where $\beta_j = B^j_{12}f_1(X_i) + B^j_{22}f_2(X_i),\ \ j = 1, 2$.

One may think that we abandon too many terms. But we assert that we are able to ignore them taking Δ so small. Now we define X_{i+1} using $R(X_i)$

$$|R(X_i) - X_{i+1}| \leq \alpha$$

where α is the error arising from roundoff errors. The value α depends on the precision of the computer.

4. PRELIMINARIES

We give some notation. Let X^0 be fixed. $\gamma = \{X \quad R^2;\ X = S_tX^0,\ 0 \leq t \leq T\}$. $F'(t,X)$ is the matrix $\left(\frac{\partial f_i(t,X)}{\partial x_j}\right)$ which is equal to $\left(\frac{\partial f_i(X)}{\partial x_j}\right)$. We consider the variational equation corresponding to γ, namely $dZ/dt = F'(t, S_t X^0)Z$ where the right-handside is equal to $F'(S_tX^0)Z$. We denote by $L(t_1,t_2)$ the fundamental matrix of solution of this system on the interval $t_1 \leq t \leq t_2$.

We put

$$C_1 = \sup_{0\leq t_1\leq t_2\leq T} \| L(t_1,t_2)\| .$$

Because $f_i(X)$ are polynomials of degree not more than two, we can find a constant C_2 for which

$$\sum_{i,j,k} \left|\frac{\partial^2 f_i(X)}{\partial x_j \partial x_k} y_j z_k\right| \le C_2|Y||Z|, \qquad x \in W_\rho(\gamma)$$

where $W_\rho(\gamma)$ is the ρ-neighbourhood of γ. Further, let

$$C_3 = \sup_{X\in W_\rho(\gamma)} \|F'(t,X)\|, \qquad C_4 = \sup_{x\in W_\rho(\gamma)} |F(t,X)|.$$

We put also $\delta_1 X(t) \equiv S_t X - S_t X^0 - L(0,t)(X-X^0)$. Then the next theorem can be proved analogously to the proof in [1].

<u>Theorem</u> A. (Ja. Sinai [1]) Let $|X-X^0| \le \rho$ where ρ satisfies the inequality $\rho \le 1/2TC_1^2C_2$. Then for $B_1(t) = 2C_1^3C_2t$

$$|\delta_1 X(t)| \le B_1(t)|X-X^0|^2 \qquad 0 \le t \le T.$$

5. ESTIMATION OF ε

Let Δ be the step and $n\Delta = T$, where T is the period of (2.1). We consider the pseudo orbits $\{X_i\}$ as before. Then we have

$$\varepsilon = |S_T X^0 - X^0| \le |S_T X^0 - X_n| + |X_n - X^0|.$$

Here $|X_n - X^0|$ is found from the result of calculation. Therefore we have only to estimate $|S_T X^0 - X_n|$. Let us put $Z_i \equiv S_{i\Delta} X^0 - X_i$. We define recurrence equations V_{i+1} such that

$$Z_{i+1} = L(t_i, t_{i+1}) Z_i + V_{i+1}.$$

Then using the property of fundamental matrix, Z_{i+1} can be rewritten as follows:

$$Z_{i+1} = \sum_{j=0}^{i+1} L(t_j, t_{i+1}) V_j.$$

Therefore it is sufficient to estimate V_j. V_{j+1} can be rewritten as follows:

$$V_{j+1} = X_{j+1} - S_\Delta X_j + \delta_1 Z_j$$

where $\delta_1 Z_j = S_\Delta X_j - S_\Delta(S_{j\Delta} X^0) - L(t_j, t_{j+1})Z_j$. First, we estimate $X_{j+1} - S_\Delta X_j$. Let us assume the following inequalities with respect to U and Δ

$$\left|\frac{A}{6}\binom{\ell_{12}}{\ell_{22}} + \frac{2}{3}(AT+B)\binom{\beta_1}{\beta_2} + \frac{1}{3}(AT+B)^2\begin{pmatrix}B^1_{22}\\ B^2_{22}\end{pmatrix}\right| \leq U$$

$$\left|\frac{A}{4}\binom{\beta_1}{\beta_2}\Delta + \frac{A}{4}(AT+B)\begin{pmatrix}B^1_{22}\\ B^2_{22}\end{pmatrix}\Delta + \frac{A^2}{20}\begin{pmatrix}B^1_{22}\\ B^2_{22}\end{pmatrix}\Delta^2\right| \leq 1$$

$$\exp(C_3\sqrt{2}\,\Delta) - [1 + C_3\sqrt{2}\,\Delta + \frac{1}{2}(C_3\sqrt{2}\,\Delta)^2] \leq \frac{1}{5}(C_3\sqrt{2})^3\Delta^3.$$

Then the following inequalities hold

$$|X_{j+1} - S\,X_j| \leq |X_{j+1} - RX_j| + |RX_j - S_\Delta X_j|$$

$$\leq \alpha + \exp(C_3\sqrt{2}\,\Delta) - [1 + C_3\sqrt{2}\,\Delta + \frac{1}{2}(C_3\sqrt{2}\,\Delta)^2] + (\frac{1}{3}C_2C_4^2 + U + 1)\Delta^3$$

$$\leq \alpha + [\frac{1}{5}(C_3\sqrt{2})^3 + \frac{1}{3}C_2C_4^2 + U + 1]\Delta^3.$$

On the other hand, under the conditions of Theorem A we have

$$|\delta_1 Z_j| \leq B_1(\Delta)|Z_j|^2 \leq B_1(\Delta)C_1^2[\sum_{i=0}^{j}|V_i|]^2.$$

Now we make the inductive hypothesis

$$|V_j| \leq \bar{A}\Delta^2 \qquad 0 \leq j \leq k,$$

and we find a condition on $\bar{A}$ under which the inequality is also valid for $j = k+1$.

$$|V_{k+1}| \le |\delta_1 Z_k| + |X_{k+1} - S_\Delta X_k|$$

$$\le B_1(\Delta) C_1^2 (k+1) \bar{A}^2 \Delta^4 + \Delta^2 [\frac{\alpha}{\Delta^2} + \Delta\{\frac{1}{5}(C_3\sqrt{2})^3 + \frac{1}{3} C_2 C_4^2 + U + 1\}]$$

$$\le \Delta^2 [B_1(\Delta) C_1^2 T^2 \bar{A}^2 + \frac{\alpha}{\Delta^2} + \Delta\{\frac{1}{5}(C_3\sqrt{2})^3 + \frac{1}{3} C_2 C_4^2 + U + 1\}].$$

Let $\bar{A}$ be the least root of the quadratic equation

$$\bar{A} - B_1(\Delta) C_1^2 T^2 \bar{A}^2 = \frac{\alpha}{\Delta^2} + \Delta\{\frac{1}{5}(C_3\sqrt{2})^3 + \frac{1}{3} C_1 C_4^2 + U + 1\}.$$

Then we have

$$|V_{k+1}| \le \bar{A}\Delta^2 .$$

Lastly we can estimate $|Z_i|$ as follows:

$$|Z_i| \le |\sum_{j=0}^{n} L(j\Delta, n\Delta) V_j| \le C_1 \sum_{j=0}^{n} |V_j| \le C_1 n \bar{A} \Delta^2 .$$

6. ESTIMATION OF L

In our case we have $\qquad L = L(0, T)$.

We define the matrix $\bar{L}(0, i\Delta)$ using the pseudo orbits $\{X_i\}$ as follows:

$$L(0, i\Delta) = [E + \Delta\bar{F}(X_{i-1})]\,\bar{L}(0, (i-1)\Delta) + \delta L_i$$

$$\|\delta L_i\| \le \beta$$

where δL_i is the error arising from roundoff errors. Then $\bar{L}(0, i\Delta)$ is the approximate value of the matrix $L(0, i\Delta)$. In order to estimate the error, we have

$$L(0,(i+1)\Delta) - \bar{L}(0,(i+1)\Delta) = [E + \Delta\bar{F}'(X_i)][L(0,i\Delta) - \bar{L}(0,i\Delta)] + \delta_1 L_{i+1}$$

where $\delta_1 L_{i+1} = [L(i\Delta,(i+1)\Delta) - (E + \Delta\bar{F}'(X_i))]L(0,i\Delta) - \delta L_{i+1}$. Now we can write

$$L(0,(i+1)\Delta) - \overline{L}(0,(i+1)\Delta) = \sum_{j=0}^{i} \prod_{j=k}^{i} [E + \Delta\overline{F}'(X_i)]\delta_1 L_k.$$

Let us take C_1 so that (see Ja. Sinai [1])

$$\| \prod_{j=k}^{i} (E + \Delta\overline{F}'(X_i))\| \leq C_1.$$

Then we have

$$\| L(0,T) - \overline{L}(0,T)\| \leq C_1 \sum_{k=0}^{n} \|\delta_1 {}_k\|.$$

Therefore it is sufficient to estimate $\|\delta_1 L_k\|$ for our purpose. The following inequalities hold.

$$\|\delta_1 L_{i+1}\| \leq \|[L(i\Delta,(i+1)\Delta) - (E + \Delta\overline{F}'(S_{i\Delta}X^0))]L(0,i\Delta)\|$$

$$+ \Delta\|(\overline{F}'(S_{i\Delta}X^0) - \overline{F}'(X_i))L(0,i\Delta)\| + \|\delta L_{i+1}\|$$

$$\|\overline{F}(S_{i\Delta}X^0) - \overline{F}'(X_i)\| \leq C_2|X_i - S_{i\Delta}X^0| \leq C_1C_2 i\overline{A}\Delta^2$$

Let us take Δ so that

$$\exp(C_3\sqrt{2}\,\Delta) - (1 + C_3\sqrt{2}\,\Delta) \leq 2C_3^2\Delta^2.$$

Then we have

$$\|\delta_1 L_{i+1}\| \leq \Delta^2(2C_1C_3^2 + C_1C_2C_4 + C_1^3C_2 i A + \frac{\beta}{\Delta^2})$$

Finally, we get the following

$$\| L(0,T) - \overline{L}(0,T)\| \leq C_1T(2C_1C_3^2\Delta + C_1C_2C_4 + C_1^3C_2TA + \frac{\beta}{\Delta}).$$

This completes the estimation of L.

7. ESTIMATION OF K_0

In order to evaluate the constant K_0, we need two lemmas.

Lemma B. (S. De Gregorio and others [2]) If $|X-X_0| \leq \rho_0$, $\rho_0 = 1/C_1^2C_2T$, then for $0 \leq t \leq T$,

$$|S_tX - S_tX^0| \leq 2C_1|X - X^0|.$$

Lemma C. ([2]) If $|X^1 - X^0| \leq \rho_0$, $|X^2 - X^0| \leq \rho_0$, then for $0 \leq t \leq T$

$$|S_tX^1 - S_tX^2| \leq 8C_1|X^1 - X^2|.$$

The proof is analogous to the proof in [2]. Now we are able to estimate K_0.

Proposition D. If $\rho \leq \rho_0$ and $|X^1 - X^0| \leq \rho$, $|X^2 - X^0| \leq \rho$, then

$$|K(Y^1) - K(Y^2)| \leq K_0\rho|Y^1 - Y^2|$$

where $Y^1 = X^1 - X^0$, $Y^2 = X^2 - X^0$ and

$$K_0 = 16TC_1^3C_2.$$

Proof. By the definition of $Q(Y)$ we have

$$S_TX^i = PX^0 - X^0 + LY^i + K(Y^i), \quad i = 1, 2.$$

From this we have

$$|K(Y^1) - K(Y^2)| = S_TX^1 - S_TX^2 - L(Y^1 - Y^2).$$

We obtain the equality

$$S_TX^i - S_TX^0 = L(0,T)Y^i + \frac{1}{2}\int_0^T ds\, L(s,T)h^i(s), \quad i = 1, 2$$

where $h^i(s) = (\bar{F}''(S_s X^i - S_s X^0), (S_s X^i - S_s X^0))$ and hence

$$S_T X^1 - S_T X^2 = L(0,T)(Y^1 - Y^2) + \frac{1}{2}\int_0^T ds\ L(s,T)(h^1(s) - h^2(s)).$$

Since $L = L(0,T)$, we have

$$K(Y^1) - K(Y^2) = \frac{1}{2}\int_0^T ds\ L(s,T)(h^1(s) - h^2(s)).$$

From this we have

$$|K(Y^1) - K(Y^2)| \leq \frac{1}{2} TC_1C_2 \sup_{0\leq s\leq T} |S_s X^1 - S_s X^2| \{ \sup_{0\leq s\leq T} |S_s X^1 - S_s X^0| + \sup_{0\leq s\leq T} |S_s X^2 - S_s X^0| \}.$$

By the above two lemmas we have

$$|K(Y^1) - K(Y^2)| \leq 16TC_1^3C_2\rho|Y^1 - Y^2|.$$

This completes the estimation of K_0.

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