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Kyoto University
BRAIDS OF PERIODIC POINTS AND A 2-DIMENSIONAL
ANALOGUE OF SHARKOVSKII'S ORDERING

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ABSTRACT
We consider a 2-dimensional analogue of Sharkovskii's theorem on the existence of periodic points for 1-dimensional maps. For a periodic point of an embedding of the plane into itself, we define its "braid type". We find some pairs (α, β) of braid types satisfying the property that any embedding of the plane with a periodic point of braid type α also has a periodic point of braid type β.

1. INTRODUCTION
In [13], Sharkovskii proved a theorem on the existence of periodic points for 1-dimensional maps. He introduced an order relation < on the set of all positive integers, and showed that if a continuous map from an interval to itself has a periodic point of period n, then it has a periodic point of period m for every m < n (for an English language version of the proof, see Štefan [14]). Kloeden [5] generalized Sharkovskii's theorem to some specific 2-dimensional maps.

Recently, Boyland [3] has presented an idea to obtain a 2-dimensional analogue of Sharkovskii's theorem. The essential points in his idea are to specify a periodic point by its "braid type" and to define an order relation on the set of braid types. However, his definition does not seem to be suitable for the problem of finding periodic solutions for systems of ordinary differential equations.
In this paper, we define an order relation \( \preceq \) on the set of braid types which is a generalization of Boyland's ordering and is suitable for the above problem. We also determine, for some sets \( V \) of braid types, the structures of the ordered sets \((V, \preceq)\).

2. ORDER RELATIONS ON BRAID TYPES

Let \( n \) be a positive integer. Let \( W_n \) be the space of unordered \( n \)-tuples of distinct points in the plane. A braid group \( B_n \) is the fundamental group of the space \( W_n \). An element of \( B_n \) is called an \( n \)-braid or simply a braid. \( B_n \) is a finitely generated group with generators \( \sigma_1, \ldots, \sigma_{n-1} \) (see Fig.1) and defining relations (see e.g. [2, Theorem 1.8]):

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if} \quad |i - j| \geq 2
\]

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \quad i = 1, \ldots, n-2.
\]

The center \( C(B_n) \) of \( B_n \) is the infinite cyclic subgroup generated by \( \theta_n = (\sigma_1 \sigma_2 \ldots \sigma_{n-1})^n \) (see e.g. [2, p.28]).

\[
\begin{array}{ccccccc}
\bullet & \cdots & \bullet & \cdots & \bullet \\
1 & i-1 & i & i+1 & i+2 & n
\end{array}
\]

\( \text{Fig. 1} \)

Define a homomorphism \( P \) from \( B_n \) to the symmetric group of degree \( n \) by \( P(\sigma_i) = \text{transposition of } i \text{ and } i+1 \).

Definition 1. An \( n \)-braid type (or simply a braid type) is the conjugacy class of an \( n \)-braid which is mapped to an \( n \)-cycle under the
homomorphism \( P \). (In the remainder of this paper, the conjugacy class
of a braid \( b \) is denoted by \([b]\).) Let \( BT_n \) denote the set of
\( n \)-braid types, and let \( BT = \bigcup_{n \geq 1} BT_n \). The center \( C(B_n) \) acts on
\( BT_n \) by \( \theta[b] = [\theta b] \ (\theta \in C(B_n)) \).

Let \( M = \mathbb{R}^2 \) or \( D^2 \) (the disk). Let \( \text{Emb}(M) \) (resp. \( \text{Diff}(M) \)) be
the set of orientation preserving \( C^1 \)-embeddings (resp. \( C^1 \)-diffeomor-
phisms) of \( M \) to itself. In the remainder of this section, we fix
an arcwise connected subspace \( E \) of \( \text{Emb}(M) \) containing \( \text{id} \), the
identity map on \( M \). For \( f \in E \), denote by \( \Omega(E,f) \) the set of isoto-
pies \( F : [0,1] \to E \) with \( F(0) = \text{id}, F(1) = f \). Let \( \text{Per}(f,n) \) be the
set of \( n \)-periodic points (i.e., periodic points of least period \( n \)) of
\( f \), and let \( \text{Per}(f) \) be the set of periodic points of \( f \).

Definition 2. Let \( f \in E, F \in \Omega(E,f) \), and \( x \in \text{Per}(f,n) \). The
conjugacy class of the braid represented by a loop

\[ [0,1] \ni t \mapsto \{F(t)(x), F(t)(f(x)), \ldots, F(t)(f^{n-1}(x))\} \]

in \( \mathcal{W}_n \) is called the braid type of \( x \) with respect to \( F \), and is
denoted by \( \beta(x,F) \). Let \( \beta \) be a braid type. Then a periodic point
\( x \) of \( f \) is called a \((\beta,F)\)-periodic point if \( \beta(x,F) = \beta \). The set
of \((\beta,F)\)-periodic points is denoted by \( \text{Per}(f,\beta;F) \).

Clearly if \( \beta \) is an \( n \)-braid type, then \( \text{Per}(f,\beta;F) \subseteq \text{Per}(f,n) \).
Definition 3. We define an order relation $\preceq$ on $BT$ as follows:

$$\beta_1 \preceq \beta_2 \quad \text{if and only if}$$

any $f \in E$ and any $F \in \Omega(E,f)$ with $\text{Per}(f,\beta_1;F) \neq \emptyset$ satisfy $\text{Per}(f,\beta_2;F) \neq \emptyset$ (i.e., any $f \in E$ with a $(\beta_1,F)$-periodic point for some $F \in \Omega(E,f)$ also has a $(\beta_2,F)$-periodic point).

Remark. It is trivial that $\preceq$ is reflexive and transitive, but it is not known whether it is antisymmetric.

Consider the case of $E = \text{Diff}(D^2, \text{rel} D^2)$, the set of diffeomorphisms of $D^2$ fixing the boundary pointwise. Then $\beta(x,F)$ is independent of the isotopy $F$. Thus $\preceq$ is precisely the ordering defined by Boyland [3]. In the general case, it is not difficult to verify the following:

Proposition 1. (1) For any $F, F' \in \Omega(E,f)$, there is an element $\theta$ of the center $C(B_n)$ such that $\beta(x,F') = \theta \beta(x,F)$.

(2) If $\beta_1 \in BT_n, \beta_2 \in BT_m$, and $\beta_1 \preceq \beta_2$, then

$$\theta_n^{k_1} \beta_1 \preceq \theta_m^{k_2} \beta_2 \quad \text{for any integer } k.$$

Remark. Boyland [3, Theorem 4] has shown that there exist some braid types $\beta$ satisfying $\beta \preceq$ infinitely many braid types, in the case of $E = \text{Diff}(D^2, \text{rel} D^2)$ (see also Kobayashi [6],[7]). For our generalized ordering, a similar result seems to hold, while we have not verified it yet.
The ordering on $BT$ has an importance in the study of periodic solutions of periodic systems of ordinary differential equations.

Let $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ be a $C^1$-map and consider the following system:

$$(2.1) \quad x' = f(t, x) \quad t \in \mathbb{R}, \ x \in \mathbb{R}^2.$$ 

We assume that

1. $f$ is periodic in $t$ of period 1,
2. every solution of (2.1) is defined on an interval $(a, \infty)$, where $a \geq -\infty$ may depend on each solution.

A solution $c$ of (2.1) is called an n-periodic solution for a positive integer $n$ if it is a periodic solution admitting the period $n$ and no smaller integral period. $c$ is a periodic solution if it is an n-periodic solution for some positive integer $n$. For an n-periodic solution $c$, define a braid type $\beta(c)$ as the conjugacy class of the braid represented by a loop $\{c(t), c(t+1), \ldots, c(t+n-1)\}$ in $W_n$.

Proposition 2. Let $E = \text{Emb}(\mathbb{R}^2)$. Suppose $c$ is a periodic solution. Then for any braid type $\beta \leq \beta(c)$, there exists a periodic solution $c'$ with $\beta(c') = \beta$.

Proof. For $t \geq 0, x \in \mathbb{R}^2$, let $F(t)(x)$ be the value at the time $t$ of the solution starting $x$ at the time 0. Then a point $x$ is an n-periodic point of the map $F(1)$ if and only if the solution $F(t)(x)$ is n-periodic. Since $F(1) \in E, F \in \Omega(E, F(1))$, $c(0) \in \text{Per}(F(1))$, and $\beta(c(0), F) = \beta(c)$, we have $\text{Per}(F(1), \beta; F) \neq \emptyset$. 

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for any $\beta \leq \beta(c)$. This completes the proof.

Let $e$ be the unique element of the trivial group $B_1$. Then we have:

Proposition 3. $[e] \leq$ any braid type.

Proof. This is trivial in the case of $E \subseteq \text{Emb}(D^2)$ (in fact, any element of $E$ always has a fixed point). In the case of $E \subseteq \text{Emb}(\mathbb{R}^2)$, this follows from Massera [12, p.460].

3. BRAID TYPES $\beta$ WITH $\beta \leq$ A GIVEN 3-BRAID TYPE

In the remainder of this paper, let $E$ be one of the spaces $\text{Emb}(M)$, $\text{Diff}(M)$, $\text{Diff}(D^2, \text{rel}\emptyset D^2)$, where $M = \mathbb{R}^2$, $D^2$. First we show that there exists a combinatorial algorithm which determines all the braid types $\leq$ a given 3-braid type. Let $\theta = \theta_3 = (\sigma_1 \sigma_2)^3$. Let $g_1$ and $g_2$ be horseshoes indicated as follows:

$g_1$ and $g_2$ have attractive periodic points $p_1, p_2, p_3$ ($p_1 = (i,0)$).
Suppose a d-tuple \( J = (j_1, \ldots, j_d) \) of integers and an integer \( m \) are given. Then, define a 3-braid \( \sigma(J) \), conjugacy classes \( \rho(J), \rho(J;m) \), and a horseshoe \( \phi_J \) by

\[
\sigma(J) = \sigma_1^{-j_1} \sigma_2^{-j_2} \cdots \sigma_1^{-j_1} \sigma_2^{-j_2},
\]

\[
\rho(J) = [\sigma(J)], \quad \rho(J;m) = \sigma^m \rho(J),
\]

\[
\phi_J = \phi_1 \cdot \phi_2 \cdots \phi_1 \cdot \phi_2.
\]

These are denoted also by \( \sigma(j_1, \ldots, j_d) \) etc. Also, let \( \phi_{J,m} \) be an element of \( \Omega(\text{Diff}(\mathbb{R}^2), \phi_J) \) such that the 3-braid represented by the loop \( \{ \phi_{J,m}(t)(p_1), \phi_{J,m}(t)(p_2), \phi_{J,m}(t)(p_3) \} \) is equal to \( \rho(J;m) \).

For an integer \( k \) not divisible by 3, let \( \rho^k = [(\sigma_1 \sigma_2)^k] \). It is clear that there are no braid types \( \rho^k \) other than \( [e] \).

Theorem 1. Let \( \beta_0 \) be a 3-braid type with \( \beta_0 \neq \rho^k \) for any \( k \). Then

(1) \( \beta_0 = \rho(J;m) \) for some integer \( m \) and some d-tuple \( J \) of non-negative integers.

(2) For a braid type \( \beta \), the following two conditions are equivalent:

(i) \( \beta \preceq \beta_0 \).

(ii) \( \beta = \beta(x, \phi_{J,m}) \) for some periodic point \( x \) of \( \phi_J \).

Since the structure of the set of periodic points of any horseshoe is completely known, this theorem gives an algorithm to determine the
braid types $\beta$ with $\beta \leq$ a given 3-braid type. (In Theorem 1, we may replace the map $\phi_J$ by an appropriate pseudo-Anosov map, as is seen by the arguments in the proof of Theorem 1, however it seems difficult to determine the braid types of periodic points of pseudo-Anosov maps.)

Remark. Suppose a 3-braid type $\beta$ satisfies $\beta \neq \rho^k$ for any $k$. Suppose $f$ is an element of $\text{Emb}(D^2)$ satisfying $f(\partial D^2) \cap \partial D^2 = \emptyset$ or $\partial D^2$. Then, if $f$ has a $(\beta,F)$-periodic point for some $F$, it has positive topological entropy and infinitely many periodic points (Kobayashi [6],[7],[8], Boyland [3]).

As applications of Theorem 1, we give some results obtained by carrying out the algorithm. (We omit the proofs.) First, for any 3-braid type $\beta$, we completely determine 2-braid types $< \beta$.

Proposition 4. Let $m$ be an integer, $i$ an odd integer, and $J = (j_1, \ldots, j_d)$ a tuple of non-negative integers. Then

$$\rho(J;m) > [\sigma_i^+] \quad \text{if and only if}$$

$$2m - d \leq i \leq 2m + \sum_{k=1}^{d} j_k.$$

Next, for some 3-braid types $\beta$, we determine all the 3-braid types $< \beta$. For a non-negative integer $d$, define subsets $\text{BT}_3,d$ and $V_d$ of $\text{BT}_3$ by

$$\text{BT}_3,d = \{ \beta \in \text{BT}_3 | \beta = \rho(J;m) \text{ for some integer } m \text{ and a } d\text{-tuple } J \text{ of non-negative integers} \}, \quad d > 0, \quad \emptyset.$$
\[ BT_{3,0} = \{ \rho^k \mid 3 \nmid k \}, \quad V_d = \bigcup_{k=0}^{d} BT_{3,k}. \]

It is not difficult to see that if \( \beta \in BT_3, \beta' \in V_2, \beta \preceq \beta' \), then \( \beta \in V_2 \). Hence for \( \beta \in V_2 \), we can determine all 3-braid types \( < \beta \) by:

Proposition 5. The order relation \( \succeq \) on the subset \( V_2 \) of \( BT \) is generated by the following inequalities:

1. \( \rho(i,j) > \rho(i-p,j-2;q) \) \( (p,q) = (0,0),(2,2),(4,3), i \geq p + 1, j \geq 3, \)
2. \( \rho(i,j) > \rho(i+j-1) \) \( i, j \geq 1, \)
3. \( \rho(i,1) > \rho(3i+1)/2 \) \( i \geq 1, \)
4. \( \rho(1,1) > \rho^{-2} \)
5. \( \rho(i) > \rho(i-2;p) \) \( p = 0,1, i \geq 3, \)
6. \( \rho(1) > \rho, \rho^{-1} \)
7. the inequalities obtained by multiplying both sides of the inequalities (1) - (6) by \( \theta^m \) (\( m \) is an integer), where \( i, j \) are odd integers.

For example, the order \( \succeq \) on the set \( \{ \beta \in BT_3 \mid \beta \preceq \rho(3,3) \} \) is generated by the inequalities indicated in Diagram 1. Also, the order on \( V_1 \) is generated by the inequalities in Diagram 2.

4. PROOF OF THEOREM 1

Proof of (1). Let \( \alpha = \sigma_1\sigma_2 \). Since \( \sigma_1^{-1}, \sigma_2, \sigma_1^{-1} \) are expressed as products of \( \theta, \theta^{-1}, \alpha, \sigma_1 \) and the equality \( \alpha^2 = \sigma_1\sigma_1 \) holds,
In these diagrams, "β → β'" means that β ≼ β'.

Diagram 1

Diagram 2

|0
$\beta_0$ is equal to $[\theta^p(\sigma_1^{i_1})(\sigma_1^{i_2})\cdots(\sigma_1^{i_s})]$, where $p$ is an integer, $i_1, \ldots, i_s$ are non-negative. This and the equality $\sigma_1^{i_1} = \theta \sigma_2^{-1} \sigma_1^{i-2} \theta^{-1}$ imply that $\beta_0 = \theta^{p+s}(i_1-2, \ldots, i_s-2)$. Therefore the proof is completed by the equalities (cf. [10, (5.15)]):

$$\sigma(j,-2,k) = \theta^{-1} \sigma(j+k+2), \quad \sigma(j,-1,k) = \sigma(j-1,k-1).$$

**Proof of (2).** Our proof is similar to that of Boyland [3, Theorem 4] and is an application of the result of Asimov and Franks [1]. We will prove the theorem only in the case of $E = \text{Emb}(\mathbb{R}^2)$. The other cases can be proved similarly.

We may assume that $m = 0$ without loss of generality. Let $\phi_j = \phi_{j,0}$. Since $\beta(p_1, \phi_j) = \rho(J)$, it is clear that (ii) follows from (i). We must prove (ii) implies (i), that is,

$$\rho(J) = \beta(x, \phi_j) \text{ for any periodic point } x \text{ of } \phi_j.$$

Fix a periodic point $x_0$ of $\phi_j$ of period $n$. Let $f \in E$, $F \in \Omega(E, \mathfrak{f})$, and assume that $f$ has a $(\rho(J), F)$-periodic point. We may assume that $p_1$ is such a point and $f^n(p_1) = \phi_j^n(p_1)$ for any $n$. For the proof, it is enough to find a $(\beta(x_0, \phi_j), F)$-periodic point. When $n = 1$, such a point exists by Proposition 3. Also, when $x_0 = p_1, p_2, \text{ or } p_3$, the existence is trivial. Thus we assume $n \geq 2$ and $x_0 \neq p_1, p_2, p_3$ in the following. Since $\beta(p_1, \phi_j) = \beta(p_1, F)$, there is an isotopy $G' : [0,1] \to E$ such that $G'(0) = \phi_j$, $G'(1) = f$, $G'(t)(p_1) = f(p_1)$ for any $t$, i.e., identify $S^2$ with $\mathbb{R}^2 \cup \{\infty\}$ and let $p_0 = \infty$. For
\[ i = 0, \ldots, 3, \text{ choose small disks } D_i, D'_i \text{ with } \text{Int} D_i \supset D'_i, \text{Int} D'_i \ni p_i. \]

Let \( X = S^2 - \bigcup_{i=0}^{3} D_i \). Let \( G(t) : S^2 \to S^2 \) be an isotopy such that

(a) \( G(t) = \phi_j \) on \( \bigcup_{i=0}^{3} D'_i \) for any \( t \),

(b) \( G(t) = G'(t) \) on \( X \) for any \( t \),

(c) \( G(0) = \phi_j \).

Let \( g = G(1) \). We denote a unique extension of \( \phi_j \) to \( S^2 \) by the same symbol \( \phi_j \). Let \( K = \{p_0, \ldots, p_3\} \) and \( Y = S^2 - K \).

Lemma. \( x_0 \) is the only point in its Nielsen class with respect to the map \( \phi_j^n \mid_Y \).

Proof. Let \( I \) be the involution on \( T^2 \) defined by \( I([(x,y)]) = [-(x,y)], x,y \in \mathbb{R} \). Define a subset \( K' \) of \( T^2/I \) to be the set of the orbits of \( [(i,j)], i,j = 0, 1/2 \). Let \( Z \) (resp. \( Z' \)) be the surface obtained from \( Y \) (resp. \( T^2/I - K' \)) by compactifying each end with a boundary circle. Let \( B : B_3 \to \text{SL}(2,\mathbb{Z}) \) be the homomorphism defined by

\[
(4.1) \quad B(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.
\]

Then \( A = B(\sigma(J))^n \) induces a homeomorphism \( f_A \) of \( T^2/I \). Let \( \phi_j' \) (resp. \( f_A' \)) be the homeomorphism of \( Z \) (resp. \( Z' \)) induced from \( \phi_j \) (resp. \( f_A \)). We may assume \( \phi_j' \) has no fixed points on the boundary of \( Z \). It is easily verified that there is a homeomorphism \( h \) from

| \_ | \_ |
$Z$ to $Z'$ such that $(\phi_j')^n$ is homotopic to $h^{-1}f_A'\circ h$. Therefore

$$N((\phi_j')^n) = N(f_A') = \text{trace } A = \# \text{Fix}(\phi_j^n) = \# \text{Fix}((\phi_j')^n),$$

where $N(\ )$ denotes the Nielsen number (for the definition, see [4]). Thus the proof is completed.

Note that the fixed point index $\text{ind}(x_0, \phi_j^n) \neq 0$ and every $p_1$ is an attracting or a repelling point of $G(t)$ for any $t$. Thus, by Lemma, using a slight modification of the Asimov–Franks result [1], we have an element $x_1$ of $\text{Per}(g, n)$ with $(x_0, \phi_j'y)$ and $(x_1, g'y)$ in the same strong Nielsen class, i.e., there exist an isotopy $H$ from $\phi_j'y$ to $g'y$ and a path $u$ in $Y$ joining $x_0$ to $x_1$ such that $u(t) \in \text{Per}(H(t), n)$.

Consider first the case $n \neq 3$. We claim that $\text{Orb}(x_1, g)$, the orbit of $x_1$ under $g$, is contained in $X$. Indeed, if not, then $x_1$ and $g^3(x_1)$ are close together and an argument similar to [1, p.27] leads to a contradiction to Lemma. Thus $x_1 \in \text{Per}(f, n)$.

Since $(x_0, \phi_j)$ and $(x_1, g)$ are in the same strong Nielsen class, $\beta(x_1, F) = \beta(x_0, \phi_j)$. So the theorem is shown in this case.

Now assume $n = 3$. Then $\text{Orb}(x_1, g) \subseteq X$, or $x_1$ and $p_j$ are close together for some $j = 1, 2, 3$. In the former case, the proof is completed by the argument given above. In the latter case, we have $\beta(p_1, F) = \beta(x_0, \phi_j)$ and the proof is completed.
5. PERIODIC SOLUTIONS OF PERIODIC SYSTEMS

As a byproduct of some arguments in the proof of Theorem 1, we have an improvement of the estimates for the number of periodic solutions of the system (2.1) given by the author [9, Theorem 1], [10, Theorems 2 and 5, Proposition 4], [11, Theorem 2].

Suppose \( c_1, c_2, c_3 \) are periodic solutions of (2.1) such that \( \{c_1(0), c_2(0), c_3(0)\} = \{c_1(1), c_2(1), c_3(1)\} \). Let \( p_i \) be the period of the periodic solution \( c_i \). Let \( \sigma_0 \) denote the 3-braid represented by a loop \( \{c_1(t), c_2(t), c_3(t)\} \) in \( W_3 \). For a positive integer \( p \), let \( M(p) \) be the number of \( p \)-periodic points of the subshift of finite type corresponding to the matrix \( B(\sigma_0) \). (\( B \) is defined by (4.1).)

Theorem 2. Suppose \( \sigma_0 \) is conjugate to \( \sigma^m \sigma(j_1, \ldots, j_d) \), where \( m \) is an integer, \( j_1, \ldots, j_d \) are non-negative, and \( j_k \geq 1 \) for some \( k = 1, \ldots, d \). Then, for any positive integer \( p \neq 1, p_1, p_2, p_3 \), the number of \( p \)-periodic solutions of (2.1) is not less than \( M(p) \).

Proof. Let \( L = \mathbb{R}^2 - \{c_1(0), c_2(0), c_3(0)\} \). Since \( p \neq 1, p_1 \), some arguments similar to those in the proof of Theorem 1 shows that there exists an injective map from \( \text{Per}(\phi_1, p) \cap Y \) to \( \text{Per}(F(1), p) \cap L \) where \( F \) is defined in the proof of Proposition 2. Thus we have the conclusion.

Remark. It is obvious that the estimate in Theorem 2 cannot be sharpened any more.
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