BRAIDS OF PERIODIC POINTS AND A 2-DIMENSIONAL

ANALOGUE OF SHARKOVSKII'S ORDERING

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ABSTRACT

We consider a 2-dimensional analogue of Sharkovskii's
theorem on the existence of periodic points for
1-dimensional maps. For a periodic point of an
embedding of the plane into itself, we define its
"braid type". We find some pairs (α, β) of braid
types satisfying the property that any embedding
of the plane with a periodic point of braid type α
also has a periodic point of braid type β.

1.   INTRODUCTION

In [13], Sharkovskii proved a theorem on the existence of periodic

points for 1-dimensional maps. He introduced an order relation $\prec$ on

the set of all positive integers, and showed that if a continuous map

from an interval to itself has a periodic point of period n, then it

has a periodic point of period m for every $m \prec n$ (for an English

language version of the proof, see Štefan [14]). Kloeden [5] genera-

lized Sharkovskii's theorem to some specific 2-dimensional maps.

Recently, Boyland [3] has presented an idea to obtain a 2-dimen-

sional analogue of Sharkovskii's theorem. The essential points in

his idea are to specify a periodic point by its "braid type" and to

define an order relation on the set of braid types. However, his

definition does not seem to be suitable for the problem of finding

periodic solutions for systems of ordinary differential equations.

In this paper, we define an order relation $\geq$ on the set of braid types which is a generalization of Boyland's ordering and is suitable for the above problem. We also determine, for some sets V of braid types, the structures of the ordered sets $(V, \geq)$.

2. ORDER RELATIONS ON BRAID TYPES

Let n be a positive integer. Let W_n be the space of unordered n-tuples of distinct points in the plane. A **braid group** B_n is the fundamental group of the space W_n. An element of B_n is called an n-**braid** or simply a **braid**. B_n is a finitely generated group with generators $\sigma_1, \ldots, \sigma_{n-1}$ (see Fig.1) and defining relations (see e.g. [2, Theorem 1.8]):

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad \text{if} \quad |i - j| \geq 2$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad i = 1, \ldots, n-2.$$

The center $C(B_n)$ of B_n is the infinite cyclic subgroup generated by

$$\theta_n = (\sigma_1 \sigma_2 \cdots \sigma_{n-1})^n \quad (\text{see e.g.} [2, \text{p.28}]).$$
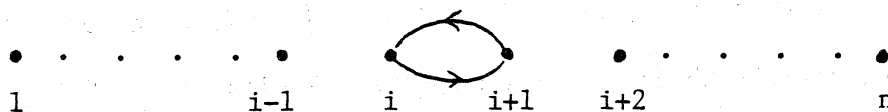

Fig. 1

Define a homomorphism P from B_n to the symmetric group of degree n by $P(\sigma_i) = $ transposition of i and $i + 1$.

Definition 1. An n-**braid type** (or simply a **braid type**) is the conjugacy class of an n-braid which is mapped to an n-cycle under the

homomorphism P. (In the remainder of this paper, the conjugacy class

of a braid b is denoted by [b].) Let BT_n denote the set of

n-braid types, and let $BT = \bigcup_{n \geq 1} BT_n$. The center $C(B_n)$ acts on

BT_n by $\theta[b] = [\theta b]$ $(\theta \in C(B_n))$.

Let $M = \mathbb{R}^2$ or D^2 (the disk). Let Emb(M) (resp. Diff(M)) be

the set of orientation preserving C^1-embeddings (resp. C^1-diffeomor-

phisms) of M to itself. In the remainder of this section, we fix

an arcwise connected subspace E of Emb(M) containing id, the

identity map on M. For $f \in E$, denote by $\Omega(E,f)$ the set of isoto-

pies $F : [0,1] \to E$ with $F(0) = id$, $F(1) = f$. Let Per(f,n) be the

set of n-periodic points (i.e., periodic points of least period n) of

f, and let Per(f) be the set of periodic points of f.

Definition 2. Let $f \in E$, $F \in \Omega(E,f)$, and $x \in Per(f,n)$. The

conjugacy class of the braid represented by a loop

$$[0,1] \ni t \longrightarrow \{F(t)(x), F(t)(f(x)),\ldots, F(t)(f^{n-1}(x))\}$$

in W_n is called the braid type of x with respect to F, and is

denoted by $\beta(x,F)$. Let β be a braid type. Then a periodic point

x of f is called a (β,F)-periodic point if $\beta(x,F) = \beta$. The set

of (β,F)-periodic points is denoted by $Per(f,\beta;F)$.

Clearly if β is an n-braid type, then $Per(f,\beta;F) \subset Per(f,n)$.

Definition 3. We define an order relation $\geq$ on BT as follows:

$\beta_1 \geq \beta_2$ if and only if

any $f \in E$ and any $F \in \Omega(E,f)$ with $\mathrm{Per}(f,\beta_1;F) \neq \emptyset$ satisfy $\mathrm{Per}(f,\beta_2;F) \neq \emptyset$ (i.e., any $f \in E$ with a (β_1,F)-periodic point for some $F \in \Omega(E,f)$ also has a (β_2,F)-periodic point).

Remark. It is trivial that $\geq$ is reflexive and transitive, but it is not known whether it is antisymmetric.

Consider the case of $E = \mathrm{Diff}(D^2, \mathrm{rel}\partial D^2)$, the set of diffeomorphisms of D^2 fixing the boundary pointwise. Then $\beta(x,F)$ is independent of the isotopy F. Thus $\geq$ is precisely the ordering defined by Boyland [3]. In the general case, it is not difficult to verify the following:

Proposition 1. (1) For any $F, F' \in \Omega(E,f)$, there is an element θ of the center $C(B_n)$ such that $\beta(x,F') = \theta\beta(x,F)$.

(2) If $\beta_1 \in BT_n$, $\beta_2 \in BT_m$, and $\beta_1 \geq \beta_2$, then

$$\theta_n^k\beta_1 \geq \theta_m^k\beta_2 \quad \text{for any integer } k.$$

Remark. Boyland [3, Theorem 4] has shown that there exist some braid types β satisfying $\beta \geq$ infinitely many braid types, in the case of $E = \mathrm{Diff}(D^2, \mathrm{rel}\partial D^2)$ (see also Kobayashi [6],[7]). For our generalized ordering, a similar result seems to hold, while we have not verified it yet.

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The ordering on BT has an importance in the study of periodic

solutions of periodic systems of ordinary differential equations.

Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be a C^1-map and consider the following system:

(2.1) $\qquad\qquad x' = f(t,x) \qquad t \in \mathbb{R}, \ x \in \mathbb{R}^2.$

We assume that

(1) f is periodic in t of period 1,

(2) every solution of (2.1) is defined on an interval (a,∞),

where $a \geq -\infty$ may depend on each solution.

A solution c of (2.1) is called an n-<u>periodic solution</u> for a

positive integer n if it is a periodic solution admitting the period

n and no smaller integral period. c is a <u>periodic solution</u> if it is

an n-periodic solution for some positive integer n. For an n-periodic

solution c, define a braid type $\beta(c)$ as the conjugacy class of the

braid represented by a loop $\{c(t), c(t+1),\ldots, c(t+n-1)\}$ in W_n.

Proposition 2. Let $E = \text{Emb}(\mathbb{R}^2)$. Suppose c is a periodic

solution. Then for any braid type $\beta \leq \beta(c)$, there exists a periodic

solution c' with $\beta(c') = \beta$.

Proof. For $t \geq 0$, $x \in \mathbb{R}^2$, let F(t)(x) be the value at the

time t of the solution starting x at the time 0. Then a point x

is an n-periodic point of the map F(1) if and only if the solution

F(t)(x) is n-periodic. Since $F(1) \in E$, $F \in \Omega(E,F(1))$,

$c(0) \in \text{Per}(F(1))$, and $\beta(c(0),F) = \beta(c)$, we have $\text{Per}(F(1),\beta;F) \neq \emptyset$

for any $\beta \leqq \beta(c)$. This completes the proof.

Let e be the unique element of the trivial group B_1. Then we have:

Proposition 3.   [e] $\leqq$ any braid type.

Proof. This is trivial in the case of $E \subset Emb(D^2)$ (in fact, any element of E always has a fixed point). In the case of $E \subset Emb(\mathbb{R}^2)$, this follows from Massera [12, p.460].

3.   BRAID TYPES β WITH $\beta \leqq$ A GIVEN 3-BRAID TYPE

In the remainder of this paper, let E be one of the spaces $Emb(M)$, $Diff(M)$, $Diff(D^2, rel\partial D^2)$, where $M = \mathbb{R}^2$, D^2. First we show that there exists a combinatorial algorithm which determines all the braid types $\leqq$ a given 3-braid type. Let $\theta = \theta_3 = (\sigma_1\sigma_2)^3$. Let g_1 and g_2 be horseshoes indicated as follows:
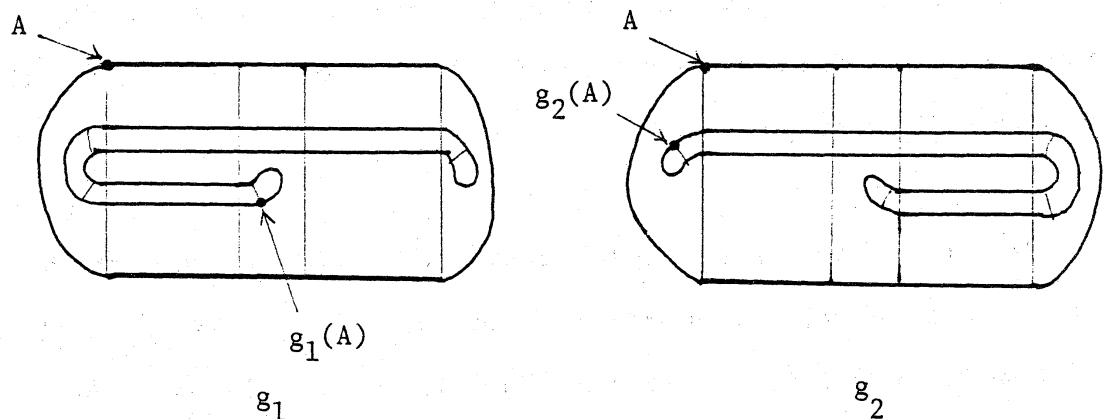
g_1 and g_2 have attractive periodic points p_1, p_2, p_3 ($p_i = (i,0)$).

Suppose a d-tuple $J = (j_1, \ldots, j_d)$ of integers and an integer m are given. Then, define a 3-braid $\sigma(J)$, conjugacy classes $\rho(J)$, $\rho(J;m)$, and a horseshoe ϕ_J by

$$\sigma(J) = \sigma_1^{j_1}\sigma_2^{-1}\sigma_1^{j_2}\sigma_2^{-1} \cdots \sigma_1^{j_d}\sigma_2^{-1} ,$$

$$\rho(J) = [\sigma(J)], \quad \rho(J;m) = \theta^m\rho(J),$$

$$\phi_J = g_1^{j_d} \circ g_2 \circ \cdots \circ g_1^{j_1} \circ g_2.$$

These are denoted also by $\sigma(j_1, \ldots, j_d)$ etc. Also, let $\Phi_{J,m}$ be an element of $\Omega(\mathrm{Diff}(\mathbb{R}^2), \phi_J)$ such that the 3-braid represented by the loop $\{\Phi_{J,m}(t)(p_1), \Phi_{J,m}(t)(p_2), \Phi_{J,m}(t)(p_3)\}$ is equal to $\rho(J;m)$.

For an integer k not divisible by 3, let $\rho^k = [(\sigma_1\sigma_2)^k]$. It is clear that there are no braid types $< \rho^k$ other than $[e]$.

Theorem 1. Let β_0 be a 3-braid type with $\beta_0 \neq \rho^k$ for any k. Then

(1) $\beta_0 = \rho(J;m)$ for some integer m and some d-tuple J of non-negative integers.

(2) For a braid type β, the following two conditions are equivalent:

    (i) $\beta \leq \beta_0$.

    (ii) $\beta = \beta(x, \Phi_{J,m})$ for some periodic point x of ϕ_J.

Since the structure of the set of periodic points of any horseshoe is completely known, this theorem gives an algorithm to determine the

braid types β with $\beta \leqq$ a given 3-braid type. (In Theorem 1, we may replace the map ϕ_J by an appropriate pseudo-Anosov map, as is seen by the arguments in the proof of Theorem 1, however it seems difficult to determine the braid types of periodic points of pseudo-Anosov maps.)

Remark. Suppose a 3-braid type β satisfies $\beta \neq \rho^k$ for any k. Suppose f is an element of $\text{Emb}(D^2)$ satisfying $f(\partial D^2) \cap \partial D^2 = \emptyset$ or ∂D^2. Then, if f has a (β, F)-periodic point for some F, it has positive topological entropy and infinitely many periodic points (Kobayashi [6],[7],[8], Boyland [3]).

As applications of Theorem 1, we give some results obtained by carrying out the algorithm. (We omit the proofs.) First, for any 3-braid type β, we completely determine 2-braid types $< \beta$.

Proposition 4. Let m be an integer, i an odd integer, and $J = (j_1, \ldots, j_d)$ a tuple of non-negative integers. Then

$$\rho(J;m) > [\sigma_1^i] \quad \text{if and only if}$$

$$2m - d \leqq i \leqq 2m + \sum_{k=1}^{d} j_k .$$

Next, for some 3-braid types β, we determine all the 3-braid types $< \beta$. For a non-negative integer d, define subsets $\text{BT}_{3,d}$ and V_d of BT_3 by

$$\text{BT}_{3,d} = \{\beta \in \text{BT}_3 | \beta = \rho(J;m) \text{ for some integer m and}$$

$$\text{a d-tuple J of non-negative integers}\}, \quad d > 0,$$

$$BT_{3,0} = \{\rho^k \mid 3 \nmid k\}, \quad V_d = \bigcup_{k=0}^{d} BT_{3,k}.$$

It is not difficult to see that if $\beta \in BT_3$, $\beta' \in V_2$, $\beta \leq \beta'$, then $\beta \in V_2$. Hence for $\beta \in V_2$, we can determine all 3-braid types $< \beta$ by:

Proposition 5. The order relation $\geq$ on the subset V_2 of BT is generated by the following inequalities:

(1) $\rho(i,j) > \rho(i-p,j-2;q)$ $(p,q) = (0,0),(2,2),(4,3),$

  $i \geq p + 1$, $j \geq 3$,

(2) $\rho(i,j) > \rho(i+j-1)$ $i, j \geq 1$,

(3) $\rho(i,1) > \rho^{(3i+1)/2}$ $i \geq 1$,

(4) $\rho(1,1) > \rho^{-2}$

(5) $\rho(i) > \rho(i-2;p)$ $p = 0,1$, $i \geq 3$,

(6) $\rho(1) > \rho, \rho^{-1}$

(7) the inequalities obtained by multiplying both sides of
   the inequalities (1) – (6) by θ^m (m is an integer),

where i, j are odd integers.

For example, the order $\geq$ on the set $\{\beta \in BT_3 \mid \beta \leq \rho(3,3)\}$ is generated by the inequalities indicated in Diagram 1. Also, the order on V_1 is generated by the inequalities in Diagram 2.

4. PROOF OF THEOREM 1

 Proof of (1). Let $\alpha = \sigma_1 \sigma_2$. Since σ_1^{-1}, σ_2, σ_2^{-1} are expressed as products of θ, θ^{-1}, α, σ_1 and the equality $\alpha^2 = \sigma_1 \alpha \sigma_1$ holds,
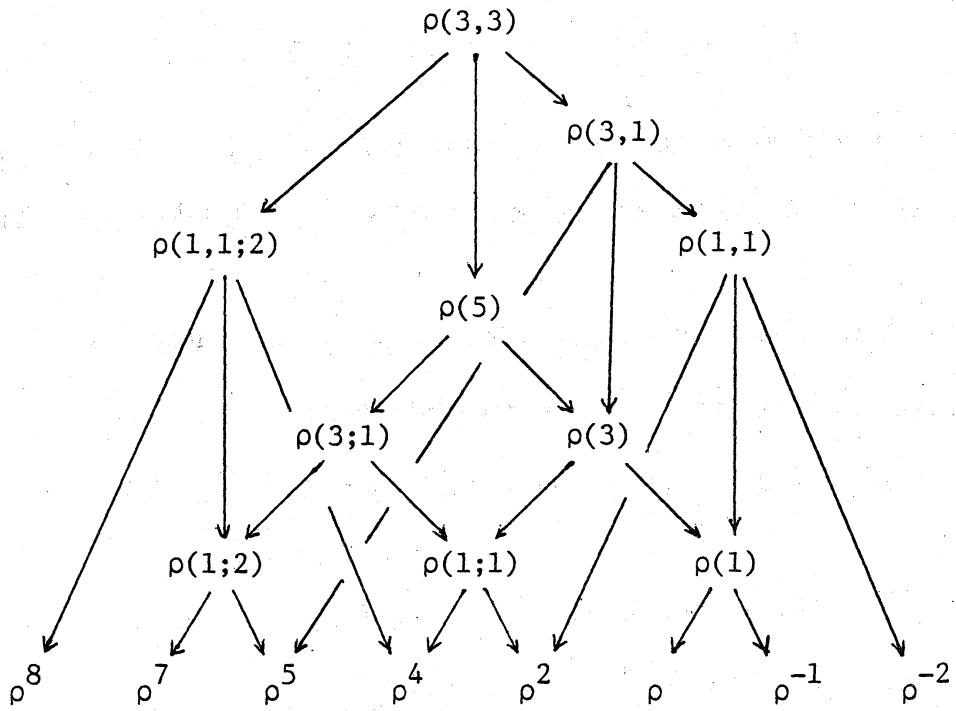
$\rho(3,3)$

$\rho(3,1)$

$\rho(1,1;2)$

$\rho(1,1)$

$\rho(5)$

$\rho(3;1)$

$\rho(3)$

$\rho(1;2)$

$\rho(1;1)$

$\rho(1)$

ρ^8 ρ^7 ρ^5 ρ^4 ρ^2 ρ ρ^{-1} ρ^{-2}

Diagram 1

$\rho(7;1)$ $\rho(7)$ $\rho(7;-1)$

$\rho(5;2)$ $\rho(5;1)$ $\rho(5)$ $\rho(5;-1)$

$\rho(3;2)$ $\rho(3;1)$ $\rho(3)$

$\rho(1;3)$ $\rho(1;2)$ $\rho(1;1)$ $\rho(1)$

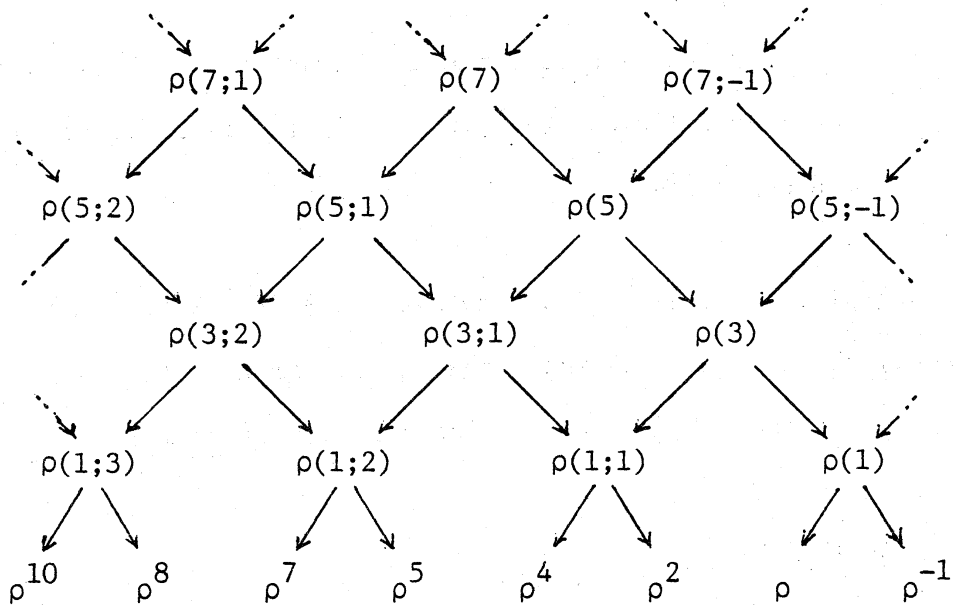ρ^{10} ρ^8 ρ^7 ρ^5 ρ^4 ρ^2 ρ ρ^{-1}

Diagram 2

In these diagrams, "$\beta \longrightarrow \beta'$" means that $\beta \geq \beta'$.

β_0 is equal to $[\theta^P(\sigma_1\alpha\sigma_1^{-1})\cdots(\sigma_1\alpha\sigma_1^{i_s})]$, where p is an integer,

$i_1,\ldots,i_s$ are non-negetive. This and the equality $\sigma_1\alpha\sigma_1^j = $

$\theta\sigma_2^{-1}\sigma_1^{j-2}$ imply that $\beta_0 = \theta^{p+s}\rho(i_1-2,\ldots, i_s-2)$. Therefore the proof

is completed by the equalities (cf. [10, (5.15)]):

$$\sigma(j,-2,k) = \theta^{-1}\sigma(j+k+2), \quad \sigma(j,-1,k) = \sigma(j-1,k-1).$$

Proof of (2). Our proof is similar to that of Boyland [3,

Theorem 4] and is an application of the result of Asimov and Franks

[1]. We will prove the theorem only in the case of $E = \mathrm{Emb}(\mathbb{R}^2)$.

The other cases can be proved similarly.

We may assume that $m = 0$ without loss of generality. Let $\Phi_J = $

$\Phi_{J,0}$. Since $\beta(p_1\cdot\Phi_J) = \rho(J)$, it is clear that (ii) follows from (i).

We must prove (ii) implies (i), that is,

$$\rho(J) \geq \beta(x,\Phi_J) \quad \text{for any periodic point } x \text{ of } \phi_J.$$

Fix a periodic point x_0 of ϕ_J of period n. Let $f \in E$, $F \in \Omega(E,f)$,

and assume that f has a $(\rho(J),F)$-periodic point. We may assume that

p_1 is such a point and $f^n(p_1) = \phi_J^n(p_1)$ for any n. For the proof,

it is enough to find a $(\beta(x_0,\Phi_J),F)$-periodic point. When $n = 1$,

such a point exists by Proposition 3. Also, when $x_0 = p_1$, p_2, or p_3,

the existence is trivial. Thus we assume $n \geq 2$ and $x_0 \neq p_1$, p_2, p_3

in the following. Since $\beta(p_1,\Phi_J) = \beta(p_1,F)$, there is an isotopy

$G' : [0,1] \to E$ such that $G'(0) = \phi_J$, $G'(1) = f$, $G'(t)(p_i) = f(p_i)$

for any t, i. Identify S^2 with $\mathbb{R}^2 \cup \{\infty\}$ and let $p_0 = \infty$. For

$i = 0,\ldots,3$, choose small disks D_i, D'_i with $\text{Int}\, D_i \supset D'_i$, $\text{Int}\, D'_i \ni p_i$.

Let $X = S^2 - \bigcup_{i=0}^{3} D_i$. Let $G(t) : S^2 \to S^2$ be an isotopy such that

(a) $G(t) = \phi_J$ on $\bigcup_{i=0}^{3} D'_i$ for any t,

(b) $G(t) = G'(t)$ on X for any t,

(c) $G(0) = \phi_J$.

Let $g = G(1)$. We denote a unique extension of ϕ_J to S^2 by the same symbol ϕ_J. Let $K = \{p_0,\ldots,p_3\}$ and $Y = S^2 - K$.

Lemma. x_0 is the only point in its Nielsen class with respect to the map $\phi_J^n \mid_Y$.

Proof. Let I be the involution on T^2 defined by $I([(x,y)]) = [-(x,y)]$, $x,y \in \mathbb{R}$. Define a subset K' of T^2/I to be the set of the orbits of $[(i,j)]$, $i,j = 0, 1/2$. Let Z (resp. Z') be the surface obtained from Y (resp. $T^2/I - K'$) by compactifying each end with a boundary circle. Let $B : B_3 \to SL(2,\mathbb{Z})$ be the homomorphism defined by

$$(4.1) \qquad B(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad B(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Then $A = B(\sigma(J))^n$ induces a homeomorphism f_A of T^2/I. Let ϕ'_J (resp. f'_A) be the homeomorphism of Z (resp. Z') induced from ϕ_J (resp. f_A). We may assume ϕ'_J has no fixed points on the boundary of Z. It is easily verified that there is a homeomorphism h from

Z to Z' such that $(\phi_J')^n$ is homotopic to $h^{-1} \circ f_A' \circ h$. Therefore

$$N((\phi_J')^n) = N(f_A') = \text{trace } A = \# \text{Fix}(\phi_J^n) = \# \text{Fix}((\phi_J')^n),$$

where $N(\ )$ denotes the Nielsen number (for the definition, see [4]). Thus the proof is completed.

Note that the fixed point index $\text{ind}(x_0, \phi_J^n) \neq 0$ and every p_i is an attracting or a repelling point of $G(t)$ for any t. Thus, by Lemma, using a slight modification of the Asimov – Franks result [1], we have an element x_1 of $\text{Per}(g,n)$ with $(x_0, \phi_J|_Y)$ and $(x_1, g|_Y)$ in the same strong Nielsen class, i.e., there exist an isotopy H from $\phi_J|_Y$ to $g|_Y$ and a path u in Y joining x_0 to x_1 such that $u(t) \in \text{Per}(H(t),n)$.

Consider first the case $n \neq 3$. We claim that $\text{Orb}(x_1,g)$, the orbit of x_1 under g, is contained in X. Indeed, if not, then x_1 and $g^3(x_1)$ are close together and an argument similar to [1, p.27] leads to a contradiction to Lemma. Thus $x_1 \in \text{Per}(f,n)$. Since (x_0, ϕ_J) and (x_1,g) are in the same strong Nielsen class, $\beta(x_1,F) = \beta(x_0, \Phi_J)$. So the theorem is shown in this case.

Now assume $n = 3$. Then $\text{Orb}(x_1,g) \subset X$, or x_1 and p_j are close together for some $j = 1,2,3$. In the former case, the proof is completed by the argument given above. In the latter case, we have $\beta(p_1,F) = \beta(x_0, \Phi_J)$ and the proof is completed.

5.   PERIODIC SOLUTIONS OF PERIODIC SYSTEMS

As a byproduct of some arguments in the proof of Theorem 1, we
have an improvement of the estimates for the number of periodic
solutions of the system (2.1) given by the author [9, Theorem 1],
[10, Theorems 2 and 5, Proposition 4], [11, Theorem 2].

Suppose c_1, c_2, c_3 are periodic solutions of (2.1) such that
$\{c_1(0), c_2(0), c_3(0)\} = \{c_1(1), c_2(1), c_3(1)\}$. Let p_i be the period
of the periodic solution c_i. Let σ_0 denote the 3-braid represented
by a loop $\{c_1(t), c_2(t), c_3(t)\}$ in W_3. For a positive integer p,
let $M(p)$ be the number of p-periodic points of the subshift of
finite type corresponding to the matrix $B(\sigma_0)$. (B is defined by
(4.1).)

Theorem 2.   Suppose σ_0 is conjugate to $\theta^m \sigma(j_1, \ldots, j_d)$, where
m is an integer, $j_1, \ldots, j_d$ are non-negative, and $j_k \geq 1$ for some
$k = 1, \ldots, d$. Then, for any positive integer $p \neq 1$, p_1, p_2, p_3, the
number of p-periodic solutions of (2.1) is not less than $M(p)$.

Proof.   Let $L = \mathbb{R}^2 - \{c_1(0), c_2(0), c_3(0)\}$. Since $p \neq 1$, p_i,
some arguments similar to those in the proof of Theorem 1 shows that
there exists an injective map from $Per(\phi_J, p) \cap Y$ to $Per(F(1), p) \cap L$
where F is defined in the proof of Proposition 2. Thus we have the
conclusion.

Remark.   It is obvious that the estimate in Theorem 2 cannot be
sharpened any more.

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