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SINGULAR PERTURBATIONS FOR CONSTRAINT SYSTEMS*

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ABSTRACT

We will show a singular perturbation theorem for constraint systems, which is a generalized version of the equation; $\dot{x} = f(x,y)$, $\varepsilon\dot{y} = g(x,y)$. At the first, we study the general properties G0 ~ G3 of constraint systems. After this we show the properties of solutions and singular perturbation theorem for constraint system satisfying G0 ~ G3.

1. INTRODUCTION

The system which we want to study here was suggested by the equations of the form

$$\begin{aligned} \dot{x} &= f(x,y) \\ 0 &= g(x,y), \end{aligned} \qquad (1.1)_0$$

$x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Many types of solutions of $(1.1)_0$ have been studied by considering $(1.1)_0$ as limit of

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$$\dot{x} = f(x,y)$$
$$\varepsilon\dot{y} = g(x,y) \qquad (1.1)_\varepsilon$$

for $\varepsilon \to 0$. For the studies of this type with $m = n = 1$, there are works of B. van der Pol [14], J. LaSalle [11], A.A. Andronov and et al. [1], and others. For the case of $m = 2$ and $n = 1$, there are works of E.C. Zeeman [15], E. Benoit [2],[3], and others. For general m and n, there are the works of L.S. Pontryagin [12], F. Takens [13], and N. Fenichel [5].

For the global version of the equation $(1.1)_\varepsilon$, we consider a vector field $\tilde{Z}_\varepsilon/\varepsilon$, where $\{\tilde{Z}_\varepsilon\}$, $\varepsilon \in [0,\varepsilon_0)$, is a family of vector field on a manifold M. The limit of $\tilde{Z}_\varepsilon/\varepsilon$ for $\varepsilon \to 0$ exists only on the set Σ of points where $\tilde{Z}_\varepsilon = 0$, (in the case of $(1.1)_\varepsilon$, Σ is the set of points where $g(x,y) = 0$). But, generically in the sense of perturbations of $\tilde{Z}$, Σ is a discrete set. To avoid this, we assume that $\tilde{Z}_0$ is tangent to the leaves of a codimension m foliation $\mathcal{F}$ on M. $\mathcal{F}$ can be considered as a generalization of the product structure $\mathbb{R}^m \times \mathbb{R}^n$. The vector field tangent to $\mathcal{F}$ is a generalization of the equation $\dot{y} = g(x,y)$ in $(1.1)_\varepsilon$.

A <u>constraint system</u> is defined as the pair $\{\{\tilde{Z}_\varepsilon\},\mathcal{F}\}$ as above (Definition 5.1). After the definition of the <u>solution</u> for a constraint system (Definition 5.4) we will define an <u>admissible solution</u>, which is a solution having useful properties (Definition 5.5). These definitions are motivated by F. Takens' definitions of constrained equations and the solutions [13]. Takens considered a fibre bundle structure, whereas we take a foliation. He considered a kind of function $M \to \mathbb{R}$ which pliyed similar role as our vector field $\tilde{Z}_0$ tangent to F.

Our main goal is Theorem E, Theorem F, and Theorem G. But these theorems are proved for systems having some generic properties. In section 4 we show generic properties G0, G1, and G2. G0 assures that the set of equilibrium points Σ of $\tilde{\Sigma}_0$ is a manifold. G1 is a regularity condition of the derivative of $\tilde{Z}_0$ on Σ. G2 assures that Σ has a stratification S, which is stratified by the number of zero-eigenvalues and the number of pure imaginary eigenvalues of the derivative of $\tilde{Z}_0|L_p$ at $p \in \Sigma$. Here L_p is a plaque of F containing p. Theorem A in section 4 asserts that G0, G1, and G2 are generic properties. We set another property G3 in section 4, which assures that the manifold Σ is in general position in the foliation F with respect to Thom-Boardman singularities. Theorem B in section 4 implies that the set of $\{\tilde{Z}_\varepsilon\}$ having property G3 is dense in the space of families of vector field on M which is a subspace of $\mathfrak{X}^r(M \times [0,\varepsilon_0))$.

Saddle-node bifurcation and Hopf bifurcation are well knows as typical codimension one bifurcations of equilibria. Theorem C in section 4 shows where these bifurcations of $\tilde{Z}_0|L_p$ appear for $p \in \Sigma$. Theorem C expresses the place in the language of the stratification S and Thom-Boardman's stratification. Theorem D determines the qualitative structure of $\tilde{Z}_0$ near the point p where saddle-node bifurcation occurs. Theorem A, .., Theorem D in section 4 are proved in [8].

Theorem E and Theorem F in section 5 shows the properties of addmissible solutions. Theorem G is the singular perturbation theorem for admissible solutions. This is an extension, in some senses, of L.S. Pontryagin [12] and N. Fenichel [5]; see Remark 5.9. Theorem E, Theorem F, and Theorem G are proved in [10].

In the case that Σ has codimension one (i.e. $m=1$), it is trivial to see that the jumping path (trace of Definition 5.7) leaving a fold point is unique. When $m>1$, the uniqueness and other properties of the jumping path are obtained by Theorem D as the properties of the stable sets.

There is an example of constraint system in the theory of LC-network perturbation (G. Ikegami [7],[9]). In this theory, there is a foliation $\mathcal{F}$ (not a trivial product structure $\mathbb{R}^m \times \mathbb{R}^n$) and a one parameter family of vector spaces, $\tilde{Z}_\varepsilon = \varepsilon X + Y$ such that Y is tangent to $\mathcal{F}$.

2. PRELIMINARIES

Let M be a smooth (C^∞) manifold with dimension $m+n$, and be a smooth foliation on M with codimension m. $\mathcal{F}$ is a disjoint decomposition of M into n dimensional injectively immersed connected smooth submanifolds (leaves) such that M is covered by C^∞ charts

$$\alpha_1 \times \alpha_2 : U \to D^m \times D^n \tag{2.1}$$

and $(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n)$ is included in the leaf through $(\alpha_1 \times \alpha_2)^{-1}$ (x,y), $y \in D^n$, where D^m and D^n are the open disks in $\mathbb{R}^m$ and $\mathbb{R}^n$, resp. We denote

$$(\alpha_1 \times \alpha_2)^{-1}(\{x\} \times D^n) = L_{(x,y)},$$

and call it the <u>plaque</u> containing the point (x,y).

Let $\tau : T\mathcal{F} \to M$ be the subbundle of the tangent bundle $TM \to M$ such that the fibre $\tau^{-1}(p)$ is an n-dimensional vector space which is tangent to the leaf of $\mathcal{F}$ through $p \in M$. Let $Y : M \to T\mathcal{F}$ be a C^r section of the vector bundle τ. Y is also a C^r-section of the tangent bundle $TM \to M$. We call such a section a C^r vector field on M <u>tangent to the foliation</u> $\mathcal{F}$. Denote by $\mathcal{Y}^r(\mathcal{F})$ the space of all C^r vector field tangent to $\mathcal{F}$ with the Whitney C^r topology.

We write Σ_Y for the subset of equilibrium points of a vector field $Y \in \mathcal{Y}^r(\mathcal{F})$. A point $p \in \Sigma_Y$ is called a <u>regular point</u>, if the derivative dY at p has the maximal rank n. $p \in \Sigma Y$ is called a <u>normally regular point</u>, if $d(Y|L_p)(p)$ is nondegenerate, where L_p is the plaque of $\mathcal{F}$ at p. We denote by Σ_r the set of normally regular points of Σ_Y. A point $p \in \Sigma_Y$ is called a <u>normally hyperbolic point</u> (resp. <u>normally stable point</u>), if p is a hyperbolic equilibrium point (resp. stable equilibrium point) of $Y|L_p$. We write Σ_h (resp. Σ_s) the set of normally hyperbolic (resp. stable) points. We have

$$\Sigma_s \subset \Sigma_h \subset \Sigma_r \subset \Sigma_Y.$$

Let $\partial\Sigma_h$ be the set of all frontiers of Σ_h; $\partial\Sigma_h = \overline{\Sigma}_h - \Sigma_h$.

A <u>stratification</u> $\mathcal{S}$ of a topological space N is a partition of N into subsets, which will be called the <u>strata</u> of $\mathcal{S}$, such that the following conditions are satisfied:

(a) Each stratum S is locally closed, i.e. each point $s \in S$ has a neighborhood U such that $U \cap S$ is closed in U.

(b) S is locally finite, i.e. each point has a neighborhood meeting only finitely many strata.

(c) If S_1 and S_2 are strata and $\overline{S}_1 \cap S_2 \neq \phi$, then $S_2 \subset \overline{S}_1$.

The relation $S_2 < S_1$ defined by $S_2 \subset \overline{S}_1$, $S_2 \neq S_1$, is an order on S. It is transitive and cannot have both $S_2 < S_1$ and $S_1 < S_2$.

Let $\tilde{N}$ be a C^1 manifold, let $N \subset \tilde{N}$, and let S be a stratification of N. We will say that S is a <u>Whitney stratification</u> if each stratum is a C^1 submanifold, and if S_1, S_2 are two strata with $S_2 < S_1$, then for all $x \in S_2$ the triple (S_1, S_2, x) satisfies the following Whitney's regularity condition.

Condition: For any sequences $\{x_i\}$ of points in S_2 and $\{y_i\}$ of points in S_1, such that $x_i \to x$, $y_i \to x$, $x_i \neq y_i$, segment $\overline{x_i y_i}$ converges (in projective space), and the tangent space $T_{x_i} S^1$ converges (in Grassmanian of $(\dim S_1)$-plane in $\mathbb{R}^n$, $n = \dim N$), we have $\ell \subset T_\infty$, where $\ell = \lim \overline{x_i y_i}$ and $T_\infty = \lim T_{x_i} S^1$.

Let S^i denote the substratification of a stratification S such that S^i consists of all strata of dimension $\leqq i$ of S. We call S^i the i-<u>skeleton</u>.

3. THOM-BOARDMAN SINGULARITIES MODULO FOLIATION

Suppose L, N are smooth manifold and f, g: $L \to N$ are C^k maps with $f(p) = g(p) = q$. f has <u>first order contact</u> with g at p if $(df)_p = (dg)_p$ as mapping $T_pL \to T_qN$ of tangent spaces. f has <u>kth order contact</u> with g at p if (df): $TL \to TN$ has (k-1)st order contact with (dg) at every point in T_pL.

Let M be a smooth manifold of dimension $m+n$, and let F be a smooth foliation on M with codimension m. Let L be a smooth manifold without boundary.

Definition 3.1. Suppose $f, g : L \to M$ are C^k maps with $f(p) = g(p) = q$. f is said to have kth order contact modulo $\mathcal{F}$ with g at p if, for some (and hence for any) chart $(U, \alpha_1 \times \alpha_2)$ of $\mathcal{F}$ with $q \in U$ given by (2.1), $\alpha_1 \circ f : L \to D^m$ has kth order contact with $\alpha_1 \circ g$ at p. This is written as $f \sim_k g \bmod \mathcal{F}$ at p. Let $J^k(L, M; \mathcal{F})_{p,q}$, $k \geqq 1$, denote the set of equivalence classes under "$\sim_k \bmod \mathcal{F}$ at p" of mappings $f : L \to M$ where $f(p) = q$. Let $J^0(L, M; \mathcal{F})_{p,q} = \{(p,q)\}$. Let $J^k(L, M; \mathcal{F}) = \cup_{(p,q)\in L\times M} J^k(L, M; \mathcal{F})_{p,q}$ (disjoint union). We call $J^k(L, M; \mathcal{F})$ a jet space modulo $\mathcal{F}$. An element σ in $J^k(L, M; \mathcal{F})$ is called a k-jet modulo $\mathcal{F}$ of mapping from L to M.

For a C^k mapping $f : L \to M$, a jet extension

$$j^k f : L \longrightarrow J^k(L, M; \mathcal{F})$$

is defined by stipulating that $j^k f(x)$ is the k-jet mod $\mathcal{F}$ of f at $x \in L$.

Our jet spaces modulo foliations follow the J.M. Boardman's theory [4]. Hence, we have the following.

Proposition 3.2. For each sequence $I = (i_1, i_2, \ldots, i_k)$ of integers, the submanifold (not necessarily closed) $\tilde{\Sigma}^I$ of the jet space modulo foliation $J^k(L, M; \mathcal{F})$ is defined. $\tilde{\Sigma}^I$ is empty unless I satisfies

$$i_1 \geqq i_2 \geqq \cdots \geqq i_{k-1} \geqq i_k \geqq 0,$$

$$\ell \geqq i_1 \geqq \ell - m,$$

$$\text{if } i_1 = \ell - m, \text{ then } i_1 = i_2 = \cdots = i_k.$$

Proposition 3.3. If $f : L \to M$ is a map whose jet section modulo F, $j^k f : L \to J(L, M; F)$ is transverse to $\tilde{\Sigma}^I$, then $\tilde{\Sigma}^I(f) \equiv (j^k f)^{-1}(\tilde{\Sigma}^I)$ is a submanifold of L. If I,i denotes the extended sequence $(i_1, i_2, \ldots, i_k, i)$, we have $\tilde{\Sigma}^{I,i}(f) = \tilde{\Sigma}^i(f|\tilde{\Sigma}^I(f))$. Also, when $I = \phi$, $\tilde{\Sigma}^i(f) = \{p \in L : \dim \operatorname{Ker} j^1 f(p) = i\}$.

Proposition 3.4. Any map $f : L \to M$ of class C^{r+1} may be C^{r+1} approximated in the C^{r+1} sense by a map $g : L \to M$ whose r-jet extension $j^r g : L \to J^r(L, M; F)$ is transverse to all submanifolds $\tilde{\Sigma}^{i_1, \ldots, i_s}$, $1 \leqq s \leqq r$.

We call $\tilde{\Sigma}^I$ the Thom-Boardman submanifold of $J^r(L, M; F)$ associated with Thom-Boardman symbol I.

These definitions and propositions in this section are described in [8].

4. GENERIC PROPERTIES OF VECTOR FIELDS TANGENT TO F.

In this section we introduce some theorems obtained by Ikegami [8].

Definition 4.1. Let $\dim M = m + n$ and $\operatorname{codim} F = m$. The following are the properties of the vector field $Y \in \mathcal{Y}^r(M, F)$.

G0: The set Σ_Y of all equilibrium points of Y is, if nonempty, an m dimensional C^r manifold.

G1: Every point of Σ_Y is regular.

G2: Y has the property G0 and there is a Whitney stratification $\mathcal{S}$ on Σ_Y having the following properties:

(i) If the differential $d(Y|L_p)(p)$ at p has ℓ eigenvalues of zero and $2(k-\ell)$ non-zero pure imaginary eigenvalues

$$0, \ldots, 0, ib_1, -ib_1, \ldots, ib_{k-\ell}, -ib_{k-\ell},$$

then p is contained in the $(m-k)$ skeleton $\mathcal{S}^{m-k}$.

(ii) The union of all $(m-1)$ dimensional strata $\cup S^{m-1}$ is a dense subset of $\partial\Sigma_h$.

(iii) $\cup S^{m-1}$ is divided into two parts, $(\partial\Sigma_h)_0$ and $(\partial\Sigma_h)_{img}$, of unions of strata such that

$$p \in (\partial\Sigma_h)_0 \implies 0 \text{ is an eigenvalue of } d(Y|L_p)(p),$$

$$p \in (\partial\Sigma_h)_{img} \implies \text{the eigenvalues of } d(Y|L_p)(p) \text{ include a pair of}$$

non-zero pure imaginary numbers.

G3: Y has the property G0, and for $k = 1, 2$, the k-jet extension $j^k\iota: \Sigma_Y \to J^k(\Sigma_Y, M; \mathcal{F})$ of the inclusion map $\iota: \Sigma_Y \to M$ is transverse to $\tilde{\Sigma}^I$ for all Thom-Boardman submanifold $\tilde{\Sigma}^I$ of length k symbol I.

Let $\mathcal{Y}^r_k$ denote the set of $Y \in \mathcal{Y}^r(M, \mathcal{F})$ satisfying the property Gk, $k = 0, 1, 2, 3$.

Theorem A. For $k = 0, 1, 2$, the set $\mathcal{Y}^r_k$ is open dense in $\mathcal{Y}^r(M; \mathcal{F})$, if $k+1 \leq r < \infty$.

Theorem B. $\mathcal{Y}^r_3$ is dense in $\mathcal{Y}^r(M; \mathcal{F})$ for $3 \leq r < \infty$.

Let $\iota : \Sigma_Y \to M$ be the inclusion map. Let $\tilde{\Sigma}^I \subset J^k(\Sigma_Y, M; F)$ be the Thom-Boardman manifold for Thom-Boardman symbol I. Denote $\tilde{\Sigma}^I(Y) \equiv (j^k\iota)^{-1}(\tilde{\Sigma}^I)$.

Let $\tau : TF \to M$ be the vector bundle of vectors tangent to F. Let $(\alpha, \alpha_1 \times \alpha_2, U)$ be a vector bundle chart of τ. Let $J^1(\tau)$ be the 1-jet space of germs of partial sections of τ. Define $\tilde{\Sigma}^i_\tau$ to be the set of 1-jet $\sigma \in J^1(\tau)$ such that, if Y represents σ at $p \in M$, then $Y(p) = 0$ and the rank of $d(Y|L_p)(p) = n - i$. Denote $\tilde{\Sigma}^i_\tau(Y) \equiv (j^1Y)^{-1}(\tilde{\Sigma}^i_\tau)$.

The following can be easily proved [8].

<u>Proposition</u> 4.2. Let $Y \in \mathcal{Y}^r(M; F)$, $r \geq 2$. Then we have the following.

(i) $\tilde{\Sigma}^i_\tau(Y) = \tilde{\Sigma}^i(Y)$, if Y satisfies G0 and G1.

(ii) If Y satisfies G3, then each point $p \in \tilde{\Sigma}^{1,0}(Y)$ is a <u>fold point</u>; i.e. there exist coordinates of class C^{r-1}, $x_1, \ldots, x_m$ centered at p in Σ_Y and $y_1, \ldots, y_m, z_1, \ldots, z_n$ centered at p in M, such that (a) $z_1, \ldots, z_n$ are the coordinates of the plaque L_p of F, (b) the inclusion map $\Sigma_Y \to M$ is given by

$$y_1 = x_1, \ \ldots, \ y_{m-1} = x_{m-1}, \ y_m = x_m^2;$$

$$z_1 = x_m, \quad z_2 = \cdots = z_n = 0.$$

This proposition is useful in the proofs of Theorem C and Theorem D below in this section.

Next, we study the bifurcations of Y at Σ_h. Suppose that

$\dim M = m+n$, $\operatorname{codim} \mathcal{F} = m$, and Y is of class C^r, $r \geqq 3$. Let p be a point in $\partial\Sigma_h$. Assume that there is a neighborhood N of p in $\partial\Sigma_h$ such that N is an $(m-1)$ dimensional manifold. Let $\alpha_1 \times \alpha_2 : U \to D^m \times D^n$ be a chart of $\mathcal{F}$ such that $(\alpha_1 \times \alpha_2)(p) = (0,0)$, (see(2.1)). Let I be a segment in D^m parametrized by μ such that $\mu = 0$ indicates the origin of D^m.

Assumption: $L \equiv (\alpha_1 \times \alpha_2)^{-1}(I \times D^n)$ is transverse to both Σ_Y and N in M.

Definition 4.3. Under the above assumption we say that Y has saddle-node bifurcation at $p \in \partial\Sigma_h$, if there is an segment I as above satisfying the following: The smooth curve $L \cap \Sigma_Y$ is tangent to L_0 at p, $\Sigma_Y \cap L_\mu = \phi$ if $\mu < 0$, and $\Sigma_Y \cap L_\mu$ consists of two points, p_μ^s and p_μ^u if $\mu > 0$. Furthermore, Y is hyperbolic at p_μ^s and p_μ^u. The dimensions of the stable manifolds at p_μ^s and p_μ^u are k and $k-1$, respectively, $1 \leqq k \leqq m$. See Figure 1.

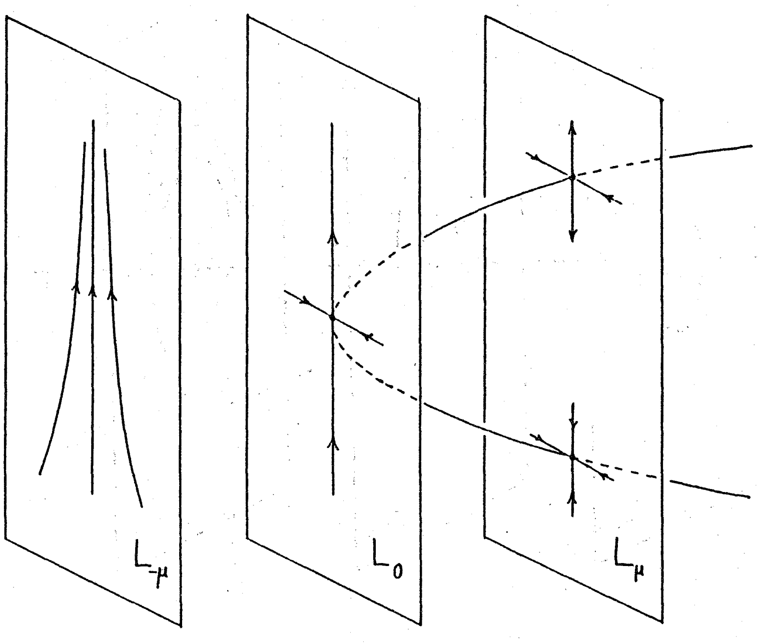

Figure 1

Definition 4.4. Under the above assumption we say that Y has Hopf bifurcation at $p \in \partial\Sigma_h$, if the following hold for every segment $I \subset D^m$ as above: There is a unique 3-dimensional center manifold C (see Guckenheimer-Holmes [6, p.127]) containing $L \cap \Sigma_Y = (\cup_\mu L_\mu) \cap \Sigma_Y$ and a system of coordinates (x, y, μ) on C, with $(x, y, \mu) \in L_\mu$, for which the Taylor expansion of degree 3 of Y on C is given by

$$\begin{cases} \dot{x} = (d\mu + a(x^2+y^2))x - (\omega + c\mu + b(x^2+y^2))y \\ \dot{y} = (\omega + c\mu + b(x^2+y^2)x + (d\mu + a(x^2+y^2))y, \end{cases}$$

which is expressed in polar coordinates as

$$\begin{cases} \dot{r} = (d\mu + ar^2)r \\ \theta = (\omega + c\mu + br^2). \end{cases}$$

See Figure 2. Consequently, if $a \neq 0$, there is a surface of periodic

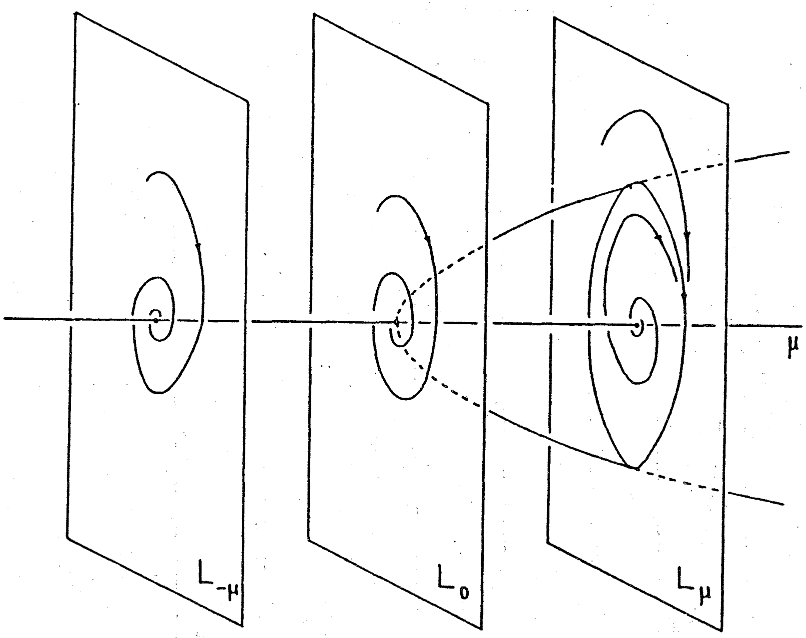

Figure 2

solutions in C which has quadratic tangency with the eigenspace of $\lambda(0)$, $\bar{\lambda}(0)$ agreeing to second order with the paraboloid $\mu = -(a/d)(x^2 + y^2)$. If $a < 0$, these solutions are stable limit cycles, while if $a > 0$, there are repelling. (See [6, Theorem 3.4.2].)

Saddle-node bifurcation and Hopf bifurcation are well known as typical codimension one bifurcations of equilibria (e.g.[6]). We want to see how these bifurcations arise in our global situation with respect to the stratifications which we defined. The stratification $\mathcal{S}$ in G2 is defined by only the first derivatives of Y. But, saddle-node bifrucation does not occur under the condition only of the first derivatives. As another condition we take the second derivatives modulo $\mathcal{F}$ of the inclusion map of the set of equilibrium points Σ_Y; while J. Guckenheimer and P. Holmes [6, Theorem 3.4.1] take the assumption for the second derivative of Y. For this purpose, we use the stratification of Thom-Boardman. In the study of constraint systems, it is natural to consider Thom-Boardman singularities (see [13] and [15]).

Let $\mathcal{S}^k$ be the k-skeleton of $\mathcal{S}$. Let $\tilde{\mathcal{S}}^k$ be the k-skeleton of the stratification determined by $\tilde{\Sigma}^i(Y) = (j^1\iota)^{-1}(\tilde{\Sigma}^i)$, $i = 0, 1, \ldots, m$. We have $\tilde{\mathcal{S}}^k = \tilde{\Sigma}^{m-k}(Y) \cup \tilde{\Sigma}^{m-k+1}(Y) \cup \ldots \cup \tilde{\Sigma}^m(Y)$. Under G1, $\mathcal{S}^k \supset \tilde{\mathcal{S}}^k$ and $\mathcal{S}^{m-1} = \partial\Sigma_h$ hold by Proposition 4.2(i) and the definition of $\mathcal{S}$. Moreover, we have that a $(m-1)$ dimensional stratum of $\mathcal{S}$ is included in a $(m-1)$ dimensional stratum of $\tilde{\mathcal{S}}$. For the sets defined in G2, we observe

$$(\partial\Sigma_h)_0 \subset \tilde{\mathcal{S}}^{m-1} \quad \text{and} \quad (\partial\Sigma_h)_{img} \cap \tilde{\mathcal{S}}^{m-1} = \phi.$$

Denote by $(\partial\Sigma_h)_f$ the set of fold points in $\partial\Sigma_h$;

$$(\partial\Sigma_h)_f \mp (\partial\Sigma_h)_0 \cap \tilde{\Sigma}^{1,0}(Y)$$

Theorem C. Let $Y \in \mathcal{Y}^r(F)$, $r \geqq 3$. Suppose that Y satisfies G1, G2, and G3. Then, there is an open dense subset $(\partial\Sigma_h)_f \cup (\partial\Sigma_h)_{img}$ of the boundary $\partial\Sigma_h$ of the normally hyperbolic domain $\Sigma_h \subset \Sigma_Y$ such that Y has saddle-node bifurcation at each point of $(\partial\Sigma_h)_f$ and has Hopf bifurcation at each point of $(\partial\Sigma_h)_{img}$.

Next, we study the qualitative structure of Y at fold points in the boundary of normally stable domain Σ_s.

Let X be a C^r vector field on an open set U in $\mathbb{R}^n$, let ϕ_t be the flow of X, and let $p \in U$ be an equilibrium point of X. Suppose that the eigenvalues $\lambda_0, \ldots, \lambda_{n-1}$ of $dX(p)$ satisfy that $\lambda_0 = 0$ and that the real parts $\mathcal{R}\lambda_1, \ldots, \mathcal{R}\lambda_{n-1} < 0$. Let E^c and E^s be the generalized eigen spaces of λ_0 and $\lambda_1, \ldots, \lambda_{n-1}$, respectively. By center manifold theorem (Guckenheimer-Holmes [6, Theorem 3.2.1]), there are an invariant C^r manifold $W^s(p)$ (called the stable manifold) tangent to E^s at p and a C^r manifold $W^c(p)$ (called the (local center manifold) tangent to E^c at p. W^c is locally invariant in the sense that, if $q \in W^c$ and $\phi_t(q) \in U$, then $\phi_t(q) \in W^c$. W^s is unique, but W^c need not be.

Let ψ_t be the flow associated to a vector field on a manifold. The subsets

$$V^s(p) = \{q : \psi_t(q) \to p \quad \text{as} \quad t \to \infty\}, \quad \text{and}$$

$$V^u(p) = \{q : \psi_t(q) \to p \quad \text{as} \quad t \to -\infty\}$$

are called the stable set and the unstable set of p, respectively.

The boundary $\partial\Sigma_s = \overline{\Sigma}_s - \Sigma_s$ of normally stable domain is included in the boundary $\partial\Sigma_h$ of normally hyperbolic domain. Suppose Y satisfies G1, G2, and G3. Then, by Theorem C, there is an open dense subset $(\partial\Sigma_h)_f \cup (\partial\Sigma_h)_{img}$ of $\partial\Sigma_h$ such that Y has saddle-node bifurcation at $(\partial\Sigma_h)_f$ and has Hopf bifurcation at $(\partial\Sigma_h)_{img}$. Define the sets as follow,

$$(\partial\Sigma_s)_f \equiv (\partial\Sigma_h)_f \cap (\partial\Sigma_s) \quad \text{and} \quad (\partial\Sigma_s)_{img} \equiv (\partial\Sigma_h)_{img} \cap (\partial I_s).$$

Theorem D. Suppose $Y \in \mathcal{Y}^r(M;\mathcal{F})$, $r \geq 3$. Let $(\partial\Sigma_s)_f \cup (\partial\Sigma_s)_{img}$ be the open dense subset of $\partial\Sigma_s$ defined as above. Let $p \in (\partial\Sigma_s)_f$.

Then, these are an open neighborhood U of p in M and a C^r embedding from the plaque, $h_p : L_p \to \mathbb{R}^1 \times \mathbb{R}^{n-1}$ such that the following are satisfied.

(i) $W^s(p) \cap L_p = h_p^{-1}(\{0\} \times \mathbb{R}^{n-1})$ and $W^c(p) \cap L_p \subset h_p^{-1}(\mathbb{R}^1 \times \{0\})$, where $W^s(p)$ and $W^c(p)$ are the stable and center manifold of $Y|L_p$, respectively.

(ii) $V^s(p) \cap L_p \subset h_p^{-1}([0,\infty) \times \mathbb{R}^{n-1})$ and $V^u(p) \cap L_p \subset h_p^{-1}((-\infty,0] \times \{0\}) \subset W^c(p)$, where $V^s(p)$ and $V^u(p)$ are the stable and unstable sets of p, respectively. (Figure 3).

(iii) The C^r embedding h_p depends C^{r-1} continuously on $p \in (\partial\Sigma_s)_f$. So that, both of the sets

$$V^u = \{q \in V^u(p) : p \in (\partial\Sigma_s)_f \cap U\}$$

and $V^u(p)$ are injectively C^{r-1} immersed submanifolds of M.

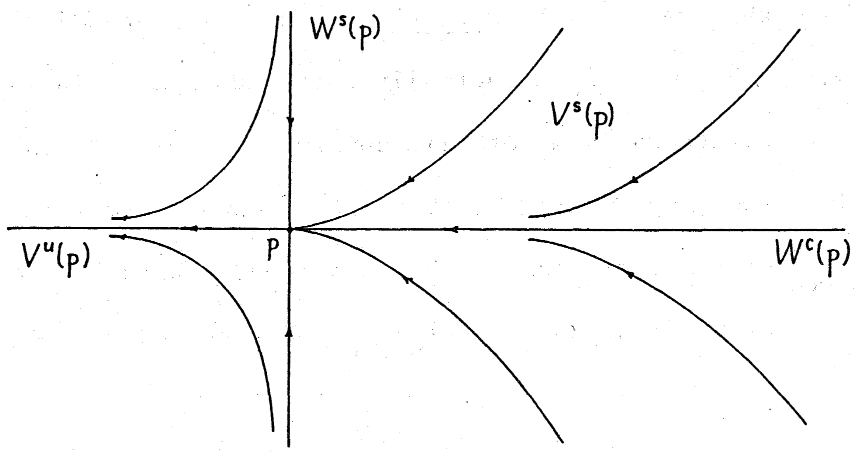

Figure 3

5. CONSTRAINT SYSTEMS AND SINGULAR PERTURBATIONS

Let M be a smooth manifold. Let $\{\tilde{Z}_\varepsilon\}$, $0 \leqq \varepsilon < \varepsilon_0$, be a family of vector fields on M. $\{\tilde{Z}_\varepsilon\}$ is called a C^r family if $\tilde{Z}_\varepsilon(p)$ is a C^r vector field on $M \times [0,\varepsilon_0)$. In this section, we assume $r \geqq 3$.

Definition 5.1. A constraint system of class C^r on M is a pair $\{ \{\tilde{Z}_\varepsilon\}, \mathcal{F} \}$ of C^r family of vector fields on M, $\{\tilde{Z}_\varepsilon\}$ $0 \leqq \varepsilon < \varepsilon_0$ and a smooth foliation $\mathcal{F}$ on M such that $\tilde{Z}_0$ $(\varepsilon = 0)$ is tangent to (the leaves of) $\mathcal{F}$. We may call the limit of $\tilde{Z}_\varepsilon/\varepsilon$ for $\varepsilon \to 0$ a constrained equation in different meaning from Takens [13]. This limit exists only at most on the subset of equilibrium points of $\tilde{Z}_0$.

Expanding $\tilde{Z}_\varepsilon$ by ε, we have

$$\left.\begin{aligned} \tilde{Z}_\varepsilon(p) &= Y(p) + \varepsilon \cdot X(p) + o(\varepsilon) \\ Y(p) &= \tilde{Z}_0(\varepsilon) \\ X(p) &= \frac{\partial}{\partial \varepsilon} \tilde{Z}_\varepsilon(p)\Big|_{\varepsilon=0}. \end{aligned}\right\} \qquad (5.1)$$

We set a following axiom for $\{\{\tilde{Z}_\varepsilon\}, \mathcal{F}\}$.

<u>Axiom</u> 5.2. $Y = \tilde{Z}_0$ satisfies G1, G2, and G3.

<u>Remark</u> 5.3. By Theorem A and Theorem B, the set of families satisfying Axiom is dense in the space $\mathcal{Z}^r$ of C^r family of vector fields $\{\tilde{Z}_\varepsilon\}$ such that $\tilde{Z}_0$ is tangent to $\mathcal{F}$. Here, $\mathcal{Z}^r$ is defined usually as a subspace of the space $\mathfrak{X}^r(M \times [0,\varepsilon_0))$ of C^r vector fields on $M \times [0,\varepsilon_0)$.

Let Σ_r be the normally regular domain of the manifold Σ_Y of equilibrium points of $Y = \tilde{Z}_0$. Hereafter, we use the simple notation Σ for Σ_Y. Let

$$\begin{array}{ccc} T_{\Sigma_r} M & \xrightarrow{\ \pi_\Sigma\ } & T_{\Sigma_r} \Sigma \\ & \circlearrowleft & \\ \text{Proj.} \searrow & & \swarrow \text{Proj.} \\ & \Sigma_r & \end{array}$$

be the bundle map obtained by the projection

$$T_pM = T_p\Sigma_r \oplus T_pL_p \longrightarrow T_p\Sigma_r$$

for each $p \in \Sigma_r$, where L_p is the plaque of $\mathcal{F}$ containing p. For a crosssection X of the bundle $T_{\Sigma_r} M \to \Sigma_r$, we define a vector field X_Σ on Σ_r by

$$X_\Sigma \overset{\text{def}}{=} \pi_\Sigma X \tag{5.2}$$

<u>Definition</u> 5.4. A curve $\gamma : (a,b) \to \Sigma_r$ is a <u>solution</u> of the constrained equation $\lim_{\varepsilon \to 0} \tilde{Z}_\varepsilon/\varepsilon$ associated with $\{\{\tilde{Z}_\varepsilon\}, \mathcal{F}\}$ if

(i) $\lim_{t \searrow t_0} \gamma(t) = \gamma(t_0)$ and there is $\lim_{t \nearrow t_0} \gamma(t) \equiv \gamma^-(t_0)$ in Σ (not necessarily in Σ_r);

(ii) whenever $\gamma^-(t_0) \neq \gamma(t_0)$, there is an orbit C (included in a leaf of F) of $\tilde{Z}_0$ such that the α limit set $\alpha(C)$ and the ω limit set $\omega(c)$ of C satisfy

$$\alpha(C) = \gamma^-(t_0) \quad \text{and} \quad \omega(C) = \gamma(t_0);$$

(iii) if $\gamma^-(t_0) = \gamma(t_0)$, then $X_\Sigma\gamma(t_0)$ is the derivative of γ at t_0; if $\gamma^-(t_0) \neq \gamma(t_0)$, then $X_\Sigma\gamma(t_0)$ is the right derivative of γ at t_0.

A curve $\gamma : [a,b) \to \Sigma_r$ is a solution if, (i) for any $a < a' < b$, $\gamma|(a',b)$ is a solution; (ii) $X_\Sigma\gamma(a)$ is the right derivative of γ at a.

A curve $\gamma : (a,b] \to \Sigma_r$ is a solution if, (i) for any $a < b' < b$, $\gamma|(a,b')$ is a solution; (ii) there is $\lim_{t \nearrow b} \gamma(t) = \gamma^-(b)$ in Σ; (iii) there is an orbit C of $\tilde{Z}_0$ such that $\alpha(C) = \gamma^-(b)$ and $\omega(C) = \gamma(b)$.

$\gamma : [a,b] \to \Sigma_r$ is a solution if $\gamma|[a,c)$ and $\gamma|(c,b]$ are solution for any $a < c < b$.

For a point $p \in \Sigma_r$, there is a solution $\gamma : (a,b) \to \Sigma_r$ such that $p = \gamma(c)$, $a < c < b$. But there may be many such solutions. See Figure 4 and 5.

Next, we consider solutions having many available properties.

Let $\tilde{Z}_\varepsilon = Y + \varepsilon X + o(\varepsilon)$. Let Σ be the set of equilibrium points of Y.

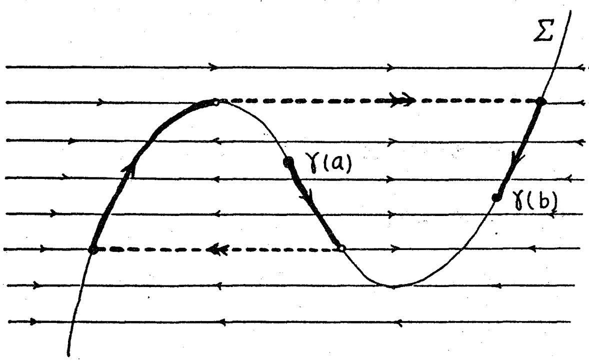

Figure 4

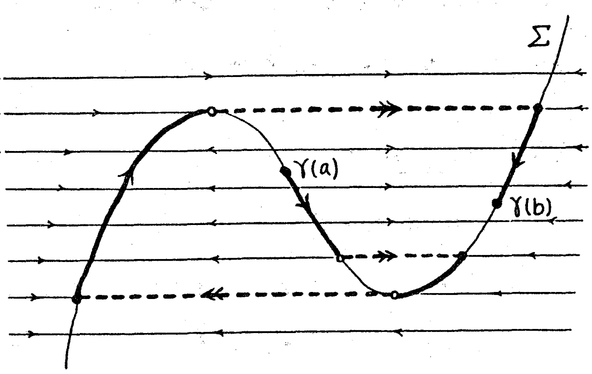

Figure 5

Definition 5.5. Let J be an interval. A solution $\gamma : J \to \Sigma_r$ of $\lim_{\varepsilon\to 0} \tilde{Z}_\varepsilon/\varepsilon$ is called to be admissible if

(i) the image $\gamma(J)$ is included in the normally stable domain Σ_s of Y,

(ii) whenever γ is not continuous at $t \in J$ then $p = \gamma^-(t)$ is contained in the fold point set $(\partial\Sigma_s)_f$ in $\partial\Sigma_s$, and furthermore

$$X(p) \notin T_p\Sigma + T_pL_p \tag{5.3}$$

is satisfied.

Remark 5.6. (5.3) is a generic condition. In fact, since $p \in (\partial\Sigma_h)_0 \subset \tilde{\Sigma}^1(Y)$, the subspace $T_p\Sigma + T_pL_p$ has codimension one in T_pM. Hence, by a perturbation of X (hence, of $\tilde{Z}$), we have $\tilde{Z}$ such that (5.3) holds for the points p in an open dense subset of $(\partial\Sigma_s)_f$.

Hereafter, we show some properties of admissible solutions. For a non-zero vector $v \in T_pM$, denote by $L(v)$ the 1-dimensional subspace of T_pM generated by v. The unstable set $V^u(p)$ of $p \in (\partial\Sigma_s)_f$ is an injectively immersed submanifold of $[0,\infty)$ in M, and it exists uniquely for p, by Theorem D.

<u>Theorem</u> E. Let $\tilde{Z}_\varepsilon = Y + \varepsilon X + o(\varepsilon)$. Suppose that X satisfies (5.3) at a point $p \in (\partial\Sigma_s)_f$. Then the following hold.

(i) For some (and hence for any) Finsler $\|\cdot\|$ on TM and $q \in \Sigma_s$, we have $\|X_\Sigma(q)\| \to \infty$ $(q \to p)$.

(ii) For $q \in \Sigma_s$, we have $L(X_\Sigma(q)) \to T_pV^u(p)$, $q \to p$.

<u>Theorem</u> F. Let $\phi_t(q)$ be the trajectory of $\pi_\Sigma X$ on Σ_s such that $\phi_0(q) = q$. Suppose that

$$\lim_{t \nearrow a} \phi_t(q) = p \in \Sigma_f, \quad a > 0.$$

Then, the following hold.

(i) For any point q' in a neighborhood U of q in Σ_s, there are $p' \in (\partial\Sigma_s)_f$ and $a' > 0$ such that

$$\lim_{t \nearrow a'} \phi_t(q') = p'.$$

(ii) The mapping $U \to (\partial\Sigma_s)_f$, defined by $q' \mapsto p'$, is continuous.

<u>Definition</u> 5.7. Let $\gamma : J \to \Sigma_s$ be a solution of $\lim_{\varepsilon\to 0} \tilde{Z}_\varepsilon/\varepsilon$. For a discontinuous point t_i, $i = 1, 2, 3, \ldots$, let C_i be the orbit of $\tilde{Z}_0$ with $\alpha(C_i) = \gamma^-(t_i)$ and $\omega(C_i) = \gamma(t_i)$. The arc

$$\Gamma(\gamma) \equiv \gamma(J) \cup C_1 \cup C_2 \cup C_3 \cup \cdots$$

is called the <u>trace</u> of γ.

Let d be a Riemannian metric on M.

Theorem G. (Singular perturbation theorem). Let $\gamma : [0,b] \to \Sigma_s$ be an admissible solution of a constrained equation $\lim_{\varepsilon\to 0} \tilde{Z}_\varepsilon/\varepsilon$ such that γ has at most finitely many discontinuous points. Let $\psi_\varepsilon : \mathbb{R}\times M \to M$ be the flow associated with the vector field $Z_\varepsilon \equiv \tilde{Z}_\varepsilon/\varepsilon$, $\varepsilon \neq 0$.

Then, for any $\delta > 0$ and $\mu > 0$, there exist $\bar{\varepsilon} > 0$ and a neighborhood U of $p = \gamma(0)$ in M such that, for any ε with $0 < \varepsilon < \bar{\varepsilon}$ and any $q \in U$ the following hold.

(i) $\psi_\varepsilon(J,q)$ is included in the δ-neighborhood of the trace $\Gamma(\gamma)$; i.e. for any $t \in J$

$$d(\psi_\varepsilon(t,q),\ \Gamma(\gamma)) < \delta.$$

(ii) If $t \in J$ and $|t - t_i| \geqq \eta$ for every discontinuous points $t_1, t_2, t_3, \ldots \in J$ of γ, then we have

$$d(\psi_\varepsilon(t,q),\ \gamma(t)) < \delta.$$

Corollary 5.8. Admissible solution $\gamma : [0,b] \to \Sigma_s$ with $\gamma(0) = p$ is unique, i.e. if $\gamma' : [0,b] \to \Sigma_s$ is another admissible solution with $\gamma'(0) = p$, then $\gamma(t) = \gamma'(t)$ for any $0 < t \leqq b$.

Remark 5.9. (i) N. Fenichel [5, Theorem 9.1] proves the singular perturbation theory for a neighborhood of a compact subset of normally hyperbolic domian Σ_h. We use this theory for the proof of Theorem G.

(ii) L.S. Pontryagin [12] shows the singular perturbation theorem in the neighborhood of a discontinuous point of γ under the condition of the derivatives of Y. This condition is slightly different to our theorem which takes the condition of $\Sigma \subsetneq M$. In the proof [10] of Theorem G, we do not use Pontryagin's results; we give another proof using center manifold theorem.

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