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NORMAL FORMS FOR CONSTRAINED SYSTEMS

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ABSTRACT

A global formulation with a coordinate-free description is given to ordinary differential equations (Abbrev. ODEs) including a small paramater ε. They are formulated as a pair composed of a vector field and a tensor field, which is an extension of the classical interpretation of autonomous ODEs as vector fields. A method to obtain normal forms of such equations is also discussed and several results of calculations are given.

1. INTRODUCTION

The object of this paper is to study the ordinary differential equations (Abbrev. ODEs) of the following type including a small parameter ε:

$$\begin{cases} \varepsilon\dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \qquad x \in \mathbf{R}^n,\ y \in \mathbf{R}^m,\ \left(\dot{} = \frac{d}{dt}\right) \tag{1}$$

A typical example is the Van der Pol equation:

$$\begin{cases} \varepsilon\dot{x} = (x-x^3/3)+y \\ \dot{y} = -x. \end{cases} \tag{2}$$

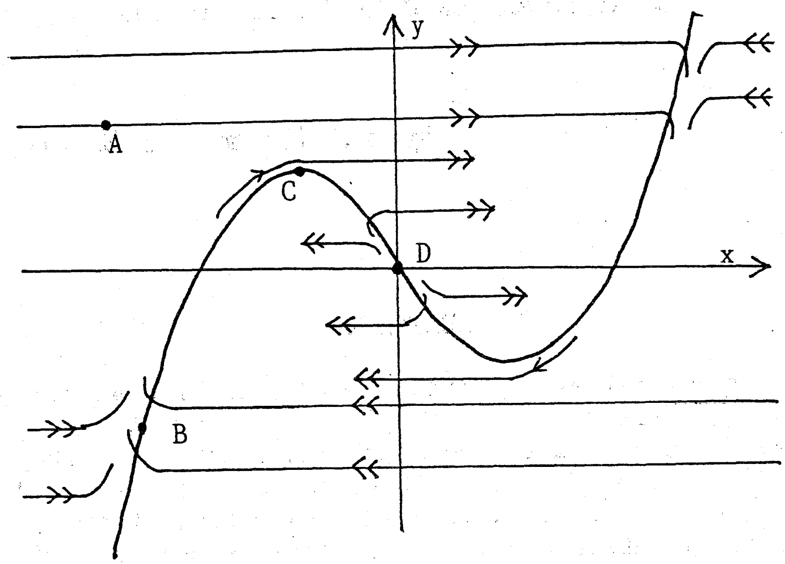

Fig.1 the phase portrait of the Van der Pol equation

The phase portrait of this system for small ε is described by the rapid motion along the x-direction and the slow motion near the curve $y = x^3/3-x$, which is obtained by setting $\varepsilon=0$ in the first equation (2). Since the orbit looks constrained on the curve for almost all time period, Eq.(1) is often called a constrained system. (See Fig.1)

For constrained systems, we must study the behavior of solutions for small ε, the convergence of such solutions when ε tends to zero, etc. The main difficulty to treat such problems is due to the fact that Eq.(1) ceases to be an ordinary vector field, when the definition of the solution of Eq.(1) with $\varepsilon=0$ loses its sense.

First of all, we propose a framework of treating constrained systems which includes the ODE of the type (1) even for $\varepsilon=0$. In our framework Eq.(1) is formulated as a pair composed of a vector field and a tensor field, which is an extension of the classical interpretation of autonomous ODEs as vector fields. See also Takens[1], Fenichel[2] and Ikegami[3),4] for other formulations.

Our next purpose is to give some local classifications of constrained systems. In other words we consider a method to classify the Taylor expansions of order k, or k-jets of constrained systems at a point in the phase space under an equivalence relation among them; the method is to obtain their normal forms. For the case of vector fields, the normal form is applied to obtain the formal part of certain linearization theorems, the elementary bifurcation theorems such as the Hopf bifurcation theorem, etc. In §4 we give corresponding results for some constrained systems.

We remark that, in our framework, the singular perturbation problem for ODEs is interpreted as the study of unfoldings of constrained systems, though they seem to be quite different. In the singular perturbation problem, we consider a family of ODEs parametrized by ε, while, in our formulation, we treat all unfoldings of a given constrained system. Our point of view is, thus, to regard the singular perturbation problem for ODEs as a sort of the bifurcation problem for constrained systems. We give one example of such treatment which is the most simple case and which depends on some tedious computations. Unfortunately we have not yet obtained a systematic method to treat this problem.

2. DEFINITION OF CONSTRAINED SYSTEM

Let M be an n-dimensional C^{∞}-manifold and TM denotes the tangent bundle of M with the bundle projection π. By $X(M)$, we denote the set of all vector fields on M, that is, the smooth sections of the vector bundle TM. A *bundle homomorphism* A of the vector bundle TM is the C^{∞}-mapping from TM to itself whose restriction on each fiber T_xM ($x \in M$) is linear endomorphism $A(x)$ of T_xM. The set of all bundle homomorphisms of TM is denoted by HOM(TM).

Definition 1 A *singular vector field* on M is the pair (A,v) of a bundle homomorphism A of TM and a vector field v on M.

Therefore the set of all singular vector fields on M, which is denoted by SX(M), is nothing but the product of HOM(TM) and X(M). On a local chart, we can identify a singular vector field (A,v) with the following local expression:

$$A(\xi)\cdot\dot{\xi} = v(\xi) \tag{3}$$

which is a generalization of the ODE including Eq.(1). Thus the singular vector field on M can be considered as a globalization of the equations of the type (1).

Let us consider a singular vector field $(A,v) \in SX(M)$. A bundle homomorphism A of TM is called *of constant rank* if, for any $x \in M$, the rank of $A(x)$ is independent of x. In this case, we say A is *of corank r* if the rank of A equals n−r where r is a non-negative integer ($0 \leq r \leq n$).

Definition 2 A *constrained system of corank r* on M is the pair (A,v) of a bundle homomorphism A of TM of corank r and a vector field v on M. A *constrained system* on M is a constrained system of corank r for an integer r. We denote the set of all constrained systems [resp. that of corank r] on M by CX(M) [resp. $CX_r(M)$].

Next we consider the notion of equivalence and transformation for singular vector fields. Let (A,v) be a singular vector field on a manifold M. We identify the singular vector field with the differential equation (3). Thus it is natural to consider that the transformed equation of (3) by a coordinate change $\zeta = \phi(\xi)$ is equivalent to (3) itself. Moreover multiplying the non-singular matrix-valued function $P(\xi)$ to both sides of (3) does not change the equation essentially. Strictly speaking, P is the *bundle isomorphism* of TM, that is, an invertible bundle homomorphism. (The set of all bundle isomorphisms on M is denoted by ISOM(TM).) Consequently we arrive at the following definition of equivalence:

Definition 3 Let (A,v) and (A',v') be singular vector fields on M. We say these singular vector fields are *equivalent* if there exist a bundle isomorphism P of TM and a diffeomorphism ϕ of M such that,

$$(A',v') = (P\circ T\phi\circ A\circ(T\phi)^{-1},\ P\circ T\phi\circ v\circ\phi^{-1}) \tag{4}$$

holds. The pair (P,ϕ) is called a *transformation* of the singular vector field.

The right hand side of (4) is denoted by $(P,\phi)_{\#}(A,v)$, which is the transformed singular vector field of (A,v) by (P,ϕ). This means that the product group of ISOM(TM) and Diff(M) acts on HOM(TM) × X(M), where (4) induces the group structure as follows:

$$(P,\phi)\cdot(Q,\psi) = (P\circ T\phi\circ Q\circ T\phi^{-1},\ \phi\circ\psi).$$

and yields the semi-direct product group $ISOM(TM) \rtimes Diff(M)$.

The set $CX(M)$ is a subset of $SX(M)$. It is easy to see that, for any constrained system (A,v) of corank r and any transformation (P,ϕ), the transformed system $(P,\phi)_{\#}(A,v)$ is again a constrained system of corank r. Thus the group $ISOM(TM) \rtimes Diff(M)$ acts on the space $CX_r(M)$ (and, as the result, on $CX(M)$).

We can also define the characteristic surface for our constrained systems.

<u>Definition 4</u> Let (A,v) be a constrained system of corank r on M. The *characteristic surface* $\mathcal{S}$ of (A,v) is given by,

$$\mathcal{S} = \{x \in M \mid v(x) \in ImA(x)\},$$

where $ImA(x)$ is a linear subspace of T_xM consisting of all images of linear endomorphism $A(x)$ of T_xM.

A standard transversality argument shows the next proposition:

<u>Proposition 5</u> For any generic (A,v) in $CX_r(M)$, the characteristic surface of (A,v) is a smooth submanifold of M of codimension r.

3. GENERAL THEORY OF NORMAL FORMS FOR CONSTRAINED SYSTEMS

In this section, we focus our attention on the study of the local structurs of constrained systems around an arbitrary point x_0 in M. If two constrained systems are transformed to each other up to order k around the point x_0, we say they are k-jet equivalent at x_0. The *k-th order normal form* of the constrained system (A,v) at x_0 is defined by a representative of the k-jet equivalence class of (A,v)

at this point.

First we explain a general framework of the normal form theory for vector fields. For simplicity we only give the idea in terms of the vector field itself, and omit some argument concerning k-jet extention in our explanation. Let v be a vector field on M. Our interest is to know how the vector field v changes by a given diffeomorphism ϕ. For this purpose, taking a one-parameter group ϕ^t of diffeomorphisms connecting the identity at $t=0$ and ϕ at $t=1$, we investigate the way of deformation $\phi^t_* v$ of v in terms of a differential equation on the space $X(M)$ of vector fields.

Recall that every vector field Y on M generates a diffeomorphism as the time-one-mapping of the flow defined by Y, that is, $\phi = \exp Y$. We call Y the *infinitesimal generator* of ϕ. In this situation we consider the one-parameter group of diffeomorphisms, $\phi^t = \exp tY$, and deform a given vector field v by ϕ^t. Then a formula in differential geometry gives,

$$\frac{d}{dt}\bigg|_{t=0} \phi^t_* v = -[Y,v] \tag{5}$$

where [,] denotes the Lie bracket for vector fields. The left hand side of (5) is called the *infinitesimal deformation* of v by Y. Since $\{\phi^t\}$ forms a one-parameter group of diffeomorphisms, (5) defines a differential equation on $X(M)$, that is,

$$\frac{d}{dt} v_t = -[Y,v_t] \qquad (v_t = \phi^t_* v), \tag{6}$$

which describes the way of deformation of v. Integrating this equation from $t=0$ to $t=1$, we obtain the transformed vector field $\phi_* v$.

Conversely, in order to simplify some terms of v, we have only to find appropriate infinitesimal generators and to solve the equation (6). This is the idea of the normal form theory for vector fields. For details and the practical method of computation, see Ushiki[5].

The normal form theory for singular vector fields is essentially the same but slightly more complicated. For any $(R,Y) \in SX(M)$ and sufficiently small t, the exponential mapping $\mathbf{exp}t(R,Y)$ of (R,Y) can be defined, which is a local one-parameter group of the transformations of singular vector fields. The infinitesimal deformation of the singular vector field (A,v) is given by,

$$\frac{d}{dt}\Big|_{t=0} \mathbf{exp}t(R,Y)_{\#}(A,v).$$

To compute the infinitesimal deformation, we identify the bundle homomorphism A with a (1,1)-type tensor field through the natural vector bundle isomorphism,

$$Hom(TM) \simeq T^{*}M \otimes TM,$$

where $Hom(TM)$ is the vector bundle over M whose fiber at $x \in M$ is the vector space $Hom(T_xM, T_xM)$, the set of all linear maps from T_xM into itself. Again, by a formula in differential geometry, the infinitesimal deformation is obtained as follows:

<u>Theorem 6</u> The infinitesimal deformation is given by,

$$\frac{d}{dt}\Big|_{t=0} \mathbf{exp}t(R,Y)_{\#}(A,v) = (R \circ A - \mathcal{L}_Y A,\ R \cdot v - [Y,v]),$$

where $\phi_t = \exp tY$ and $\mathcal{L}_Y A$ denotes the Lie derivative of the tensor field A with respect to the vector field Y. With a local coordinate representation,

$$R\circ A-\mathcal{L}_Y A = \sum_{i,j,k}\left(R_{ik}A_{kj} - \frac{\partial A_{ij}}{\partial x_k}Y_k + \frac{\partial Y_i}{\partial x_k}A_{kj} - A_{ik}\frac{\partial Y_k}{\partial x_j}\right)\partial/\partial x_i\otimes dx_j$$

$$R\triangleright v-[Y,v] = \sum_{i,k}\left(R_{ik}v_k - \frac{\partial v_i}{\partial x_k}Y_k + \frac{\partial Y_i}{\partial x_k}v_k\right)\partial/\partial x_i$$

where $R = \Sigma R_{ij}\partial/\partial x_i\otimes dx_j$, $A = \Sigma A_{ij}\partial/\partial x_i\otimes dx_j$, $Y = \Sigma Y_i\partial/\partial x_i$ and $v = \Sigma v_i\partial/\partial x_i$.

Using this theorem, we can calculate normal forms for singular vector fields. For the detail, see Oka and Kokubu[6) 7)]. The normal form problem for constrained systems is solved by reducing it to that for singular vector fields by a natural inclusion of CX(M) into SX(M). See Oka[8)].

4 LOCAL CLASSIFICATION OF CONSTRAINED SYSTEMS

We begin with the classificasion of the leading part of a constrained system. Let (A,v) be a constrained system of corank r. For a point $x_0 \in M$, we put,

$$(A_0,v_0) = (A(x_0),v(x_0)),$$

and call (A_0,v_0) the *leading part* of (A,v) at x_0. Since (A,v) is of corank r, A_0 is a linear map of rank $n-r$. An elementary linear algebra argument leads the following,

<u>Proposition 7</u> Every 0th order normal form of a constrained system (A,v) of corank r is one of the following forms:

$$\text{(a)} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} e_r \\ 0 \end{pmatrix} \qquad \text{(b)} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} 0 \\ e_{n-r} \end{pmatrix} \qquad \text{(c)} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-r} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

where I_{n-r} is the unit matrix of order n-r and e_r is the r-dimensional vector ${}^t(1,0,\ldots,0)$.

For each 0th order normal form, we continue to calculate normal forms of higher order. For simplicity we restrict ourselves to the case of corank 1, and we suppose that $M = \mathbf{R}^n$ and x_0 is the origin.

<u>Theorem 8</u> Suppose that the 0-jet of a constrained system (A,v) of corank 1 is equivalent to (a) in Prop.7, that is,

$$\text{(a)} \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} . \tag{7}$$

Then the infinite order normal form of (A,v) is given by (7) itself. In other words, every finite order part, except for the leading part, of (A,v) at the origin can be supressed by a suitable transformation.

<u>Theorem 9</u> Suppose that the 0-jet of a constrained system (A,v) of corank 1 is equivalent to (b) in Prop.7. Then the generic infinite order normal form of (A,v) is given by,

$$\begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix} \begin{pmatrix} \pm x + \sum_{i,I} s_{i,I} y^i z^I \\ e_{n-1} \end{pmatrix}, \qquad (x,y,z) \in \mathbf{R}\times\mathbf{R}\times\mathbf{R}^{n-2}$$

where I is the multi-index with $|I| = i$ and $s_{i,I} = \pm 1$.

If we further assume the constrained system is two-dimensional, we can calculate more degenerate cases.

<u>Theorem 10</u> Suppose that the 0-jet of a two-dimensional constrained system (A,v) of corank 1 is equivalent to (b) in Prop.7. Then its 1st order normal form is one of the followings:

$$(b_1)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}\pm x\\1\end{pmatrix}\quad(b_2)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}\pm y\\1\pm x\end{pmatrix}\quad(b_3)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}\pm y\\1\end{pmatrix}$$

$$(b_4)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0\\1\pm x\end{pmatrix}\quad(b_5)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}.$$

Moreover if the 1-jet of (A,v) is equivalent to (b_1), then its infinite order normal form is also given by (b_1) itself. If the 1st order part is equivalent to (b_2), then the generic 2nd order normal form is given by,

$$\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}\pm y+ax^2\\1\pm x\end{pmatrix}.$$

<u>Theorem 11</u> Suppose that the 0-jet of a two-dimensional constrained system (A,v) of corank 1 is equivalent to (c) in Prop.7. Then its 1st order normal form is one of the followings:

$$(c_1)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}x\\ay\end{pmatrix}\quad(c_2)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}y\\x\end{pmatrix}\quad(c_3)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}y\\0\end{pmatrix}$$

$$(c_4)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0\\x\end{pmatrix}\quad(c_5)\begin{pmatrix}0&0\\0&1\end{pmatrix}\begin{pmatrix}0\\ay\end{pmatrix}$$

Moreover if the 1st order part is equivalent to (c_1) [resp. (c_2)], then the generic 2nd order normal form is given by,

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ ay \pm x^2 \end{pmatrix} \qquad \left[\text{resp.} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \pm x^2 \\ x \end{pmatrix} \right].$$

In contrast with the case of singular vector fields, we cannot obtain, at present, versal unfoldings for constrained systems in a systematic manner. The main difficulty is that the space $CX(M)$ is neither a vector space nor a manifold. We give below an example of versal unfolding of a constrained system, which is based on a cumbersome computation for this special case.

<u>Proposition 12</u> The versal unfolding of Eq.(7) in Th.8 is given by,

$$\begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{8}$$

where ε is an unfolding parameter.

This versal unfolding can be expressed in terms of differential equations as follows:

$$\varepsilon \dot{x} = 1, \quad \dot{y} = 0. \tag{9}$$

which describes a rapid motion in the phase space, and it corresponds to the local structures around the point A in Fig.1. See Fig.2-A.

For other cases, versal unfoldings is not obtained generally. Nevertheless it is remarkable to consider unfoldings of them of the following forms:

$$\varepsilon \dot{x} = \alpha \pm x, \quad \dot{y} = 1 \tag{10}$$

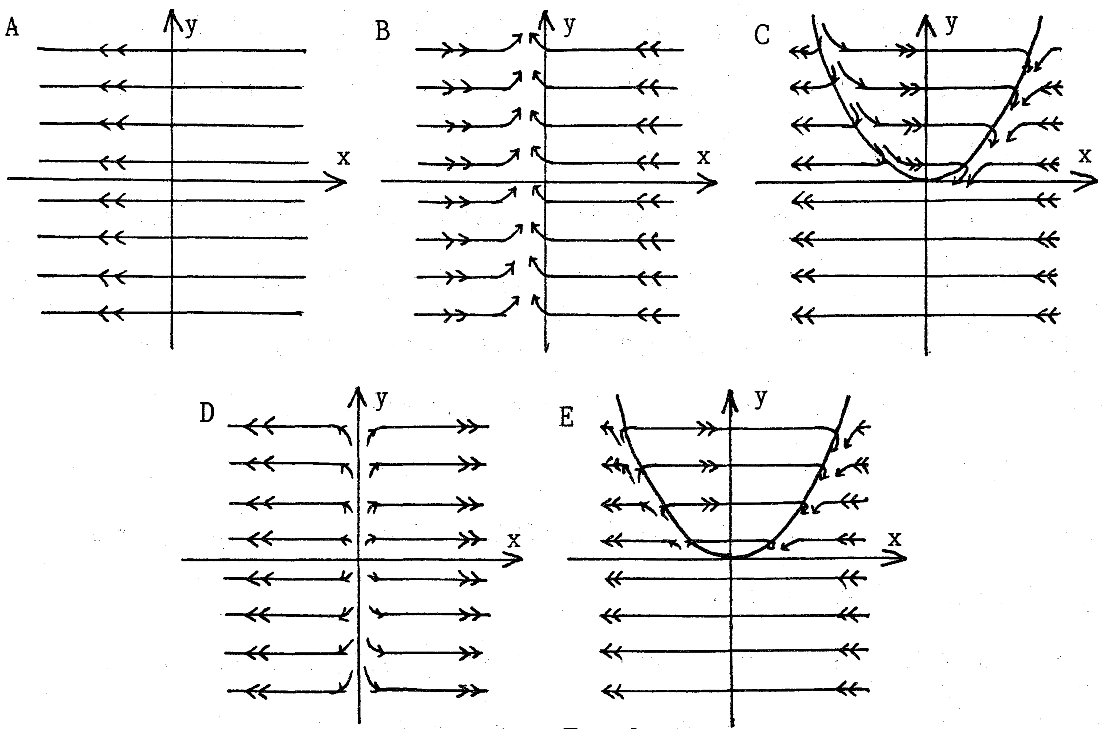

Fig.2

$$\varepsilon\dot{x} = \alpha\pm y+\beta x+ax^2, \quad \dot{y} = 1\pm x \tag{11}$$

$$\varepsilon\dot{x} = x, \quad \dot{y} = ay\pm x^2 \tag{12}$$

$$\varepsilon\dot{x} = \pm y+\alpha x+ax^2, \quad \dot{y} = \beta\pm x \tag{13}$$

where ε,α,β are unfolding parameters. Each of them, similarly to Eq.(9), seems to describe a typical local orbit structure of the equation of the form (1). For instance, in the phase portrait of the Van der Pol equation (2) given in Fig.1, the local orbit structures around the point A, B, C and D resemble to those for Eq.(9), (10), (11) and (12) respectively. See Fig.2-A ∿ D. The phase portrait of (13) presents a 'canard' effect proposed and studied by a French group of non-standard analysis[9]. See Fig.2-E.

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