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DYNAMICAL SYSTEMS ON DRAGON DOMAINS

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ABSTRACT

Dynamical systems on fractal domains are studied. These domains are called twindragon, tetradragon and cross dragon respectively.

1. INTRODUCTION

We can see the following fact in Knuth[1]: For any complex number there exists the zero-one sequence $a_k.a_{k-1},\ldots a_0.a_{-1},\ldots$ such that

$$z = \Sigma_{-\infty\leqq j\leqq k} a_j(i-1)^j,$$

that is, evry complex number has a "binary" representation with base i-1. This fact suggests an existence of a number theoretic dynamical system $(\hat{X}_{i-1},\hat{T}_{i-1},\hat{\mu})$ which induces the binary expansion. Actually if there exists a domain $\hat{X}_{i-1}$ and its partition $\{\hat{X}_{i-1,0},\hat{X}_{i-1,1}\}$ such that

(i) $\hat{X}_{i-1} = \hat{X}_{i-1,0} \cup \hat{X}_{i-1,1}$ and $\mathrm{int}(\hat{X}_{i-1,0}) \cap \mathrm{int}(\hat{X}_{i-1,1}) = \phi$

(ii) $\hat{X}_{i-1} = (i-1)\hat{X}_{i-1,0} = (i-1)\hat{X}_{i-1,1} - 1$,

then the transformation $\hat{T}_{i-1}$ on $\hat{X}_{i-1}$ such that

$$\hat{T}_{i-1}z = (i-1)z - \hat{a}((i-1)z)$$

where $\hat{a}(z)=j$ if $z \in j+\hat{X}_{i-1,j}$, $j=0,1$, induces the binary expansion.

On the other hand we can see also the followings in Davis and Knuth[2]: for any complex integer $m+in$, there exists a revolving sequence of finite length $\delta_1, \delta_2, \ldots \delta_k$ such that

$$m + in = \sum_{j=1}^{k} \delta_j (1+i)^{k-j}$$

where the revolving sequence $(\delta_1, \delta_2, \ldots)$ is defined by the following conditions:

(i) $\delta_j \in \{0, 1, -i, -1, i\}$

(ii) if $(\delta_1, \ldots, \delta_j) \neq (0, \ldots, 0)$

then $\delta_{j+1} = 0$ or $(-i)\delta_{k_0}$ for all $j \in \mathbb{N}$

where $k_0 = \max\{k; \delta_k \neq 0, 1 \leqq k \leqq j\}$

(iii) if $(\delta_1, \ldots, \delta_j) = (0, \ldots, 0)$

then $\delta_{j+1} \in \{0, \pm 1, \pm i\}$.

This fact also suggests an existence of a number theoretic dynamical system (X, T, ν) which induces the revolving expansion

$$z = \sum_{k=1}^{\infty} \delta_k (1+i)^{-k}$$

We consider the existence problem of above dynamical systems $(\hat{X}_{1-i}, \hat{T}_{1-i}, \hat{\mu})$ and (X, T, ν) and show that the boundaries of these domains $\hat{X}_{i-1}$ and X, called the twindragon and the tetradragon respectively, are indeed fractal sets [3].

Moreover we propose a new construction of the dragon different from the paper folding process and consider a dynamical system (Y, S, λ) on a domain tiled by four dragon which is not the tetradragon, called a cross dragon . Suprisingly we can show that this cross dragon system[4] is actually the dual system for a very simple group endomorphism T_L on $\mathbb{T}^2$ such that

$$T_L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} [x-y] \\ [x+y] \end{pmatrix} .$$

2. BINARY EXPANSION ON TWINDRAGON

Firstly consider the binary expansion with base $(1+i)$;

$$z = \Sigma_{k=1}^{\infty} \varepsilon_k (1+i)^{-k},$$

where $\varepsilon_k \in \{0, i\}$ for all $k \in \mathbb{N}$. If there exist a dynamical system (X_{1+i}, T_{1+i}, μ) which induces this representation, then the domain must be the limit points of Q_n such that

$$Q_n = \{\Sigma_{k=1}^{n} \varepsilon_k (1+i)^{-k} ; \varepsilon_k \in \{0, i\} \}$$

and also $X_{1+i, \varepsilon}$, $\varepsilon = 0, i$, must be the limit point of $Q_{n, \varepsilon} = \{\Sigma_{k=1}^{n} \varepsilon_k (1+i)^{-k} ; \varepsilon_1 = \varepsilon \}$ in the Hausdorff metric space $(\mathcal{F}, d)$. For after discussions we put

$$P_{n+1} = (1+i) Q_n \quad \text{for } n \geqq 1,$$

that is,

$$P_{n+1} = \{ \Sigma_{k=0}^{n-1} \varepsilon_k (1+i)^{-k} ; \varepsilon_k \in \{0, i\} \}.$$

We consider the shape and properties of X_{1+i} such that $d(X_{1+i}, P_n) \to 0$ as $n \to \infty$. Let U be a closed square with vertices 0, 1, 1-i and -i, and for each point $x(\varepsilon_0, \dots, \varepsilon_{n-1}) \in P_{n+1}$ we prepare the neighborhood of a point $x(\varepsilon_0, \dots, \varepsilon_{n-1})$ such that

$$U_{x(\varepsilon_0, \dots, \varepsilon_{n-1})} = x(\varepsilon_0, \dots, \varepsilon_{n-1}) + (1+i)^{-(n-1)}U,$$

and let F_{n+1} and B_{n+1} be

$$F_{n+1} = \bigcup_{x \in P_{n+1}} U_{x(\varepsilon_0, \dots, \varepsilon_{n-1})}$$

and

$$B_{n+1} = \partial F_{n+1}$$

respectively. We call B_{n+1} a (n+1)-step Bernoulli boundary (Fig.1(a)). We give the names for each side of B_{n+1} as a following way: For each $n \geqq 1$ we name each side of the square $(1+i)^{-(n-1)}U$ A, B, A^{-1} and B^{-1} respectively, then we obtain names of each side of the neighborhood of point $x(\varepsilon_0, \dots, \varepsilon_{n-1})$ according to above namings. Therefore we can read a sequence of names for B_{n+1} as to be $A_{n+1,1}, A_{n+1,2}, \dots, A_{n+1,m(n)}$ where $A_{n+1,1}$ is a first name of a side $[0, (1+i)^{-(n-1)}(-i)]$ and $A_{n+1,k} \in \{A, A^{-1}, B, B^{-1}\}$ is a name of k-th side of B_{n+1}.

<u>Lemma(2.1)</u>

The names of each side of B_{n+1} are obtained from these of B_n by the substitution $\Theta: A \to AB,\ B \to A^{-1}B$, that is, the names of each side of $B_{n+1} = \Theta^n(ABA^{-1}B^{-1})$.

By the way, recall the notation by Dekking[5),6)] for our purpose. Let G be a finite set of symbols, G^* the free

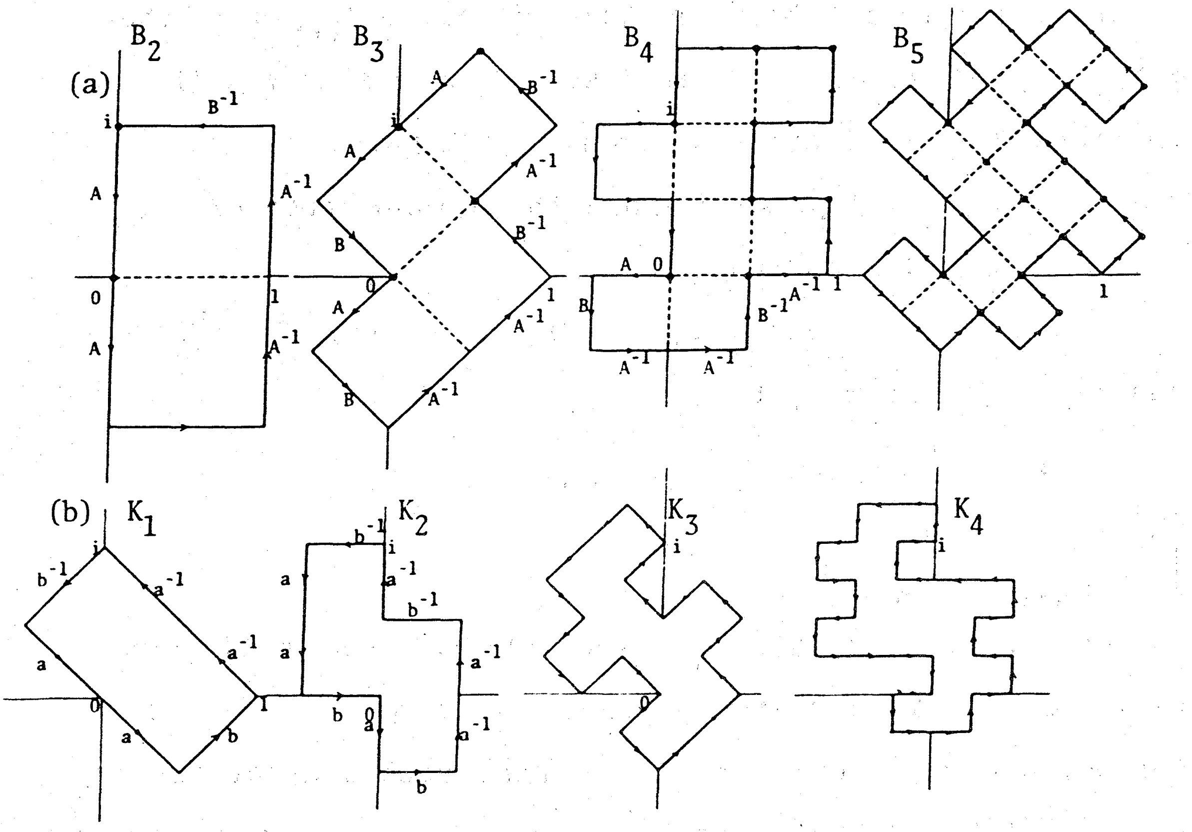

Fig.1: Bernoulli boundary B_n and Dragon boundary K_{n-1}.

semigroup generated by G and $\theta: G^* \to G^*$ a semigroup endomorphism. Let $f: G^* \to \mathbb{C}$ be a homeomorphism which satisfies

$$f(VW) = f(V) + f(W), \quad f(V^{-1}) = -f(V)$$

for all words $V, W \in G^*$. Define a map $K: S^* \to \mathbb{C}$, which satisfies

$$K[VW] = K[V] \cup (K[W] + f(V))$$

for all reduced words $V, W \in G^*$, by

$$K[s] = \{tf(s);\ 0 \leqq t \leqq 1\} \quad \text{for } s \in G.$$

This makes $K[s_1 \ldots s_m]$ the polygonal line with vertices at $0, f(s_1), f(s_1)+f(s_2), \ldots, f(s_1)+\ldots+f(s_m)$.

Especially we consider here a following case.

$G = \{a, b\}$, $f(a) = 1$, $f(b) = i$,

and

θ: $\theta(a) = ab$, $\theta(b) = a^{-1}b$.

Then the following relation holds

$$f\theta = (1+i)f.$$

We put

$$K_n = (1+i)^{-n} K[\theta^n(aba^{-1}b^{-1})],$$

and call K_n a n-step dragon boundary (Fig.1(b)).

<u>Theorem</u> (Dekking[5),6)])

(1) There exists a closed curve K_θ such that

$$(1+i)^{-n} K[\theta^n(aba^{-1}b^{-1})] \to K_\theta \quad \text{as } n \to \infty$$

in the Hausdorff metric,

(2) $\dim_H K_\theta = 2\log\beta_0/\log 2$, where β_0 is a unique real root of $\beta^3 - \beta^2 - 2 = 0$.

K_θ is called a dragon boundary or a twindragon skin because of lemma(3.2).

We obtain a following relation between B_n and K_n.

<u>Lemma(2.2)</u>

$$B_{n+1} = 2(1+i)^{-1}(K_{n-1}).$$

<u>Corollary(2.3)</u>

Let X_{1+i} and $X_{1+i,\varepsilon}$, $\varepsilon = 0, i$, be convergent sets of Q_n and $Q_{n,\varepsilon}$ ($\varepsilon = 0, i$) in the Hausdorff metric (Fig.2), then

(1) ∂X_{1+i} is similar to the dragon boundary.

(2) $X_{1+i} = X_{1+i.0} \cup X_{1+i.i}$.

(3) $X_{1+i} = (1+i)X_{1+i,0} = (1+i)X_{1+i.i} - i$.

(4) $\dim_H(X_{1+i.0} \cap X_{1+i,i}) = 2\log\beta_0/\log 2$.

Thus we can define a transformation T_{1+i} on X_{1+i} by

$$T_{1+i}z = (1+i)z - [(1+i)z]_{1+i}$$

where a digit $[z]_{1+i}$ be

$$[w]_{1+i} = \begin{cases} 0 & \text{if } w \in X_{1+i.0} \\ i & \text{if } w \in i + X_{1+i.i}. \end{cases}$$

Then we obtain

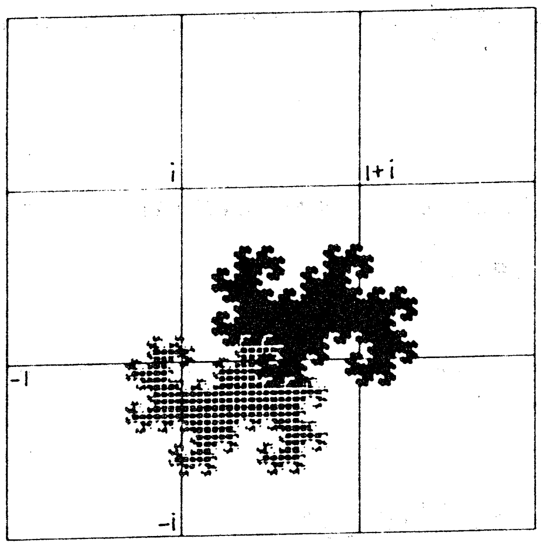

Fig.2: Domain X_{1+i}.

Theorem(2.1)

(1) The transformation (X_{1+i}, T_{1+i}) induces the complex binary expansion for a.e. $z \in X_{1+i}$ such that

$$z = \Sigma_{k=1}^{\infty} a_k(z)(1+i)^{-k}$$

where $a_k(z) = [(1+i)T_{1+i}^{k-1}z]_{1+i}$.

(2) The Lebesgue measure μ is invariant with respect to (X_{1+i}, T_{1+i}) and the dynamical system (X_{1+i}, T_{1+i}, μ) is isomorphic to the two states Bernoulli system.

Remark:

(i) Put

$$X_{1-i} = \overline{X_{1+i}}, \quad [w]_{1-i} = \overline{[w]_{1+i}} ,$$

where $\overline{\quad}$ means to take a complex complex conjugate, and

$$T_{1-i}z = (1-i)z - [(1-i)z]_{1-i} \quad \text{for } z \in X_{1-i}.$$

Then dynamical system (X_{1-i}, T_{1-i}, μ) induces the complex binary expansion with base $(1-i)$.

(ii) Putting

$$X_{i-1} = \frac{1-2i}{5} + X_{1-i},$$

$$X_{i-1,\varepsilon} = \frac{1-2i}{5} + X_{1-i,\varepsilon}, \quad \varepsilon = 0, -i,$$

and

$$T_{i-1}z = (i-1)z - [(i-1)z]_{i-1},$$

where $[w]_{i-1} = \varepsilon$ if $w \in \varepsilon + X_{i-1}$, then (X_{i-1}, T_{i-1}, μ) is well defined and induces the complex binary expansion with base $(i-1)$.

(iii) Taking a complex conjugate of (X_{i-1}, T_{i-1}, μ), then the dynamical system $(X_{-1-i}, T_{-1-i}, \mu)$ is obtained and induces the complex binary expansion with base $(-1-i)$.

We remark that the sets X_{i-1} and X_{-1-i} include the origin as an internal point respectively.

(iv) The set of the twin dragons $\{X_{1+i}+m+in; m+in\in Z(i)\}$ tiles the whole plane, that is,

$$\bigcup_{m+in} X_{1+i}+m+in = \mathbb{C}$$

and

$$\mu\left(\bigcup_{m+in}\partial(X_{1+i}+m+in)\right) = 0.$$

3. REVOLVING EXPANSION ON TETRADRAGON

Let $M=(M_{j,k})$, $j,k\in\{0,1,2,3\}$, be a 0-1 matrix such that

$$M = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

and (X_M, σ_M) a Markov subshift (topological Markov chain) for the structure matrix M. Define a coding function Ψ_0 and a isomorphism Ψ on X_M by

$$\Psi_0(\varepsilon_1,\varepsilon_2)=\delta_1=\begin{cases} 0 & \text{for } \varepsilon_1-\varepsilon_2=0 \\ 1 & \text{for } \varepsilon_1=0 \\ -i & \text{for } \varepsilon_1=1 \\ -1 & \text{for } \varepsilon_1=2 \\ i & \text{for } \varepsilon_1=3 \end{cases} \left.\vphantom{\begin{matrix}1\\1\\1\\1\end{matrix}}\right\} \text{ and } \varepsilon_1-\varepsilon_2\neq 0,$$

and for each $\omega\in X_M$

$$\Psi(\omega) = \{\Psi_0(\sigma_M{}^{n-1}\omega)\}_{n=1}^{\infty}.$$

Then we obtain,

Proposition(3.1)

Let W be a set of the revolving sequences. Then the map Ψ is one-one onto from $X_M \setminus \{\varepsilon_1, \varepsilon_2, \ldots) : \varepsilon_j = a$ for all j and $a \in \{0,1,2,3\}\}$ to $W \setminus \{(0,0,\ldots.)\}$, and satisfies a commutative relation

$$\sigma \circ \Psi = \Psi \circ \sigma_M.$$

Now denote a set of all finite revolving sequences with length n by $W^{(n)}$ and the decomposition of $W^{(n)}$ by

$$W_\varepsilon{}^{(n)} = \{ \{\Psi_0(\varepsilon_j, \varepsilon_{j+1})\}_{j=1}^{n} : \varepsilon_1 = \varepsilon \text{ and } (\varepsilon_1, \ldots, \varepsilon_{n+1}) \text{ is M-admissible } \},$$

and

$$W^{(n)}_{(\varepsilon,\delta)} = \{ (\delta_1, \ldots, \delta_n) \in W_\varepsilon{}^{(n)} ; \delta_1 = \delta \}.$$

Then we obtain.

Proposition(3.2)

(1) $\quad W^{(n)} = \bigcup_{\varepsilon \in \{0,1,2,3\}} W_\varepsilon{}^{(n)}$.

(2) $\quad W_\varepsilon{}^{(n)} = W^{(n)}_{(\varepsilon,0)} \cup W^{(n)}_{(\varepsilon,(-i)^\varepsilon)}$,

(3) $\quad \sigma W^{(n)}_{(\varepsilon,0)} = W_\varepsilon{}^{(n-1)}$ and

$\quad \sigma W^{(n)}_{(\varepsilon,(-i)^\varepsilon)} = W^{(n-1)}_{\varepsilon+1(\mathrm{mod}\ 4)}$

(4) $\quad (-i)W_\varepsilon{}^{(n)} = W^{(n)}_{\varepsilon+1(\mathrm{mod}\ 4)}$,

$\quad (-i)W^{(n)}_{(\varepsilon,0)} = W^{(n)}_{(\varepsilon+1(\mathrm{mod}\ 4),0)}$, and

$$(-i)W^{(n)}_{(\varepsilon,(-i)^{\varepsilon})} = W^{(n)}_{(\varepsilon+1(\text{mod } 4),(-i)^{\varepsilon+1})}.$$

Let ℓ be a map from $W^{(n)}$ to a line segment such that

$\ell(\delta_1,\ldots,\delta_n)$ = segment which connects $p(\delta_1,\ldots,\delta_n,0)$ and $p(\delta_1,\ldots,\delta_n,\delta_{n+1}\neq 0)$, where

$$p(\delta_1,\ldots,\delta_n) = \Sigma_{k=1}^{n}\delta_k(1+i)^{-k}.$$

By the way, define a n-step twindragon curve D_n and a n-step dragon (paper folding) curve H_n (Fig.3(a)(b)) by

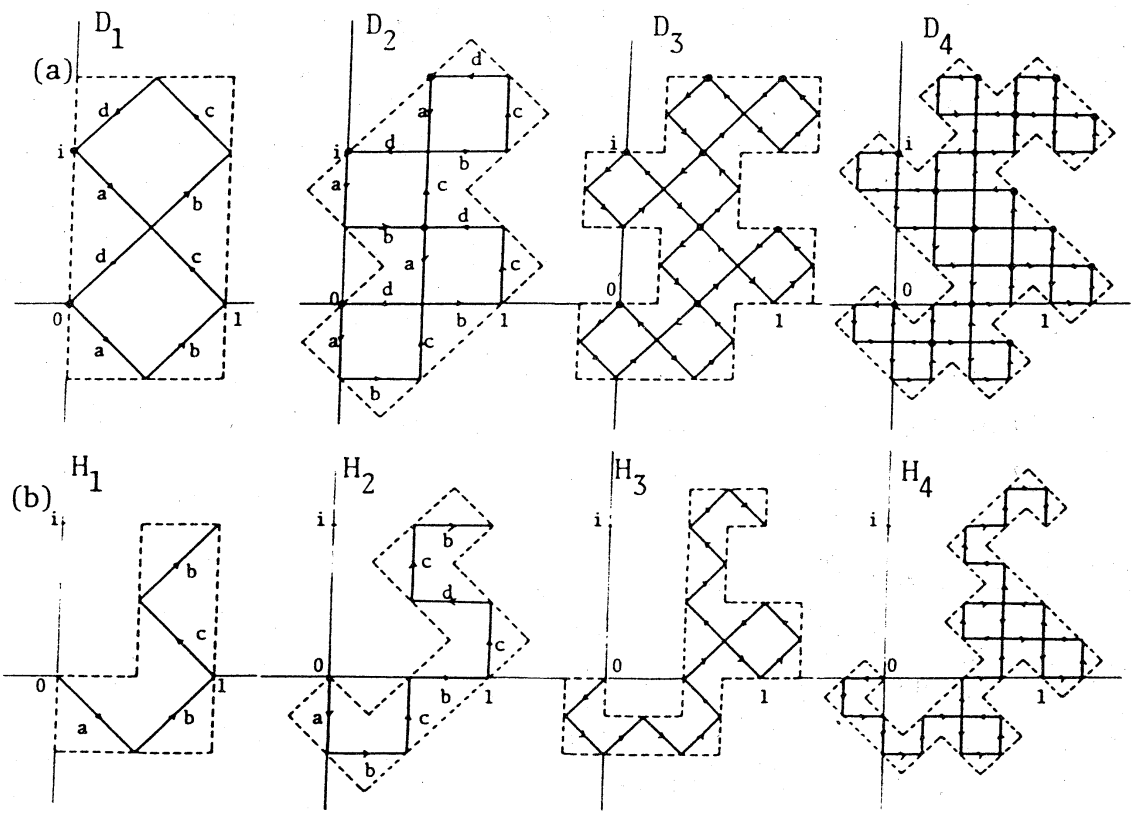

Fig.3: Twindragon D_n and Dragon H_n and their boundaries.

$$D_n = (1+i)^{-n} K[\theta_T{}^n(abcd)]$$

and

$$H_n = (1+i)^{-n} K[\theta_T{}^n(ab)],$$

where G={abcd} and a homeomorphism f is such that

$$f(a)=1=-f(c) \quad \text{and} \quad f(b)=i=-f(d),$$

and an endomorphism θ_T is defined by

$$\theta_T: a\to ab,\ b\to cb,\ c\to cd,\ d\to ad.$$

We notice the twindragon curve is tiled by two dragon curves, that is,

$$D_n = H_n \cup (-H_n+1+i).$$

<u>Lemma(3.1)</u>

Let $\ell_\varepsilon^{(n)}$ and $\ell_{(\varepsilon,\delta)}^{(n)}$ be defined by

$$\ell_\varepsilon^{(n)} = \bigcup_{(\delta_1,\dots,\delta_n)\in W_\varepsilon^{(n)}} \ell(\delta_1,\dots,\delta_n)$$

and

$$\ell_{(\varepsilon,\delta)}^{(n)} = \bigcup_{(\delta_1,\dots,\delta_n)\in W_{(\varepsilon,\delta)}^{(n)}} \ell(\delta_1,\dots,\delta_n),$$

then $\ell_{(\varepsilon,\delta)}^{(n)}$ and $\ell_\varepsilon^{(n)}$ are similar to the (n-2)-step and (n-1)-step dragon curve respectively (Fig.4(a)(b)).

Let U be a closed square in section 2 , U' a closed square such that U'=U+i/2 and B'_{n+1} defined by

$$B'_{n+1} = \partial\left(\bigcup_{x\in P_{n+1}} x(\varepsilon_0,\dots,\varepsilon_{n-1})+(1+i)^{-(n-1)}U' \right),$$

then

<u>Lemma(3.2)</u>

(1) The n-step twindragon curve D_n is covered by a closed curve B'_{n+1} as an envelope (Fig.3(a)), that is,

$$d_0(D_n, B'_{n+1}) = \sup_{x\in B'_{n+1}} \inf_{y\in D_n} |x-y| = \left(\frac{1}{\sqrt{2}}\right)^{n+1},$$

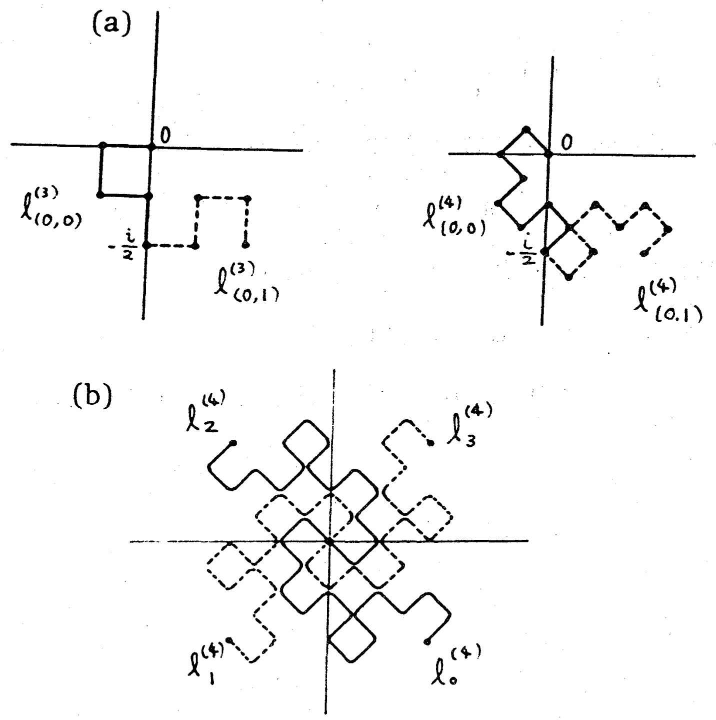

Fig.4: (a) Line segments $\ell^{(n)}_{(0,0)}$ and $\ell^{(n)}_{(0,1)}$ for n=3,4.
(b) Line segments $\{\ell^{(n)}_{\varepsilon}\}$ for n=4.

(2) The limit set D_T of $\{D_n\}_{n=1}$ has a dragon boundary as its boundary.

Moreover using above lemma we can prove that

<u>Lemma(3.3)</u>

Let H_T be the limit set of the paper folding curve H_n. Then the boundary of H_T consists of three parts of the dragon boundary. Therefore $\dim_H \partial H_T = \dim_H \partial D_T = 2\log\beta_0/\log 2$.

Put

$$X_{(\varepsilon,\delta)}^{(n)} = \{\Sigma_{k=1}^{n}\delta_k(1+i)^{-k};(\delta_1,\ldots,\delta_n)\in W_{(\varepsilon,\delta)}^{(n)}\}$$

$$X_{\varepsilon}^{(n)} = \{\Sigma_{k=1}^{n}\delta_k(1+i)^{-k};(\delta_1,\ldots,\delta_n)\in W_{\varepsilon}^{(n)}\},$$

and let $X_{(\varepsilon,\delta)}$ and X_ε be limit sets of $X_{(\varepsilon,\delta)}^{(n)}$ and $X_\varepsilon^{(n)}$ respectively (Fig.5). Thus we can prove that

<u>Lemma(3.4)</u>

(1) $(1+i)X_{(\varepsilon,0)} = X_\varepsilon$

(2) $(1+i)X_{(\varepsilon,(-i)^\varepsilon)} = X_{\varepsilon+1(\mathrm{mod}\ 4)} + (-i)^\varepsilon$,

(3) $\mathrm{int}(X_{(\varepsilon,\delta)}) \cap \mathrm{int}(X_{(\varepsilon',\delta')})=\phi$ for $(\varepsilon,\delta)\neq(\varepsilon',\delta')$,

and $\partial X_{(\varepsilon,\delta)}\cap\partial X_{(\varepsilon',\delta')}$ consists of parts of the dragon boundary.

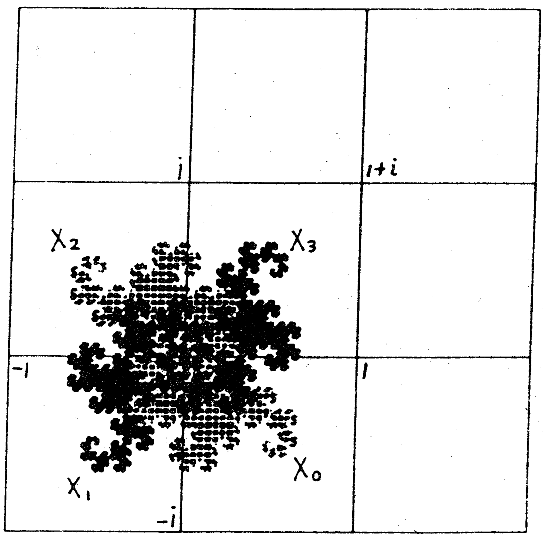

Fig.5: Tetradragon X.

Then putting

$$\hat{U}_0=X \text{ and } \hat{U}_\delta=\delta + X_{\varepsilon+1(\mathrm{mod}\ 4)} \text{ for } \delta=(-i)^\varepsilon,$$

and let a map T on X be

$$Tz = (1+i)z - [z]_D,$$

where $[z]_D=\delta$ if $w\in\hat{U}_\delta$ for $\delta\in\{0,1,-i,-1,i\}$,

then a transformation (X,T) is well defined and induces the revolving expansion.

<u>Theorem(3.1)</u>

(1) The Lebesgue measure ν is invariant with respect to (X,T).

(2) the dynamical system (X,T,ν) is isomorphic to (X_M,σ_M,μ_M), where μ_M is a stationary Markov measure such that

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}, \qquad \Pi = (1/4,1/4,1/4,1/4),$$

<u>Remark 1</u>:

The dual algorithm of (X,T,ν) is constructed by taking a complex conjugate, $X^*=\overline{X}$, and putting

$$\hat{U}^*_0 = X^*, \ \hat{U}^*_\delta = \delta + X^*_{\varepsilon-1(\mathrm{mod}\ 4)} \text{ for } \delta\in\{0,1,-i,-i,i\},$$

and

$$T^*z = (1-i)z -[(1-i)z]_{D^*},$$

where $[w]_{D^*}=\delta$ if $w\in\hat{U}^*_\delta$. Then a dynamical system (X^*,T^*,ν) is the dual system for the system (X,T,ν) and induces the "converse" revolving expansion,

$$z = \Sigma_{k=1}^{\infty} \delta^*_k (1-i)^{-k}.$$

Remark 2:

If we choose formally the dual domain $X^\#$ as

$$X^\# = \cup_\varepsilon X^\#_\varepsilon,$$

where

$$X^\#_\varepsilon = \{ \Sigma_{k=1}^{\infty} \delta^*_k (1+i)^{-k} ; (\delta^*_1, \delta^*_2, \ldots) \in W^*_\varepsilon \}.$$

Then we obtain an interesting picture (Fig.6). This selfsimilar fractal curve is already studied by P. Lévy in 1938[8].

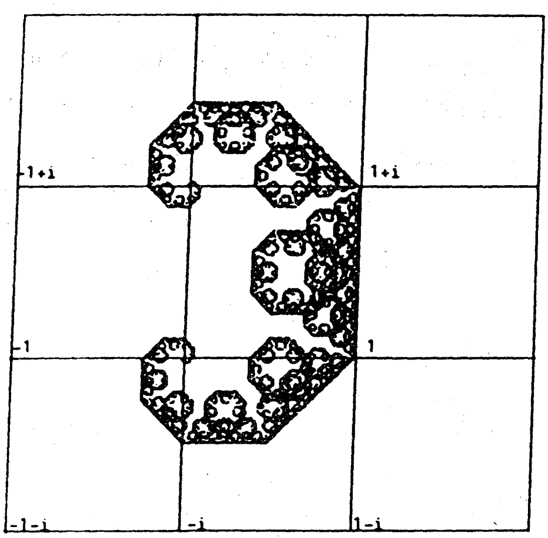

Fig.6: $X^\#_0$.

4. DUAL SYSTEM ON CROSS DRAGON

Let $E=(E_{j,k})$, $1\leqq j,k\leqq 4$, be a matrix such that

$$E = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

We consider E as the structure matrix for states $\Gamma=\{0,i,-1+i,-1\}$ by a correspondence $\tau:\{1,2,3,4\}\to\Gamma$ such that $\tau[1]=0$, $\tau[2]=i$, $\tau[3]=-1+i$ and $\tau[4]=-1$, that is, let V be a set of infinite sequences generated by the structure matrix E,

$$V = \{(\gamma_1,{}_2,\ldots.);E_{\gamma_j,\gamma_{j+1}}=1,\gamma_j\in\Gamma \text{ for all } j\in N\}$$

and σ a shift on V. Then the system (V,σ) is a Markov subshift. Let $V^{(n)}$ be a set of E-admissible sequences with length n and $V_\gamma^{(n)}$ be

$$V_\gamma^{(n)} = \{(\gamma_1,\ldots,\gamma_n)\in V^{(n)} \;;\; \gamma_1=\gamma\}.$$

Notice that nonzero entries of the structure matrix can be written as $E_{\tau[k],\tau[(k+1)\mathrm{mod}\ 4]}=E_{\tau[k],\tau[(k+2)\mathrm{mod}\ 4]}$ and denote these two admissible states after $\gamma=\tau[k]$ by $\gamma[1]=\tau[(k+1)\mathrm{mod}\ 4]$ and $\gamma[2]=\tau[(k+2)\mathrm{mod}\ 4]$ respectively.

<u>Property(4.1)</u>

(1) $V^{(n)} = \bigcup_{\gamma\in\{0,i,-1+i,-1\}} V_\gamma^{(n)}$,

(2) $\sigma V_\gamma^{(n)} = V_{\gamma[1]}^{(n-1)} \cup V_{\gamma[2]}^{(n-1)}$,

(3) $iV_\gamma^{(n)} + i = V_{\gamma[1]}^{(n)}$

and

$$-V_\gamma^{(n)} + (-1+i) = V_{\gamma[2]}^{(n)}.$$

We realize a sequence $(\gamma_1,\ldots,\gamma_n)$ to a point $p(\gamma_1,\ldots,\gamma_n)$ by

$$p(\gamma_1,\ldots,\gamma_n) = \Sigma_{k=1}^{n}\gamma_k(1+i)^{-k}.$$

According to the set of sequence $V^{(n)}$ and $V_\gamma^{(n)}$ let $Y^{(n)}$ and $Y_\gamma^{(n)}$ be sets of points $\{p(\gamma_1,\ldots,\gamma_n)\}$.

It is verified that

$$d(Y^{(n)},Y^{(n+1)}) \leqq (\frac{1}{\sqrt{2}})^n ,$$

in the Hausdorff metric. So $Y^{(n)}$ and $Y_\gamma^{(n)}$ converge to Y and Y_γ respectively as $n\to\infty$ (Fig.7).

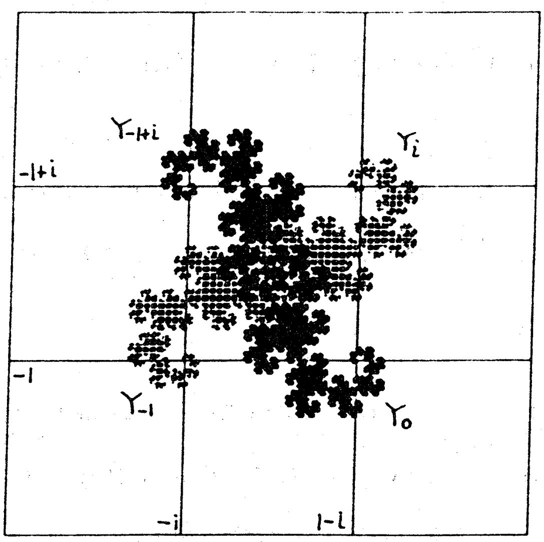

Fig.7: Cross dragon Y.

<u>Lemma(4.1)</u>

Let $Y=\{\Sigma_{k=1}^{\infty}\gamma_k(1+i)^{-k}:(\gamma_1,\gamma_2,\ldots)\in V\}$ and $Y_\gamma=\{\Sigma_{k=1}^{\infty}\gamma_k(1+i)^{-k}:(\gamma_1,\gamma_2,\ldots)\in V_\gamma\}$. Then the sets Y and Y_γ, $\gamma\in\Gamma$, satisfy following properties.

(1) $Y = \cup_{\gamma\in\{0,i,-1+i,-1\}}Y_\gamma$.

(2) $(1+i)Y_\gamma - \gamma = Y_{\gamma[1]} \cup Y_{\gamma[2]}$.

(3) $iY_\gamma + 1 = Y_{\gamma[1]}$

and

$$-Y_\gamma + 1+i = Y_{\gamma[2]}.$$

(4) $Y_\gamma = F_{0,\gamma}(Y_\gamma) \cup F_{1,\gamma}(Y_\gamma)$,

where $F_{0,\gamma}$ and $F_{1,\gamma}$ are contraction maps such that

$$F_{0,\gamma}(z) = (1+i)^{-1}(iz+\gamma+1)$$

and

$$F_{1,\gamma}(z) = (1+i)^{-1}(-z+\gamma+1+i) \quad \text{for each } \gamma\in\Gamma.$$

Recall another approach for selfsimilar fractal sets proposed by Hutchinson[7] using a set of contraction maps.

<u>Theorem</u> (Hutchinson[7])

Let $\mathcal{L}$ be a finite set of contraction maps $\{S_1,\ldots,S_M\}$ on a metric space. Then there exists a unique closed bounded set K such that $K = \cup_{j=1}^{M}S_j(K)$. Moreover, let $\mathcal{L}(A)=\cup_{j=1}^{M}S_j(A)$ and $\mathcal{L}^p(A)=\mathcal{L}(\mathcal{L}^{p-1}(A))$ for arbitrary set A, then $\mathcal{L}^p(A)\to K$ in the Hausdorff metric as $p\to\infty$ for closed bounded set A.

Thus we can say that the limit sets $\{Y_\gamma\}$ are invariant sets for the contraction maps $\{F_{0,\gamma}, F_{1,\gamma}\}$. Notice that the set $\{X^*_\varepsilon\}$ in section 3 are the invariant set for the contraction maps $\{G^*_{0,\varepsilon}, G^*_{1,\varepsilon}\}$ for each $\varepsilon \in \{0,1,2,3\}$, where

$$G^*_{0,\varepsilon}(z)=(1-i)^{-1}z \quad \text{and} \quad G^*_{1,\varepsilon}(z)=(1-i)^{-1}(iz+i^{\varepsilon}),$$

that is,

$$X^*_\varepsilon = G^*_{0,\varepsilon}(X^*_\varepsilon) \cup G^*_{1,\varepsilon}(X^*_\varepsilon)$$

and for $\mathcal{L}=\{G^*_{0,\varepsilon}, G^*_{1,\varepsilon}\}$

$$G^*_{0,\varepsilon}(\ \mathcal{L}^n(0)\ ) = X^{*(n+1)}_{(\varepsilon,0)}$$

and

$$G^*_{1,\varepsilon}(\ \mathcal{L}^n(0)\ ) = X^{*(n+1)}_{(\varepsilon,i^{\varepsilon})}.$$

Then we obtain

<u>Theorem(4.1)</u>

Let $\{Y_\gamma\}$ satisfy $Y_\gamma = F_{0,\gamma}(Y_\gamma) \cup F_{1,\gamma}(Y_\gamma)$ for each $\gamma \in \Gamma$, and $Y=\bigcup_{\gamma \in \{0,i,-1+i,-1\}} Y_\gamma$. Then

(1) Each set Y_γ is a dragon with end points 0 for Y_{-1}, 1 for Y_0, $1+i$ for Y_i, i for Y_{-1+i} and $(1+i)/2$ in common.

(2) The set Y is tiled by four dragons $\{Y_\gamma\}$, that is,

$$Y=\bigcup_{\gamma \in \{0,i,-1+i,-1\}} Y_\gamma$$

and

$$\lambda(Y_\gamma \cap Y_{\gamma'})=0 \qquad \text{for } \gamma \neq \gamma'.$$

We call the set Y a cross dragon.

Let a map S on Y be

$$Sz = (1+i)z - [(1+i)z]_C,$$

where $[w]_C = \gamma$ if $w \in \gamma + (Y_{\gamma[1]} \cup Y_{\gamma[2]})$. Then (Y.S) is well defined and induces an expansion

$$z = \Sigma_{k=1}^{\infty} \gamma_k (1+i)^{-k} \qquad \text{for a.e. } z \in Y.$$

Now let $Y^* = \{x+iy : 0 \leqq x, y < 1\}$ and a map S^* be

$$S^* z = (1+i)z - [(1+i)z],$$

where $[w] = [Re(w)] + i[Im(w)]$ for $z \in \mathbb{C}$. This system is equivalent to a group endomorphism T_L on the torus $\mathbb{T}^2$ such that

$$T_L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} [x-y] \\ [x+y] \end{pmatrix}.$$

<u>Theorem(4.2)</u>

(1) The Lebesgue measure λ is invariant with respect to (Y,S).

(2) The cross dragon system (Y, S, λ) is actually the dual system for (Y^*, S^*, λ).

<u>Remark</u>:

The cross dragon system (Y, S, λ) is isomorphic to a following map on the torus.

$$T^{\dagger} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \begin{pmatrix} [x+y-1] \\ [-x+y+1] \end{pmatrix}.$$

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