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On a Bernoulli property for multi-dimensional mappings with finite range structure

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ABSTRUCT

A class of multi-dimensional mappings whose element has a finite range structure is proposed. In previous work [2], sufficient conditions for ergodicity and the existence of σ-finite invariant measures on this class were given. In this paper, we show that such mappings are weak Bernoulli if they admit finite invariant measures.

1. INTRODUCTION

In previous paper [2], we introduced a mapping T on a bounded domain $X \subset R^n$ which is characterized by a partition $Q = \{X_a : a \in I\}$ of X and a finite number of subsets $\{U_0, \cdots, U_N\}$ of X as follows.

(1) Each X_a is measurable connected subset with piecewise smooth boundary.

(2) Each U_k has a positive measure.

(3) For each X_a the mapping $T|_{X_a}$ restricted on X_a is injective, of class C^1, and det $DT|_{X_a} \neq 0$.

(4) If $\mathrm{int}(X_{a_1}) \cap \mathrm{int}(T^{-1}X_{a_2}) \cap \cdots \cap \mathrm{int}\,(T^{-(n-1)}X_{a_n}) \neq \emptyset$, then $T^n(X_{a_1} \cap \cdots \cap T^{-(n-1)}X_{a_n}) = U_k$ for some $k \in \{0, \cdots, N\}$.

Various mappings which are associated with the expansions of points in R^n have the above structure. In this paper, we call this mapping T a multi-dimensional mapping with a finite range structure. The ergodicity and properties of invariant measure of T were investigated in (2) .

In this work , we prove that the partition Q becomes a weak Bernoulli partition with respect to T under some conditions on T.

For mappings on an interval having a finite partition, it is known that the weak mixing property implies the weak Bernoulli property if the mapping is uniformly expansive. (See Bowen [1] for details). When the mapping is not uniformly expansive Ledrappier [4] recently proved the same statemant as above under the existence of an absolutely continuous invariant measure with a positive entropy. The main ingredient of his proof, which is patterned after the work of Sinai [7] and Ratner [6], is the use of Rohlin's formula for proving the absolute continuity of some conditional measures.

For multi-dimensional mappings with a countable partition, we should point out in this connection that a regular condition ("Renyi condition") on T allows us to have the following: T is an exact endomorphism with an absolutely continuous ergodic invariant measure whose density is bounded, and Q is a weak Bernoulli partition [5), 9)]. However, if Renyi condition is not satisfied, then it is very difficult to establish these properties. Our main purpose of this work is to give a sufficient condition for T to have the weak Bernoulli property when they do not necessarily satisfy the Renyi condition. We do need, however, to make several assumptions on the transformation. Under the assumptions, we will show that T is exact, and that "Rohlin's formula" holds for T. Next, we construct a natural extension for T and conditional measures with respect to the extension. By using Rohlin's formula, we prove the absolutely continuity of the conditional measures, and this will leads us to the desired conclusion along the line of argument used in (4) and (6).

2. NOTATIONS AND RESULTS

Throughout this paper, T denotes a multi-dimensional mapping with a finite range structure. If $\mathrm{int}(X_{a_1}) \cap \mathrm{int}(T^{-1}X_{a_2}) \cap \cdots \cap \mathrm{int}(T^{-(n-1)}X_{a_n}) \neq \emptyset$, we denote $X_{a_1} \cap T^{-1}X_{a_2} \cap \cdots \cap T^{-(n-1)}X_{a_n}$ by $X_{a_1 \cdots a_n}$ and call it a cylinder of rank n with respect to T. $\mathcal{L}^{(n)}$ denotes the family of all cylinders $X_{a_1 \cdots a_n}$ of rank n, and $\mathcal{L} = \bigcup \mathcal{L}^{(n)}$. If $X_{a_1 \cdots a_n} \in \mathcal{L}^{(n)}$, we call the sequence $(a_1 \cdots a_n)$ T-admissible. Denote the set of all T-admissible sequences of length n by A(n). We write Ψ_a for $(T|_{X_a})^{-1}$ and define inductively

$$\Psi_{a_1 \cdots a_n} = \Psi_{a_1 \cdots a_{n-1}} \circ \Psi_{a_n} .$$

For a constant $C \geq 1$, we call a cylinder $X_{a_1 \cdots a_n}$ an "R.C-cylinder" if it satisfies "Renyi condition", i.e.

$$\sup_{x \in T^n X_{a_1 \cdots a_n}} |\det D\Psi_{a_1 \cdots a_n}(x)| \leq C \inf_{x \in T^n X_{a_1 \cdots a_n}} |\det D\Psi_{a_1 \cdots a_n}(x)|$$

Let R(C.T) denote the set of all R.C-cylinders.

We define for $C \geq 1$

$$\mathcal{D}_n = \left\{ X_{a_1 \cdots a_n} \in \mathcal{L}^{(n)} : X_{a_1 \cdots a_j} \in \mathcal{L} \setminus R(C.T) \text{ for } 1 \leq j \leq n \right\}$$

$$D_n = \bigcup_{X_{a_1 \cdots a_n} \in \mathcal{D}_n} X_{a_1 \cdots a_n},$$

$$\beta_n = \left\{ X_{a_1 \cdots a_n} \in \mathcal{L}^{(n)} : X_{a_1 \cdots a_{n-1}} \in \mathcal{D}_{n-1},\ X_{a_1 \cdots a_n} \in R(C.T) \right\}$$

$$B_n = \bigcup_{X_{a_1 \cdots a_n} \in \beta_n} X_{a_1 \cdots a_n}.$$

Let $\lambda(.)$ be the normalized Lebesgue measure on X.

In previous paper [2], we showed that T is ergodic and has a finite invariant measure μ equivalent to λ under the following conditions.

(C.1) (generator condition)

$$\bigvee_{m=1}^{\infty} T^{-m}Q = \mathcal{E} \text{ , i.e. the partition into points.}$$

Assume that there exists a constant $C \geqq 1$ such that

(C.2) (transitivity condition)

for each j with $0 \leqq j \leqq N$, there exists a cylinder $X_{a_1 \cdots a_{s_j}}$ contained in U_j such that $X_{a_1 \cdots a_{s_j}} \in R(C.T)$ and $T^{s_j} X_{a_1 \cdots a_{s_j}} = X$.

(C.3) If $X_{a_1 \cdots a_n} \in R(C.T)$, then $X_{b_1 \cdots b_k a_1 \cdots a_n} \in R(C.T)$ for any $(b_1 \cdots b_k a_1 \cdots a_n) \in A(k+n)$,

(C.4) $$\sum_{n=1}^{\infty} \lambda(D_n) < +\infty.$$

Under the same conditions (C.1) $\sim$ (C.4), we can show that

Theorem 1. T is exact.

In general, the density of μ is not bounded. For this reason, we need some technical conditions.

(C.5) For all $n>0$,

$$W_n = \sum_{m=0}^{\infty} \sum_{X_{k_1 \cdots k_m} \in \mathcal{D}_m} \left(\sup_{y \in T^m X_{k_1 \cdots k_m} \cap \left(\bigcup_{j=1}^{n} B_j\right)} \left|\det D\Psi_{k_1 \cdots k_m}(y)\right| \right)$$

is finite.

Remark 1. The following important properties follow from this condition: The density of μ is bounded on $(D_n)^c$ for each $n>0$, and therefore for all $n>0$, there exists $M(n)$ such that

$$\frac{\mu(X_{a_1\cdots a_n})}{\lambda(X_{a_1\cdots a_n})} < M(n) \quad \text{for any } X_{a_1\cdots a_n} \in \mathcal{L}^{(n)}.$$

(C.6) $\#\mathcal{D}_1$ is finite.

The next condition is weaker Renyi condition.

(C.7) there exists a positive integer l such that for all $n>0$ and all $(a_1\cdots a_n) \in \mathcal{D}_n$

$$\frac{\sup_{x \in T^n X_{a_1\cdots a_n}} \left|\det D\Psi_{a_1\cdots a_n}(x)\right|}{\inf_{x \in T^n X_{a_1\cdots a_n}} \left|\det D\Psi_{a_1\cdots a_n}(x)\right|} = O(n^l).$$

Remark 2. This condition allows us to have $\inf_{x \in X} \det|DT(X)|>0$.

We also suppose

(C.8) $\log\left|\det DT(\cdot)\right| \in \mathcal{L}^1(\lambda, X)$.

Then we have

Theorem 2. Rohlin's formula (R) is true.

(R): $h(\mu) = \int_X \log\left|\det DT(x)\right| \, d\mu(x)$.

For main theorem, we assume further

(C.4)* $\sum_{n=1}^{\infty} \lambda(D_n) \log n < +\infty$.

(C.9) there exists a positive integer k_0 which satisfies the following; if $X_{a_1\cdots a_n} \in \mathcal{D}_n^c$ and $X_{a_2\cdots a_n}$

$\in \mathcal{D}_{n-1}$, then $X_{a_1 \cdots a_n} \subset \bigcup_{j=1}^{k_0} B_j$.

Theorem 3. Let T be a multi-dimensional mapping with a finite range structure satisfying (C.1) $\sim$ (C.9). Then Q is a weak Bernoulli partition with respect to T.

3. APPLICATIONS

As applications of our theorem, we give two examples of multi-dimensional mapping with a finite range structure. These examples arise from the metrical theory of numbers: an inhomogeneous diophantine approximation problem and complex continued fractions. Neither of these transformations satisfies the Renyi condition, but both of them do satisfy a local Renyi condition. We can show that these transformations satisfy all of the assumptions, and therefore, are weak Bernoulli.

Example 1.[3] $X = \{(x,y) \in R^2 : 0 \leqq y \leqq 1,\ -y \leqq x < -y+1\}$

$$T(x,y) = \left(\frac{1}{x} - \left[\frac{1-y}{x}\right] - \left[-\frac{y}{x}\right],\ -\left[-\frac{y}{x}\right] - \frac{y}{x} \right).$$

Example 2.[8] $X = \left\{ z = x\alpha + y\bar{\alpha} : -\frac{1}{2} \leqq x, y \leqq \frac{1}{2} \right\}$

$(\alpha = 1 + i)$.

$T(z) = \frac{1}{z} - \left[\frac{1}{z}\right]_1$, where $[z]_1$ denotes

$\left[x + \frac{1}{2}\right]\alpha + \left[y + \frac{1}{2}\right]\bar{\alpha}$ for a complex number

$z = x\alpha + y\bar{\alpha}$.

The proofs of these theorems and examples will be published in Tokyo Journal of Mathematics[10]. Hence we state results only here.

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