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POINCARÉ MAPS OF THE DOUBLE SCROLL

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ABSTRACT

A family of picewise linear vector fields of $\boldsymbol{R}^3$ is discussed. A detailed analysis is given of the linearly conjugate classes and the Poincaré maps for the family. A result of the analysis is applied to the study of bifurcation of an attractor.

§ 0 Double-Scroll System

The double-scroll system is a picewise linear ordinary differential equation on $\boldsymbol{R}^3$ defined by

$$\begin{cases} \dot{x} = S(y - f(x)) \\ \dot{y} = x - y + z \\ \dot{z} = -Ty \end{cases}$$

$$f(x) = \begin{cases} M_{p(x-1)} + M_0, & x \geq 1 \\ M_0 x & , |x| \leq 1 \\ M_{p(x+1)} + M_0, & x \leq -1 \end{cases}$$

where $S > 0$, $T > 0$, $M_0 < 0$ and $M_p > 0$ are parameters. The double-scroll system is derived from an extremely simple autonomus electrical circuit [1]. When $(S, T, M_0, M_p) = (9, 14{\cdot}2/7, -1/7, 2/7)$, a chaotic attractor, which is called the double scroll [1], is observed. In this paper, we study a large family of picewise linear vector fields of $\boldsymbol{R}^3$ which contains the double-scroll system. Moreover the family contains the various system with chaotic attractors studied by several authors, including C. Sparrow [5], A. Arneodo et al. [3], R. Brockett [4] and B. Uehleke et al. [6]. A detailed analysis is given of the linearly conjugate classes and the Poincaré maps for the family. A result of the analysis is applied to the study of bifurcations of the double scroll.

§ 1 Linearly Conjugate Classes

Definition 1. Let $\mathscr{L}$ be the set of all continuous vector fields on $\boldsymbol{R}^3$ which satisfy the following (1) – (6): for each $\xi\in\mathscr{L}$,

(1) is symmetric with respect to the origin, i.e.

$$\xi(-\underset{\sim}{x})=-\xi(\underset{\sim}{x}),\ \underset{\sim}{x}\in\boldsymbol{R}^3$$

(2) There are two planes U_1 and U_{-1} which are symmetric with respect to the origin, and which divide $\boldsymbol{R}^3$ into three regions D_1, D_0 and D_{-1}.

(3) On each region D_i ($i = 0 \pm 1$), the vector field $\xi|_{D_i}$ is linear.

(4) An equlibrium point 0 (resp. $P\pm$) is in the interior of D_0 (resp. $D\pm 1$).

(5) Eigenvalues of $\xi|D_0$ (resp. $\xi|D_{\pm1}$) are a complex conjugate pair $\tilde{\alpha}_0\pm\sqrt{-1}\,\tilde{\beta}_0$, $\tilde{\beta}_0>0$ (resp. $\tilde{\alpha}_1\pm\sqrt{-1}\,\tilde{\beta}_1$, $\tilde{\beta}_1>0$) and a real $\tilde{\gamma}_0\neq0$ (resp. $\tilde{\gamma}_1\neq0$).

(6) Each eigenspace is not parallel to $U_{\pm1}$.

Definition 2. For each $\xi\in\mathscr{L}$, define (see Fig. 1)

$E^c(0)=$ the eigenspace corresponding to $\tilde{\alpha}_0\pm\sqrt{-1}\,\tilde{\beta}_0$ at 0,

$E^r(0)=$ the eigenspace corresponding to $\tilde{\gamma}_0$ at 0,

$E^c(P^+)=$ the eingenpace corresponding to $\tilde{\alpha}_1\pm\sqrt{-1}\,\tilde{\beta}_1$ at P^+,

$E^r(P^+)=$ the eingenpace corresponding to $\tilde{\gamma}_1$ at P^+,

$$\begin{aligned}
&L_0=U_1\cap E^c(0),\\
&L_1=U_1\cap E^c(P^+),\\
&L_2=\{\underset{\sim}{x}\in U_1\,|\,\xi(\underset{\sim}{x})/\!\!/U_1\},\\
&A=L_0\cap L_1,\\
&B=L_1\cap L_2,\\
&C=U_1\cap E^r(0),\\
&D=U_1\cap E^r(P^+),\\
&E=L_0\cap L_2,\\
&F=\{\underset{\sim}{x}\in L_2\,|\,\xi(\underset{\sim}{x})/\!\!/L_2\}.
\end{aligned}$$

The points A, B, E and P^+ are called the fundamental points of ξ.

Definition 3. Define a map $H:\mathscr{L}\to\boldsymbol{R}^5$ by

$$H(\xi)=(\alpha_0,\ \gamma_0,\ \alpha_1,\ \gamma_1,\ \kappa)$$

where

$$\begin{aligned}
&\alpha_0=\tilde{\alpha}_0/\tilde{\beta}_0,\quad \gamma_0=\tilde{\gamma}_0/\tilde{\beta}_0\quad (\tilde{\beta}_0>0)\\
&\alpha_1=\tilde{\alpha}_1/\tilde{\beta}_1,\quad \gamma_1=\tilde{\gamma}_1/\tilde{\beta}_1\quad (\tilde{\beta}_1>0)\\
&\kappa=-\tilde{\gamma}_0/\tilde{\gamma}_1.
\end{aligned}$$

Theorem 1

(A) For $\xi_1,\ \xi_2\in\mathscr{L}$, the following is equivalent:

(1) $H(\xi_1)=H(\xi_2)$

(2) ξ_1 and ξ_2 are linearly conjugate preserving time-orientation, i.e. there exist a real

$\nu>0$ and a linear transformation

$G: \boldsymbol{R}^3 \to \boldsymbol{R}^3$ such that $DG \circ \xi_1 = \nu \xi_2 \circ G$

(B) Put $\mathcal{R}=\{(\alpha_0, r_0, \alpha_1, r_1, \kappa) \in \boldsymbol{R}^5 \mid r_0 r_1<0, \kappa>0\}$, then

$H(\mathcal{L})=\mathcal{R}$.

(C) For any $\mu \in \mathcal{R}$, there exist real numbers

$l=l(\mu)$, $m=m(\mu)$, $n=n(\mu)$ such that, for any $\xi \in H^{-1}(\mu)$,

$$\overrightarrow{OP^+}= l\overrightarrow{OA}+m\overrightarrow{OB}+n\overrightarrow{OE},$$

where A, B, E and P^+ are the fundamental points of ξ .

Remark 1. It is easy to obtain a linearly conjugate class (not necessarily time-orientation preserving) from the theorem.
Indeed, define $(\alpha_0, r_0, \alpha_1, r_1, \kappa) \sim (\alpha_0', r_0', \alpha_1', r_1', \kappa')$ by $(\alpha_0, r_0, \alpha_1, r_1, \kappa)=(\alpha_0', r_0', \alpha_1', r_1', \kappa)$ or $(-\alpha_0', -r_0', -\alpha_1', -r_1', \kappa')$, then ξ_1 and ξ_2 are linearly conjugate if and only if $H(\xi_1) \sim H(\xi_2)$.

Remark 2. For $\mu=(\alpha_0, r_0, \alpha_1, r_1, \kappa)$, the l, m, n in the statement (C) is explicitly given as follows:

$$l=-(r_1\kappa+\alpha_1)^2-\kappa^2 r_1^2(r_0/\kappa+2\alpha_0)/r_0(\alpha_1^2+1)$$
$$m=(r_1\kappa+\alpha_1)^2+1$$
$$n=\kappa^3 r_1^2\{(r_0/\kappa+\alpha_0)^2+1\}/r_0^2(\alpha_1^2+1)$$
$$s=l+m+n=1+\kappa^3 r_1^2(\alpha_0^2+1)/r_0^2(\alpha_1^2+1).$$

Remark 3. For $\mu=(\alpha_0, \nu_0, \alpha_1, r_1, \kappa) \in \mathcal{R}$, a vector field $\xi \in \mathcal{L}$ with $H(\xi)=\mu$ is explicitly given as follows:

$$\xi(x, y, z)=(a_{ij})(x, y, z)^T+(b_1, b_2, b_3)^T\{|z-1|-|z+1|\},$$

$$\begin{array}{lll}
a_{11}=\lambda(c_1\bar{n}+r_1) & a_{12}=\lambda c_1\bar{m} & a_{13}=\lambda\bar{s}(c_1\bar{l}-r_1) \\
a_{21}=c_0\bar{n} & a_{22}=c_0\bar{m}+r_0 & a_{23}=\bar{s}(c_0\bar{l}-r_0) \\
a_{31}=c_0\bar{n} & a_{32}=c_0\bar{m} & a_{33}=\bar{s}c_0\bar{l} \\
b_1=\lambda\bar{r}(c\bar{l}-r_1) & b_2=\bar{r}(c_0\bar{l}-r_0) & b_3=\bar{r}c_0\bar{l}
\end{array}$$

where

$$\lambda=-r_0/r_1\kappa, \quad c_0=-\kappa(\alpha_0^2+1)/r_0, \quad c_1=-(\alpha_1^2+1)/r_1\kappa$$
$$\bar{l}=l/s, \quad \bar{m}=m/s, \quad \bar{n}=n/s$$
$$\bar{s}=1/(1-s), \quad \bar{r}=s/2(1-s)$$

(l, m, n, s are as in Remark 2).

Fundamental points:

$$A=(1, 1, 1), \quad B=(1, -(l+n)/m, 1), \quad E=(-(l+m)/n, 1, 1)$$
$$P^{\pm}=(0, 0, \pm s), \quad U_{\pm 1}=\{(x, y, z) \mid z=\pm 1\}.$$

Definition 4. Let $\xi \in \mathscr{L}$ with $H(\xi)=(\alpha_0, r_0, \alpha_1, r_1, \kappa)$ be given. We can take two affine transformations $\psi_0 : D_0 \to \boldsymbol{R}^3$ and $\psi_0 : D_1 \to \boldsymbol{R}^3$ such that (see Fig. 2)

a) $\Psi_0(0)=0$

$\Psi_0(U_1)=V_0 \triangleq \{(x, y, z) | x+z=1\}$

$\Psi_0(U_{-1})=V_0^- \triangleq \{(x, y, z) | x+z=-1\}$

$$\frac{1}{\tilde{\beta}_0} D\Psi_0(\xi(\Psi_0^{-1}\underset{\sim}{x}))=\xi_0(\underset{\sim}{x}) \triangleq \begin{bmatrix} \alpha_0 & -1 & 0 \\ 1 & \alpha_0 & 0 \\ 0 & 0 & r_0 \end{bmatrix} \underset{\sim}{x},$$

b) $\Psi_1(P^+)=0.$

$\Psi_1(U_1)=V_1 \triangleq \{(x, y, z) | x+z=1\},$

$$\frac{1}{\tilde{\beta}_1} D\Psi_1(\xi(\Psi_1^{-1}\underset{\sim}{x}))=\xi_1(\underset{\sim}{x}) \triangleq \begin{bmatrix} \alpha_1 & -1 & 0 \\ 1 & \alpha_1 & 0 \\ 0 & 0 & r_1 \end{bmatrix} \underset{\sim}{x}.$$

Define the connection map $\Phi : V_1 \to V_0$ by $\Phi=(\Psi_0 | U_1) \circ (\Psi_1 | U_1)^{-1}$

Let us denote

$A_i=\Psi_i(A), \quad B_i=\Psi_i(B), \quad E_i=\Psi_i(F), \quad F_i=\Psi_i(F)$

$p_i=\alpha_i+(\alpha_i^2+1)K_i/r_i, \quad Q_i=(\alpha_i-r_i)^2+1 \quad (i=0, 1),$

$K_0=K, \quad K_1=1/K.$

Then the following holds:

i) $A_0=(1, p_0, 0)$

$B_0=(r_0(r_0-\alpha_0-p_0)/Q_0, \quad r_0\{1-p_0(\alpha_0-r_0)\}/Q_0, \quad 1-r_0(r_0-\alpha_0-p_0)/Q_0)$

$E_0=(1, \alpha_0, 0)$

$F_0=(r_0(r_0-2\alpha_0)/Q_0, \quad r_0\{1-\alpha_0(\alpha_0-r_0)\}/Q_0, \quad (\alpha_0^2+1)/Q_0)$

ii) $A_1=(1, p_1, 0)$

$B_1=(1, \alpha_1, 0)$

$E_1=(r_1(r_1-\alpha_1-p_1)/Q_1, \quad r_1\{1-p_1(\alpha_1-r_1)\}/Q_1, \quad 1-r_1(r_1-\alpha_1-p_1)/Q_1)$

$F_1=(r_1(r_1-2\alpha_1)/Q_1, \quad r_1\{1-\alpha_1(\alpha_1-r_1)\}/Q_1, \quad (\alpha_1^2+1)/Q_1)$

iii) $\Phi : V_1 \to V_0$ is obtained by

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}=\Phi\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}=(L_{ij})\begin{pmatrix} x_1-1 \\ y_1-p_1 \end{pmatrix}+\begin{pmatrix} 1 \\ p_0 \end{pmatrix},$$

$L_{11}=-r_0(K_1+1)[Q_1+r_1(\alpha_1-r_1)(K_0+1)]R$

$L_{12}=r_0 r_1(K_0+1)(K_1+1)R$

$L_{21}=\{-r_0(K_1+1)(\alpha_0-r_0)[\alpha_1(\alpha_1-r_1)+1]-r_1(K_0+1)(\alpha_1-r_1)[\alpha_0(\alpha_0-r_0)+1]\}R$

$L_{22}=r_1(K_0+1)[Q_0+r_0(\alpha_0-r_0)(K+1)]R$

$R=(\alpha_0^2+1)K_0/(\alpha_1^2+1)(K_1+1)Q_0 Y_0$

where we identify $(x_i, y_i) \equiv (x_i, y_i, z_i) \in V_i$ because $z_i=1-x_i$ holds (i=0, 1).

§ 2 Poincaré maps

In this section, we assume that

$$\alpha_0<0,\ r_0>0,\ \alpha_1>0,\ r_1<0,\ K>0.$$

Definition 5. We induce a new coordinate system, say (u, v)-coordinate, to angular region $\angle A_i B_i E_i$ on V_i. Define

$$\underset{\sim}{x}_i(u, v)=u(vA_i+(1-v)E_i)+(1-u)(vB_i+(1-v)F_i)\in V_i,\ (u, v)\in[0, \infty)\times[0, 1]$$

$$\angle A_i B_i E_i=\{\underset{\sim}{x}_i(u, v)\mid(u, v)\in[0, \infty)\times[0, 1]\},$$

$$\Delta A_i B_i E_i=\{\underset{\sim}{x}_i(u, v)\mid(u, v)\in[0, 1]\times[0, 1]\},\ (i=0, 1).$$

Let φ_i^t be a flow of ξ_i (i=0, 1). (See Fig. 2.)

(a) A return map for φ_0^t, $\pi_0^+: \Delta A_0 B_0 E_0 \to V_0$ is defined by

$$\pi_0^+(\underset{\sim}{x})=\varphi_0^T(\underset{\sim}{x}),\ T=T(\underset{\sim}{x})=\inf\{t>0\mid\varphi_0^t(\underset{\sim}{x})\in V_0\}.$$

(b) A return map for $\varphi_0^t : \pi_0^-: \angle A_0 B_0 E_0 \setminus \Delta A_0 B_0 E_0 \to V_0^-$ is defined by

$$\pi_0^-(\underset{\sim}{x})=\varphi_0^T(\underset{\sim}{x}),\ T=T(\underset{\sim}{x})=\inf\{t>0\mid\varphi_0^t(\underset{\sim}{x})\in V_0^-\}.$$

(c) In consideration of the symmetry of ξ, we define a return map $\pi_0: \angle A_0 B_0 E_0 \to V_0$ by

$$\pi_0(\underset{\sim}{x})=\begin{cases}\pi_0^+(\underset{\sim}{x}), & \underset{\sim}{x}\in\Delta A_0 B_0 E_0\\ \pi_0^-(\underset{\sim}{x}), & \underset{\sim}{x}\in\angle A_0 B_0 E_0\setminus\Delta A_0 B_0 E_0.\end{cases}$$

(d) A return map for φ_1^{-t}, $\pi_1: \angle A_1 B_1 E_1 \to V_1$ is defined by

$$\pi_1(\underset{\sim}{x})=\varphi_1^{-T}(\underset{\sim}{x}),\ T=T(\underset{\sim}{x})=\inf\{t>0\mid\varphi_1^{-t}(\underset{\sim}{x})\in V_1\}.$$

We can identify a point of V_i with a complex number:

$$(x_i,\ y_i,\ z_i)\equiv(x_i,\ y_i)\equiv x_i+\sqrt{-1}\ y_i\in \boldsymbol{C}.$$

Then the return maps are represented as follows:

Theorem 2. Put $A_{0v}=\underset{\sim}{x}_0(1, v)$, $B_{0v}=\underset{\sim}{x}_0(0, v)$, $A_{1u}=\underset{\sim}{x}_1(u, 1)$, $E_{1u}=\underset{\sim}{x}_1(u, 0)$ and $h=(1, 0, 1)$. We consider that $\underset{\sim}{x}_i(u, v)$ is a complex number (i=0, 1) except the points A_{0u}, B_{0u}, A_{1u} and E_{1u}, which are considered vectors in $\boldsymbol{R}^3$. The usual inner product in $\boldsymbol{R}^3$ is denoted by $\langle\ ,\ \rangle$.

(a) $\pi_0^+(\underset{\sim}{x}_0(u, v))=x_0(u, v)\exp[(\alpha_0+\sqrt{-1})t]$.

where $u=u(v, t)=\{\langle\varphi_0^t(B_{0v}), h\rangle-1\}/\langle\varphi_0^t(B_{0v}-A_{0v}), h\rangle$

for $t\in\{t>0\mid\partial u/\partial t>0$ on $\{v\}\times(0, t)\}$.

(b) $\pi_0^-(\underset{\sim}{x}_0(u, v))=\underset{\sim}{x}_0(u, v)\exp[(\alpha_0+\sqrt{-1})t]$

where $u=u(v, t)=\{\langle\varphi_0^t(B_{0v}), h\rangle+1\}/\langle\varphi_0^t(B_{0v}-A_{0v}), h\rangle$

for $t\in\{t>0\mid\partial u/\partial t<0$ on $\{v\}\times(0, t)\}$.

(c) $\pi_1(\underset{\sim}{x}_1(u, v))=\underset{\sim}{x}_1(u, v)\exp[-(\alpha_1+\sqrt{-1})t]$

where $v=v(u, t)=\{\langle\varphi_1^{-t}(E_{1u}), h\rangle-1\}/\langle\varphi_1^{-t}(E_{1u}-A_{1u}), h\rangle$

for $t\in\{t>0\mid\partial v/\partial t>0$ on $\{u\}\times(0, t)\}$.

§ 3 Birth and Death of the Double Scroll

(1) Birth of the double scroll

Observations of the double scroll bifurcations [2] indicate that the double scroll is born out of a collision of a pair of Rössler's screw type attractors. We call such a phenomenon the birth of the double scroll.

Now we assume that $\pi_1(\overline{A_1E_1})$ and $l=\{(x, y): x=1\}$ have a point of intersection, say A_1', as in Fig. 3(b). Then, in order for a pair of screw type attractors to collide with each other, it is necessary for the arc $\overset{\frown}{E_1A_1'}=\pi_1(\overline{A_1E_1})$ to intersect the spiral $\overset{\frown}{B_1C_1}=\Phi^{-1}\pi_0\Phi(\overline{A_1B_1})$. Therefore, the parameter value at which $\overset{\frown}{E_1A_1'}$ and $\overset{\frown}{B_1C_1}$ touch each other, is an approximation of the value at which the double scroll is born. This approximation turned out to be in an excellent agreement with the observations of the double-scroll system using Runge-Kutta iterations.

Remark Note that an intersection of a screw type attractor and U_1 must be between the spirals $\overset{\frown}{BC}=\Psi_1^{-1}(\overset{\frown}{B_1C_1})$ and $\overset{\frown}{FC}=\Psi_1^{-1}(\overset{\frown}{F_1C_1})$, except for a part included in ΔABE. Therefore the parameter value at which $\overset{\frown}{E_1A_1'}$ and $\overset{\frown}{B_1C_1}$ touch each other is before the birth of the double scroll, while the parameter value at which $\overset{\frown}{E_1A_1'}$ and $\overset{\frown}{F_1C_1}$ touch each other is after the birth of the double scroll.

(2) Death of the double scroll

It is known that there is a saddle type closed orbit around the double scroll [1]. In the double-scroll system, for instance, when M_O, M_P and T are fixed and S is increased, the distance between the attractor and the saddle type closed orbit decreases, and they touch each other, finally the attractor disappears [2]. We call such a phenomenon the death of the double scroll.

Let H^+ and H^- be the points of intersection of the saddle type closed orbit Γ and the plane U_1, where H^- is the point which belongs to $\angle ABE$. Put $H_1^-=\Psi_1(H^-)$ and $H_1^+=\Psi_1(H^+)$. Then $H_1^+=\pi_1(H_1^-)=\Phi^{-1}\pi_0\Phi(H_1^-)$. Define $\pi=\pi_1^{-1}\Phi^{-1}\pi_0\Phi$ and $W^s(H_1^-)=\{\underset{\sim}{x}\in \angle A_1B_1E_1 \mid \pi^n(\underset{\sim}{x})\rightarrow H_1^-(n\rightarrow\infty)\}$, $W^s(H_1^+)=\pi_1(W^s(H_1^-))$.

For the death of the double scroll, it is necessary for $W^s(H_1^+)$ to intersect $\overset{\frown}{B_1C_1}=\Phi^{-1}\pi_0\Phi(\overline{A_1B_1})$. Therefore the parameter value at which $W^s(H_1^+)$ and $\overset{\frown}{B_1C_1}$ touch each other, is an approximation of the value at which the double scroll dies. Since computation of $W^s(H_1^+)$ is difficult, we further approximate $W^s(H_1^+)$ by $\pi_1(\overline{A_{1u0}E_{1u0}})$, where

$$H_1^+=\underset{\sim}{x}_1(u_0, v_0)$$

$$A_{1u0}=u_0A_1+(1-u_0)B_1,\quad E_{1u0}=u_0E_1+(1-u_0)F_1.$$

Again, this is in an excellent agreement with the observation of the double-scroll system by the Runge-Kutta iterations.

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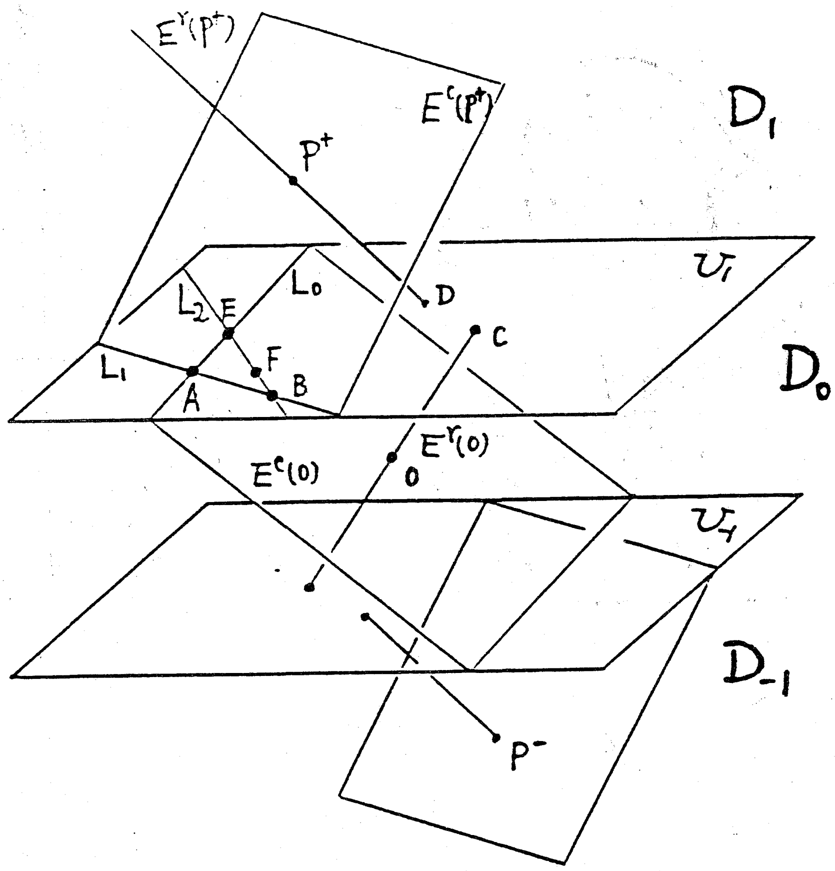
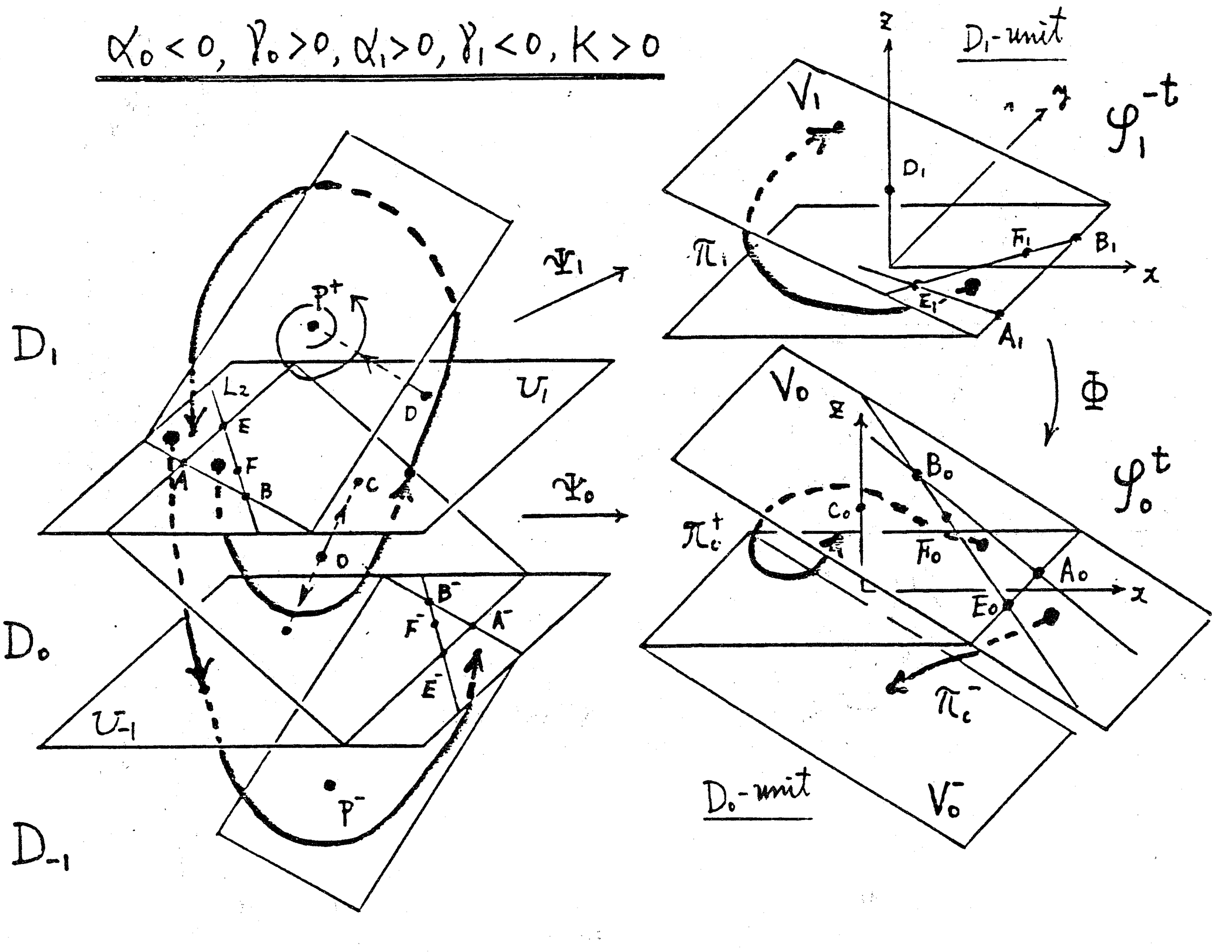


Eγ(p+)
Ec(p+)
D1
p+
v1
L0
L2
E
D
C
D0
L1
F
A
B
Eγ(0)
Ec(0)
0
v-1
D-1
p-

Fig. 1



α0 < 0, γ0 > 0, α1 > 0, γ1 < 0, K > 0
D1-unit
φ1 -t
V1
π1
D1
F1
B1
E1
A1
Ψ1
Φ
D1
p+
L2
E
F
A
B
D
C
v1
Ψ0
V0
B0
C0
F0
A0
E0
π+c
π-c
φ0 t
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v-1
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p-
D-1
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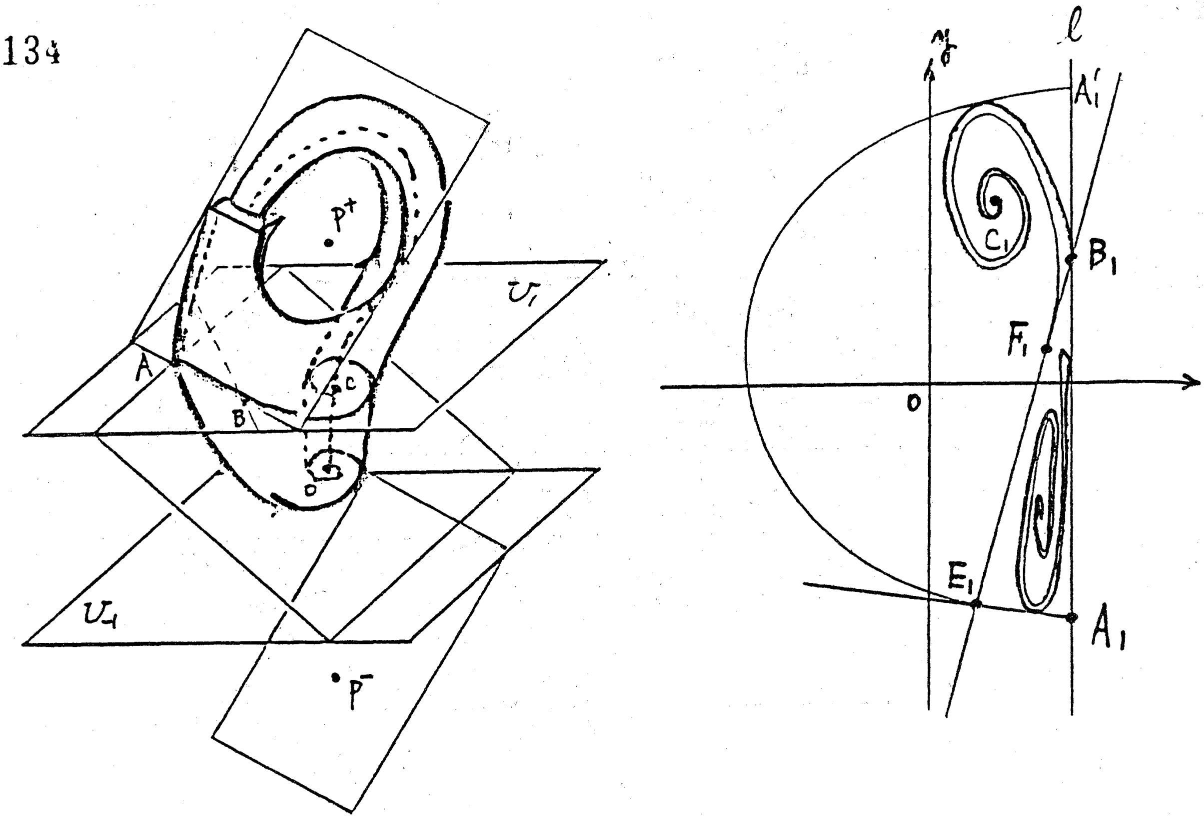
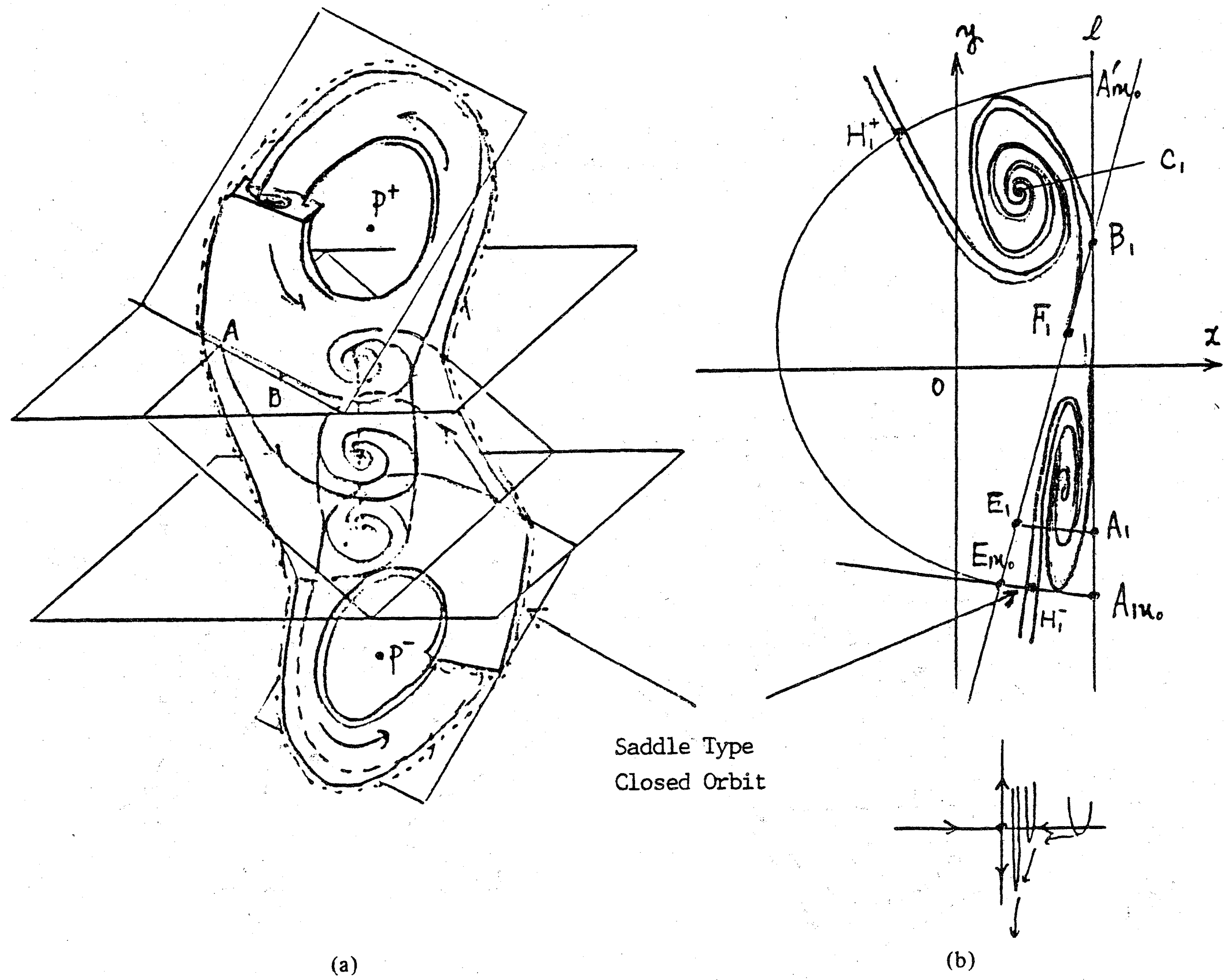
V-0

Fig. 2



(a) (b)

Fig. 3



(a) (b)

Fig. 4