POINCARÉ MAPS OF THE DOUBLE SCROLL

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ABSTRACT

A family of piecewise linear vector fields of $\mathbb{R}^3$ is discussed. A detailed analysis is given of the linearly conjugate classes and the Poincaré maps for the family. A result of the analysis is applied to the study of bifurcation of an attractor.

§ 0 Double-Scroll System

The double-scroll system is a piecewise linear ordinary differential equation on $\mathbb{R}^3$ defined by

\[
\begin{align*}
\dot{x} &= S(y - f(x)) \\
\dot{y} &= x - y + z \\
\dot{z} &= -Ty
\end{align*}
\]

\[
f(x) = \begin{cases} 
M_p(x-1) + M_0, & x \geq 1 \\
M_0 x, & |x| \leq 1 \\
M_p(x+1) + M_0, & x \leq -1
\end{cases}
\]

where $S > 0$, $T > 0$, $M_0 < 0$ and $M_p > 0$ are parameters. The double-scroll system is derived from an extremely simple autonomous electrical circuit [1]. When $(S, T, M_0, M_p) = (9, 14-2/7, -1/7, 2/7)$, a chaotic attractor, which is called the double scroll [1], is observed. In this paper, we study a large family of piecewise linear vector fields of $\mathbb{R}^3$ which contains the double-scroll system. Moreover the family contains the various system with chaotic attractors studied by several authors, including C. Sparrow [5], A. Arneodo et al. [3], R. Brockett [4] and B. Uehleke et al. [6]. A detailed analysis is given of the linearly conjugate classes and the Poincaré maps for the family. A result of the analysis is applied to the study of bifurcations of the double scroll.
§ 1 Linearly Conjugate Classes

Definition 1. Let \( \mathcal{F} \) be the set of all continuous vector fields on \( \mathbb{R}^3 \) which satisfy the following

(1) - (6): for each \( \xi \in \mathcal{F} \),

(1) is symmetric with respect to the origin, i.e.
\[ \xi(-z) = -\xi(z), \quad z \in \mathbb{R}^3 \]

(2) There are two planes \( U_1 \) and \( U_{-1} \) which are symmetric with respect to the origin, and which divide \( \mathbb{R}^3 \) into three regions \( D_1, D_0 \) and \( D_{-1} \).

(3) On each region \( D_i \) (\( i = 0 \pm 1 \)), the vector field \( \xi|_{D_i} \) is linear.

(4) An equilibrium point \( 0 \) (resp. \( P^{+} \)) is in the interior of \( D_0 \) (resp. \( D_{1} \)).

(5) Eigenvalues of \( \xi|_{D_0} \) (resp. \( \xi|_{D_{1}+} \)) are a complex conjugate pair \( \tilde{a}_0 \pm \sqrt{-1} \tilde{p}_0, \tilde{p}_0 > 0 \) (resp. \( \tilde{a}_1 \pm \sqrt{-1} \tilde{p}_1, \tilde{p}_1 > 0 \)) and a real \( \tilde{r}_0 \neq 0 \) (resp. \( \tilde{r}_1 \neq 0 \)).

(6) Each eigenspace is not parallel to \( U_{1} \).

Definition 2. For each \( \xi \in \mathcal{F} \), define (see Fig. 1)

\[ E^c(0) = \text{the eigenspace corresponding to } \tilde{a}_0 \pm \sqrt{-1} \tilde{p}_0 \text{ at } 0, \]
\[ E^r(0) = \text{the eigenspace corresponding to } \tilde{r}_0 \text{ at } 0, \]
\[ E^c(P^+) = \text{the eigenspace corresponding to } \tilde{a}_1 \pm \sqrt{-1} \tilde{p}_1 \text{ at } P^+, \]
\[ E^r(P^+) = \text{the eigenspace corresponding to } \tilde{r}_1 \text{ at } P^+, \]

\[ L_0 = U_1 \cap E^c(0), \]
\[ L_1 = U_1 \cap E^c(P^+), \]
\[ L_2 = \{ z \in U_1 | \xi(z)/U_1 \} \]
\[ A = L_0 \cap L_1, \]
\[ B = L_1 \cap L_2, \]
\[ C = U_1 \cap E^r(0), \]
\[ D = U_1 \cap E^r(P^+), \]
\[ E = L_0 \cap L_2, \]
\[ F = \{ z \in L_2 | \xi(z)/L_2 \} \]

The points \( A, B, E \) and \( P^+ \) are called the fundamental points of \( \xi \).

Definition 3. Define a map \( H: \mathcal{F} \rightarrow \mathbb{R}^5 \) by
\[ H(\xi) = (a_0, r_0, a_1, r_1, \xi) \]
where
\[ a_0 = \frac{\tilde{a}_0}{\tilde{p}_0}, \quad r_0 = \frac{\tilde{r}_0}{\tilde{p}_0} \quad (\tilde{p}_0 > 0) \]
\[ a_1 = \frac{\tilde{a}_1}{\tilde{p}_1}, \quad r_1 = \frac{\tilde{r}_1}{\tilde{p}_1} \quad (\tilde{p}_1 > 0) \]
\[ \xi = -\frac{\tilde{r}_0}{\tilde{r}_1}. \]

Theorem 1

(A) For \( \xi_1, \xi_2 \in \mathcal{F} \), the following is equivalent:

(1) \( H(\xi_1) = H(\xi_2) \)

(2) \( \xi_1 \) and \( \xi_2 \) are linearly conjugate preserving time-orientation, i.e. there exist a real
$\nu > 0$ and a linear transformation
\[ G : R^3 \rightarrow R^3 \] such that \[ DG \circ \xi_1 = \nu \xi_2 \circ G \]
(B) Put \( \mathcal{R} = \{ (a_0, r_0, a_1, r_1, \kappa) \in R^5 \mid r_0 r_1 < 0, \kappa > 0 \} \), then
\[ H(\mathcal{R}) = \mathcal{R} \).

(C) For any \( \mu \in \mathcal{R} \), there exist real numbers
\[ \ell = \ell(\mu), m = m(\mu), n = n(\mu) \] such that, for any \( \xi \in H^{-1}(\mu) \),
\[ \vec{OP}^+ = \ell \vec{OA} + m \vec{OB} + n \vec{OE} - \]
where \( A, B, E \) and \( P^+ \) are the fundamental points of \( \xi \).

Remark 1. It is easy to obtain a linear conjugate class (not necessarily time-orientation preserving) from the theorem.
Indeed, define \( (a_0, r_0, a_1, r_1, \kappa) \sim (a_0', r_0', a_1', r_1', \kappa') \) by \( (a_0, r_0, a_1, r_1, \kappa, a_0', r_0', a_1', r_1', \kappa') \) or \( (-a_0', -r_0', -a_1', -r_1', \kappa') \), then \( \xi_1 \) and \( \xi_2 \) are linearly conjugate if and only if \( H(\xi_1) \sim H(\xi_2) \).

Remark 2. For \( \mu = (a_0, r_0, a_1, r_1, \kappa) \in \mathcal{R} \), a vector field \( \xi \in \mathcal{Z} \) with \( H(\xi) = \mu \) is explicitly given as follows:
\[ \xi(x, y, z) = (a_{ij})(x, y, z)^T + (b_1, b_2, b_3)^T \}

\[ a_{11} = \lambda(c_1\overline{n} + \gamma_1) \]
\[ a_{12} = \lambda c_1 \overline{m} \]
\[ a_{13} = \lambda s(c_1 \overline{l} - \gamma_1) \]
\[ a_{21} = c_0 \overline{n} \]
\[ a_{22} = c_0 \overline{m} + r_0 \]
\[ a_{23} = s(c_0 \overline{l} - \gamma_0) \]
\[ a_{31} = c_0 \overline{n} \]
\[ a_{32} = c_0 \overline{m} \]
\[ a_{33} = s c_0 \overline{l} \]
\[ b_1 = \lambda \overline{r}(c \overline{l} - \gamma_1) \]
\[ b_2 = \overline{r}(c_0 \overline{l} - \gamma_0) \]
\[ b_3 = \overline{r} c_0 \overline{l} \]

where
\[ \lambda = -r_0/r_1 \kappa \], \( c_0 = -\kappa(a_0^2 + 1)/r_0 \kappa \), \( c_1 = -(a_1^2 + 1)/r_1 \kappa \]
\[ \overline{l} = \ell/s, \quad \overline{m} = m/s, \quad \overline{n} = n/s \]
\[ \overline{s} = 1/(1 - s), \quad \overline{r} = s/2(1 - s) \]

(F) Fundamental points:
\[ A = (1, 1, 1), \quad B = (1, -(\ell + n)/m, 1), \quad E = (-(\ell + m)/n, 1, 1) \]
\[ P^\pm = (0, 0, \pm s), \quad U_{\pm 1} = [(x, y, z)| z = \pm 1]. \]
Definition 4. Let $\xi \in \mathcal{L}$ with $H(\xi) = (a_0, r_0, a_1, r_1, \kappa)$ be given. We can take two affine transformations $\psi_0: D_0 \rightarrow \mathbb{R}^3$ and $\psi_1: D_1 \rightarrow \mathbb{R}^3$ such that (see Fig. 2)

a) $\psi_0(0) = 0$
$\psi_0(U_1) = V_0 \triangle (x, y, z) | x + z = 1$
$\psi_0(U_{-1}) = V_0^{-} \triangle (x, y, z) | x + z = -1$

\[
\frac{1}{\tilde{\beta}_0} D\psi_0(\xi(\psi_0^{-1}x)) = \xi_0(x) = \begin{bmatrix}
    a_0 & -1 & 0 \\
    1 & a_0 & 0 \\
    0 & 0 & r_0
\end{bmatrix} x.
\]

b) $\psi_1(P^+) = 0$.
$\psi_1(U_1) = V_1 \triangle (x, y, z) | x + z = 1$,

\[
\frac{1}{\tilde{\beta}_1} D\psi_1(\xi(\psi_1^{-1}x)) = \xi_1(x) = \begin{bmatrix}
    a_1 & -1 & 0 \\
    1 & a_1 & 0 \\
    0 & 0 & r_1
\end{bmatrix} x.
\]

Define the connection map $\Phi: V_1 \rightarrow V_0$ by $\Phi = (\psi_0|U_1) \circ (\psi_1|U_1)^{-1}$

Let us denote

$A_i = \psi_i(A), \quad B_i = \psi_i(B), \quad E_i = \psi_i(E), \quad F_i = \psi_i(F)$

$p_i = a_i + \frac{a_i^2 + 1}{Q_i}$, $Q_i = (a_i^2 + 1)$ (i = 0, 1),

$K_0 = K$, $K_1 = 1/K$.

Then the following holds:

i) $A_0 = (1, p_0, 0)$
$B_0 = (r_0(a_0 - a_0 - p_0)/Q_0, \quad r_0(1-p_0(a_0-a_0))/Q_0, \quad 1-r_0(a_0-a_0)/Q_0)$
$E_0 = (1, a_0, 0)$
$F_0 = (r_0(a_0-a_0)/Q_0, \quad r_0(1-a_0(a_0-a_0))/Q_0, \quad (a_0^2 + 1)/Q_0)$

ii) $A_1 = (1, p_1, 0)$
$B_1 = (1, a_1, 0)$
$E_1 = (r_1(a_1-a_1-p_1)/Q_1, \quad r_1(1-p_1(a_1-a_1))/Q_1, \quad 1-r_1(a_1-a_1)/Q_1)$
$F_1 = (r_1(a_1-a_1)/Q_1, \quad r_1(1-a_1(a_1-a_1))/Q_1, \quad (a_1^2 + 1)/Q_1)$

iii) $\Phi: V_1 \rightarrow V_0$ is obtained by

\[
\begin{pmatrix}
    x_0 \\
    y_0
\end{pmatrix} = \Phi \begin{pmatrix}
    x_1 \\
    y_1
\end{pmatrix} = \begin{pmatrix}
    L_{11} & L_{12} & L_{13} \\
    L_{21} & L_{22} & L_{23}
\end{pmatrix} \begin{pmatrix}
    x_1^{-1} \\
    y_1^{-1} - p_1
\end{pmatrix} + \begin{pmatrix}
    1 \\
    p_0
\end{pmatrix},
\]

$L_{11} = -r_0(K_1+1) [Q_1 + r_1(a_1-a_1)(K_0+1)] R$
$L_{12} = r_0 r_1 (K_0+1)(K_1+1) R$
$L_{13} = -r_0(K_1+1)(a_0-r_0)(a_1-a_1+1)-r_1(K_1+1)(a_1-a_1)(a_0(a_0-r_0)+1)] R$
$L_{21} = r_1(K_0+1)[Q_0+r_0(a_0-r_0)(K+1)] R$
$L_{22} = r_1(K_0+1)(a_0^2+1)(K_1+1) Q_0 R$
$L_{23} = r_1(K_0+1)(a_0^2+1)(K_1+1) Q_0 Y_0$

where we identify $(x_i, y_i) \equiv (x_i, y_i, z_i) \in V_i$ because $z_i = 1-x_i$ holds (i = 0, 1).
§ 2 Poincaré maps

In this section, we assume that

\[ a_0 < 0, \quad r_0 > 0, \quad a_1 > 0, \quad r_1 < 0, \quad K > 0. \]

Definition 5. We induce a new coordinate system, say \((u,v)\)-coordinate, to angular region \(\angle A_i B_i E_i\) on \(V_i\). Define

\[ x_i(u, v) = \{ x_i(u, v) | (u, v) \in [0, \infty) \times [0, 1] \}, \]

\[ \triangle A_i B_i E_i = \{ x_i | (u, v) \in [0, 1] \times [0, 1] \}, \]

\((i = 0, 1)\).

Let \(\varphi_i\) be a flow of \(\xi_i\) \((i = 0, 1)\). (See Fig. 2.)

(a) A return map for \(\varphi_0^{-t}\) is defined by

\[ \pi_0(x(u, v)) = \begin{cases} \varphi_0^T(x), & x \in \Delta A_0 B_0 E_0 \\ \varphi_0(x(u, v)), & x \in \angle A_0 B_0 E_0 \setminus \Delta A_0 B_0 E_0 \end{cases} \]

for \(t > 0\) \(\partial u/\partial t > 0\) on \([0, \infty) \times (0, t)\).

(b) A return map for \(\varphi_0^{-t}\) is defined by

\[ \pi_0(x(u, v)) = \begin{cases} \varphi_0^T(x), & x \in \Delta A_0 B_0 E_0 \\ \varphi_0(x(u, v)), & x \in \angle A_0 B_0 E_0 \setminus \Delta A_0 B_0 E_0 \end{cases} \]

for \(t > 0\) \(\partial u/\partial t < 0\) on \((0, 1) \times (0, t)\).

(c) We can identify a point of \(V_i\) with a complex number:

\[ (x_i, y_i, z_i) \equiv (x_i, y_i, z_i) = (x_i, y_i) \in \mathbb{C}. \]

Then the return maps are represented as follows:

Theorem 2. Put \(A_0u = x_0(1, v), \quad B_0v = x_0(0, v), \quad A_1u = x_1(u, 1), \quad E_1u = x_1(u, 0)\) and \(h = (1, 0, 1)\). We consider that \(x_i(u, v)\) is a complex number \((i = 0, 1)\) except the points \(A_0u, B_0v, A_1u\) and \(E_1u\), which are considered vectors in \(\mathbb{R}^3\). The usual inner product in \(\mathbb{R}^3\) is denoted by \(\langle, \rangle\).

(a) \(\pi_0^+(x_0(u, v)) = x_0(u, v) \exp[(a_0 + \sqrt{-1})t]\).

where \(u = u(v, t) = \{ (\varphi_0^t(B_0v), h) - 1/\langle\varphi_0^t(B_0v - A_0u), h\rangle \}_{t > 0} \) \(\partial u/\partial t > 0\) on \((v) \times (0, t)\).

(b) \(\pi_0^-(x_0(u, v)) = x_0(u, v) \exp[(a_0 + \sqrt{-1})t]\).

where \(u = u(v, t) = \{ (\varphi_0^t(B_0v), h) + 1/\langle\varphi_0^t(B_0v - A_0u), h\rangle \}_{t > 0} \) \(\partial u/\partial t < 0\) on \((v) \times (0, t)\).

(c) \(\pi_1(x_1(u, v)) = x_1(u, v) \exp[-(a_1 + \sqrt{-1})t]\).

where \(v = v(u, t) = \{ (\varphi_1^t(E_1u), h) - 1/\langle\varphi_1^t(E_1u - A_1u), h\rangle \}_{t > 0} \) \(\partial v/\partial t > 0\) on \((u) \times (0, t)\).
§ 3 Birth and Death of the Double Scroll

(1) Birth of the double scroll

Observations of the double scroll bifurcations [2] indicate that the double scroll is born out of a collision of a pair of Rössler's screw type attractors. We call such a phenomenon the birth of the double scroll.

Now we assume that \( \pi_1(\overline{A_1E_1}) \) and \( \ell = \{ (x, y): x = 1 \} \) have a point of intersection, say \( A_1' \), as in Fig. 3(b). Then, in order for a pair of screw type attractors to collide with each other, it is necessary for the arc \( \overline{E_1A_1'} = \pi_1(\overline{A_1E_1}) \) to intersect the spiral \( \overline{B_1C_1} = \Phi^{-1}\pi_0\Phi(\overline{A_1B_1}) \). Therefore, the parameter value at which \( \overline{E_1A_1'} \) and \( \overline{B_1C_1} \) touch each other, is an approximation of the value at which the double scroll is born. This approximation turned out to be in an excellent agreement with the observations of the double-scroll system using Runge-Kutta iterations.

Remark Note that an intersection of a screw type attractor and \( U_1 \) must be between the spirals \( \overline{BC} = \Psi_1^{-1}(\overline{B_1C_1}) \) and \( \overline{FC} = \Psi_1^{-1}(\overline{F_1C_1}) \), except for a part included in \( \angle ABE \). Therefore the parameter value at which \( \overline{E_1A_1'} \) and \( \overline{B_1C_1} \) touch each other is before the birth of the double scroll, while the parameter value at which \( \overline{E_1A_1'} \) and \( \overline{F_1C_1} \) touch each other is after the birth of the double scroll.

(2) Death of the double scroll

It is known that there is a saddle type closed orbit around the double scroll [1]. In the double-scroll system, for instance, when \( M_0, M_p \) and \( T \) are fixed and \( S \) is increased, the distance between the attractor and the saddle type closed orbit decreases, and they touch each other, finally the attractor disappears [2]. We call such a phenomenon the death of the double scroll.

Let \( H^+ \) and \( H^- \) be the points of intersection of the saddle type closed orbit \( \Gamma \) and the plane \( U_1 \), where \( H^- \) is the point which belongs to \( \angle ABE \). Put \( H_1^- = \Psi_1(H^-) \) and \( H_1^+ = \Psi_1(H^+) \). Then \( H_1^+ = \pi_1(H_1^-) = \Phi^{-1}\pi_0\Phi(H_1^-) \). Define \( \pi = \pi_1^{-1}\Phi^{-1}\pi_0\Phi \) and \( W^+(H_1^-) = \{ x \in \angle A_1B_1E_1 | \pi(x) \rightarrow H_1^- (n \rightarrow \infty) \} \), \( W^+(H_1^+) = \pi_1(W^+(H_1^-)) \).

For the death of the double scroll, it is necessary for \( W^+(H_1^+) \) to intersect \( \overline{B_1C_1} = \Phi^{-1}\pi_0\Phi(\overline{A_1B_1}) \). Therefore the parameter value at which \( W^+(H_1^+) \) and \( \overline{B_1C_1} \) touch each other, is an approximation of the value at which the double scroll dies. Since computation of \( W^+(H_1^+) \) is difficult, we further approximate \( W^+(H_1^+) \) by \( \pi_1(\overline{A_{1uo}E_{1uo}}) \), where

\[
H_1^+ = \Delta(x_0, y_0) \\
A_{1uo} = u_0 A_1 + (1-u_0) B_1, \quad E_{1uo} = u_0 E_1 + (1-u_0) F_1.
\]

Again, this is in an excellent agreement with the observation of the double-scroll system by the Runge-Kutta iterations.
References


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$\alpha_0 < 0, \gamma_0 > 0, \alpha_1 > 0, \gamma_1 < 0, K > 0$

Fig. 1

Fig. 2
Fig. 3

(a) Saddle Type
Closed Orbit

Fig. 4