数理解析研究所講究録
574巻 1985年 114-125

ON PERSISTENT HOMEOMORPHISMS

K. Sakai 都立大・理・酒井一博

Department of Mathematics
Tokyo Metropolitan University
Tokyo, Japan

and

H. Kobayashi 新潟大・理・小林久永

Department of Mathematics
Niigata University
Niigata, Japan

Abstruct

In this note we prove that a solenoidal group automorphism is persistent if and only if topologically stable.

§ 0. Introduction.

In [3] Lewowicz introduced the notion of persistency for a homeomorphism of a compact connected Riemannian manifold. Then he showed that every pseudo-Anosov map is persistent and by using this notion, that is structurally stable under some conditions.

In this note we define as in [3] a persistency for a homeomorphism of a compact metric space, and study a topological property of a persistent homeomorphism.

The following is proved.

Theorem. Let X be a solenoidal group, and let $\sigma : X \to X$ be a group automorphism. Then the following (1) and (2) are equivalent;

(1) (X, σ) is persistent,

(2) (X, σ) is topologically stable.

In [1] Aoki proved that (X, σ) is topologically stable if and only if (X, σ) has the pseudo-orbit tracing property. Further, there exist solenoidal automorphisms with the pseudo-orbit tracing property such that one of the following conditions holds:

(a) (X, σ) is not expansive,

(b) (X, σ) is not densely periodic.

Since every finite-dimensional torus is a solenoidal group, we have the following corollary.

<u>Corollary</u>. Let T^r be the r-dimensional torus, and let σ be a group automorphism of T^r. Then the following conditions are mutually equivalent;

(i) σ is persistent,

(ii) σ is topologically stable,

(iii) σ has the pseudo-orbit tracing property,

(iv) σ is expansive,

(v) σ is hyperbolic,

(vi) σ is structurally stable.

The statement is true for a group automorphism of $\mathbb{R}^r$, where $\mathbb{R}^r$ is the r-dimensional vector space (cf. [4]).

§ 1. <u>Definitions and Examples</u>.

Let $f : X \to X$ be a homeomorphism of a compact metric space (X, d). We denote by $\mathscr{H}(X)$ the set of all homeomorphisms of X with metric $d(f, g) = \max\{d(f(x), g(x)) : x \in X\}$ $(f, g \in \mathscr{H}(X))$. We say that an f-invariant subset $K \subset X$ is <u>persistent</u> if for each $\varepsilon > 0$ there is $\delta > 0$ with the property that for every $g \in \mathscr{H}(X)$ with $d(f, g) < \delta$ and for every $x \in K$, there is $y \in X$ such that $d(f^n(x), g^n(y)) < \varepsilon$ for every $n \in \mathbb{Z}$. When $K = X$ we say that f is <u>persistent</u>. We remark that this notion is independent of the metric for X. We call f to be <u>topologically stable</u> if for each $\varepsilon > 0$ there is $\delta > 0$ with the property that for every $g \in \mathscr{H}(X)$ with $d(f, g) < \delta$ there is a continuous map $h : X \to X$ such that $f \circ h = h \circ g$ and $d(h, id) < \varepsilon$. If X is a compact manifold and $\varepsilon > 0$ is small enough, then $d(h, id) < \varepsilon$ implies that h maps X onto itself. Therefore it is easy to see that every topologically stable homeomorphism of a compact manifold is persistent. In general case there is an example that is not true.

<u>Example 1</u>. The finite set $X_i = \{0, 1\}$ is fixed with the discrete topology for $i \in \mathbb{Z}$. Consider $X = \Pi_{i=-\infty}^{\infty} X_i$, equipped with the product topology, and the shift homeomorphism $\sigma : X \to X$ defined by $(\sigma(x))_j = x_{j+1}$ for all $j \in \mathbb{Z}$. Let d be the metric on X defined by $d(x, y) = 2^{-n}$ if n is the largest natural number with $x_j = y_j$ for all $|j| < n$, and $d(x, y) = 1$ if $x_0 \neq y_0$. It is well known that σ is topologically stable. Now we show that σ is not persistent. Put $\varepsilon = 1/4$ and fix any $\delta > 0$. Then there is $n > 0$ such that $1/2^n < \delta$. Define $g \in \mathscr{H}(X)$ by $(g(x))_j = x_j$ if $j < -n$

or $j > n$, $(g(x))_j = x_{j+1}$ if $-n \leqq j < n$, and $(g(x))_n = x_{-n}$. Obviously, $d(g, \sigma) < \delta$ and $g^{2n+1}(y) = y$ for all $y \in X$. Consider

$$x' = (\cdots, 0, 0, 1, 0, 0, 1, 0, \overset{\downarrow}{1}, 0, 1, 0, 0, 1, 0, 0, \cdots) \in X.$$

Then for all $y \in X$ with $d(x', y) < \varepsilon$, it is easy to see that $d(\sigma^{2n+1}(x'), g^{2n+1}(y)) \geqq \varepsilon$. Therefore σ is not persistent.

Let (X, d) and f be as above. Given $\delta > 0$, a sequence $\{x_j\}_{j=a}^{b}$ $(-\infty \leqq a < b \leqq \infty)$ is called a δ-pseudo-orbit of f if $d(f(x_j), x_{j+1}) < \delta$ for $a \leqq j \leqq b-1$. Given $\varepsilon > 0$, a sequence $\{x_j\}_{j=a}^{b}$ is said to be ε-traced by a point y in X if $d(f^j(y), x_j) < \varepsilon$ for $a \leqq j \leqq b$. We say that f has the pseudo-orbit tracing property (POTP) if for each $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo-orbit of f can be ε-traced by some point in X.

We say that X is solenoidal if X is a compact connected finite-dimensional metric abelian group.

Finally, we give two examples of persistent homeomorphisms of compact totally disconnected metric spaces.

Example 2. Let X be the Cantor set in $[0, 1]$: i. e. X is the set of the numbers $x \in [0, 1]$ with $x = 3^{-1}a_1 + 3^{-2}a_2 + \cdots$ ($a_i = 0$ or 2 for $i \geqq 1$). For $r \geqq 1$, we call the set $X \cap [3^{-r}i, 3^{-r}(i+1)]$ $(0 \leqq i \leqq 3^r - 1)$ a Cantor subinterval with rank r if $X \cap (3^{-r}i, 3^{-r}(i+1)) \neq \phi$ (see [5]). We denote by $I(i, r)$ $(i = 1, 2, 3, \cdots, 2^r)$, the i-th Cantor subinterval with rank r from the left. We show that if $f \in \mathscr{H}(X)$ is an isometry, then f is persistent. To do this, for any $\varepsilon > 0$, fix $r > 0$ with $3^{-r} < \varepsilon$. Choose $0 < \delta < 3^{-r}$ such that if $d(f, g) < \delta$ $(g \in \mathscr{H}(X))$, then $d(f^{-1}, g^{-1}) < 3^{-r}$. For every $x \in X$ and every $j \in \mathbb{Z}$, define $i_j \in \{1, 2, 3, \cdots, 2^r\}$ by $f^j(x) \in I(i_j, r)$. Obviously, $g(x) \in I(i_1, r)$. Since f is an isometry, $d(f^2(x), fg(x)) < 3^{-r}$ and so $fg(x) \in I(i_2, r)$. On the other hand, we have that $d(fg(x), g^2(x)) < 3^{-r}$ (since $d(f, g) < \delta$), and so $g^2(x) \in I(i_2, r)$: i. e. $d(f^2(x), g^2(x)) < 3^{-r} < \varepsilon$. Continuing in this fashion, we can see that $d(f^n(x), g^n(x)) < \varepsilon$ for all $n \geq 0$. A similar way shows that $d(f^n(x), g^n(x)) < \varepsilon$ for all $n \leq 0$. Thus f is persistent.

Example 3. Let (X, d) be a compact totally disconnected metric group, and let $\sigma : X \to X$ be a group automorphism. The group operation is written by multiplicative form. We show that if (X, σ) has zero-topological entropy, then (X, σ) is persistent. It is known that every group automorphism of X has the POTP (see Application 2 of [2]). Since (X, σ) has zero-topological entropy, X contains a sequence $X = X_0 \supset X_1 \supset X_2 \supset \cdots$ of completely σ-invariant normal subgroups such that $\bigcap X_n$ is trivial and for every $n \geq 0$, X/X_k is a finite group (cf. Lemma 14 of [2]). For each $\varepsilon > 0$, there is $k > 0$ such that $\mathrm{diam}(X_k) < \varepsilon/2$. Since X/X_k is finite, there is an integer $\ell_k > 0$ such that $X = \bigcup_{i=1}^{\ell_k} h_i X_k$ $(h_i \in X)$ and $h_i X_k \cap h_j X_k = \phi$ for $1 \leq i \neq j \leq \ell_k$. Thus we have that $d(h_i X_k, h_j X_k) = \inf\{d(a, b) : a \in h_i X_k, b \in h_j X_k\} > 0$ if $1 \leq i \neq j \leq \ell_k$ (since each $h_i X_k$ is open and closed in X). Let us put $\delta_k = \min\{\varepsilon/2, \min\{d(h_i X_k, h_j X_k) : 1 \leq i \neq j \leq \ell_k\}\}$. Choose $\delta = \delta(\delta_k) > 0$

as in the definition of the POTP of σ and fix $f \in \mathscr{H}(X)$ with $d(\sigma, f) < \delta$. Then for every $x \in X$, $\{f^n(x)\}_{n=-\infty}^{\infty}$ is a δ-pseudo-orbit of σ. Since σ has the POTP, there is a point $y \in X$ such that $d(\sigma^n(y), f^n(x)) < \delta_k$ for $n \in \mathbb{Z}$. Putting $n = 0$ gives $d(x, y) < \delta_k$ and so $xy^{-1} \in X_k$ (the metric d is translation invariant). Hence, we get that $d(\sigma^n(x), \sigma^n(y)) < \varepsilon/2$ for $n \in \mathbb{Z}$ since $\sigma(X_k) = X_k$. Therefore we have that

$$d(f^n(x), \sigma^n(x)) \leq d(f^n(x), \sigma^n(y)) + d(\sigma^n(y), \sigma^n(x)) < \varepsilon$$

for all $n \in \mathbb{Z}$, and so $\sigma : X \to X$ is persistent.

§ 2. Proof of Theorem.

Hereafter X is an r-dimensional solenoidal group with the invariant metric d and σ is a group automorphism of X. We write the group operation by additive form. First of all we prepare lemmas that we need. The following lemmas 1 and 2 are known (see § 1, [1]).

Lemma 1. There exist the r-dimensional vector space $\mathbb{R}^r$, a group automorphism $\gamma : \mathbb{R}^r \to \mathbb{R}^r$, a group homomorphism $\psi : \mathbb{R}^r \to X$ and a totally disconnected subgroup of X such that

(i) $\psi \circ \gamma = \sigma \circ \psi$,

(ii) $X = \psi(\mathbb{R}^r) + F$ and $\overline{\psi(\mathbb{R}^r)} = X$,

(iii) $\psi^{-1}\{\psi(\mathbb{R}^r) \cap F\} = \mathbb{Z}^r$,

(iv) there is a closed neighbourhood U of 0 in $\mathbb{R}^r$ so that $\psi : U \to X$ is an into homeomorphism, $\psi(U) \cap F = \{0\}$ and $\psi(U) + F$ is a closed neighbourhood of 0 in X (we shall write $\psi(U) \oplus F$ such a neighbourhood $\psi(U) + F$).

We call $(\mathbb{R}^r, \gamma)$ the <u>lifting system</u> of (X, σ).

<u>Lemma 2</u>. Let F be as in Lemma 1. Then F contains subgroups F^+, F^- and H such that

(i) $\sigma(H) = H$,

(ii) $F^+ \supset \sigma F^+ \supset \cdots \supset \bigcap_{n=0}^{\infty} \sigma^n(F^+) = \{0\}$,

(iii) $F^- \supset \sigma^{-1} F^- \supset \cdots \supset \bigcap_{n=0}^{\infty} \sigma^{-n}(F^-) = \{0\}$,

(iv) $\sigma F^- / F^-$ and $F^+ / \sigma F^+$ are finite,

(v) $F = F^- \oplus F^+ \oplus H$.

The following lemma is well known.

<u>Lemma 3</u>. Let $h : \mathbb{R}^r \to \mathbb{R}^r$ be a continuous map, and let $\varepsilon > 0$ be any real number. If $\|h(v) - v\|_{\mathbb{R}^r} < \varepsilon$ for all $v \in \mathbb{R}^r$, then h is a surjection. Here $\|\cdot\|_{\mathbb{R}^r}$ denotes a usual norm of $\mathbb{R}^r$.

<u>Proof</u>. Assuming that $\mathbb{R}^r \setminus h(\mathbb{R}^r) \neq \phi$, we derive a contradiction. If we take $u \in \mathbb{R}^r \setminus h(\mathbb{R}^r)$, then $u \notin h(\mathbb{R}^r)$. Hence we may assume that $0 \notin h(\mathbb{R}^r)$. For, put $h'(v) = h(v+u) - u$ for $v \in \mathbb{R}^r$. Then $h' : \mathbb{R}^r \to \mathbb{R}^r$ is a continuous map such that $0 \notin h'(\mathbb{R}^r)$ and $\|h'(v) - v\|_{\mathbb{R}^r} < \varepsilon$ for $v \in \mathbb{R}^r$. Let $H_t(v) = (1-t)v + th(v)$ for $0 \leqq t \leqq 1$ and $v \in \mathbb{R}^r$. Then $H_t : \mathbb{R}^r \to \mathbb{R}^r$ is a homotopy from h to $id_{\mathbb{R}^r}$. Define

$$F_t^{(m)}(v) = H_t(mv) / \|H_t(mv)\|_{\mathbb{R}^r}$$

for $m > 0$, $0 \leqq t \leqq 1$ and $v \in \mathbb{R}^r$ with $H_t(mv) \neq 0$, then for a sufficiently large $m' > 0$, $F_t^{(m')} : S^{r-1} \to S^{r-1}$ $(0 \leqq t \leqq 1)$ is a homotopy from $F_1^{(m')}$ to $id_{S^{r-1}}$ (since $\|h(v) - v\|_{\mathbb{R}^r} < \varepsilon$ for $v \in \mathbb{R}^r$). Since degree is homotopy invariant, we have that $\deg(F_1^{(m')}) = 1$. On the other hand, since $h(0) \neq 0$, if we choose $m'' > 0$ small enough, then $F_1^{(m'')}(S^{r-1}) \subsetneq S^{r-1}$ and so $\deg(F_1^{(m'')}) = 0$. This is contradictory to the fact that $F_1^{(m')}$ is homotopic to $F_1^{(m'')}$.

Now we give a proof of Theorem. It was showed in [1] that (X, σ) is topologically stable if and only if the lifting system $(\mathbb{R}^r, \gamma)$ of (X, σ) is hyperbolic (see Theorems 1 and 2 of [1]). Hence, to see that (1) → (2), assuming that $(\mathbb{R}^r, \gamma)$ is not hyperbolic, we prove that (X, σ) is not persistent.

As usual $\mathbb{R}^r = E^s \oplus E^c \oplus E^u$ where E^s, E^c and E^u are the subspaces corresponding to the eigenvalues of γ with modulus less than one, equal to one and greater than one respectively. Let $|\cdot|_s$ and $|\cdot|_u$ be some norms on E^s and E^u respectively. Since $E^c \neq \{0\}$, by using Jordan's normal form in the real field for (E^c, γ), we get a finite direct sum $E^c = E^{c0} \oplus \cdots \oplus E^{ck}$ of the subspaces E^{ci} satisfying the following conditions; for $0 \leqq i \leqq k$, the dimension of E^{ci} is 1 or 2, and

$$\gamma_{E^c} = \begin{pmatrix} \gamma_0 & I_1 & & 0 \\ & \gamma_1 & \ddots & \\ & & \ddots & I_k \\ 0 & & & \gamma_k \end{pmatrix}$$

where $\gamma_i : E^{ci} \to E^{ci}$ is an isometry under some norm $|\cdot|_{c_i}$ of E^{ci}

and each $I_i : E^{c_i} \to E^{c_{i-1}}$ is either a zero map or a map corresponding to the identity matrix. Define a norm $|\cdot|_c$ of E^c by

$$|v|_c = \max\{|v^i|_{c_i} : 0 \leq i \leq k\} \quad (v = v^0 + \cdots + v^k \in \oplus_{i=0}^{k} E^{c_i}).$$

Clearly

$$\|v\| = \max\{|v^s|_s, |v^c|_c, |v^u|_u\} \quad (v = v^s + v^c + v^u \in \mathbb{R}^r)$$

is equivalent to the usual norm of $\mathbb{R}^r$. If $B(\alpha) = \{v \in \mathbb{R}^r : \|v\| \leq \alpha\}$ for $\alpha > 0$, then there is $\alpha_1 > 0$ such that $\psi(B(\alpha_1)) \oplus F$ is a closed neighbourhood of 0 in X (by Lemma 1 (iv)). For $x = x_1 + x_2$ with $x_1 \in \psi(B(\alpha_1))$ and $x_2 \in F$, put

$$\rho(x) = \max\{\alpha_1, \max\{\|\psi^{-1}(x_1)\|, d(x_2, 0)\}\}$$

and define a metric d_1 for X by

$$d_1(x, y) = \begin{cases} \rho(x, y) & \text{if } x - y \in \psi(B(\alpha_1)) \oplus F \\ \alpha_1 & \text{otherwise.} \end{cases}$$

The metric d_1 is compatible with the original topology of X and in particular $d_1(\psi(v), 0) = \|v\|$ for $v \in B(\alpha_1)$. For $\alpha \in (0, \alpha_1)$, we define $F(\alpha) = \{x \in F : d_1(x, 0) \leq \alpha\}$. Since

$$F' = \bigcap_{n=-1}^{1} \sigma^n(F^+) \oplus \bigcap_{n=-1}^{1} \sigma^n(F^-) \oplus H$$

is an open subgroup of F (by Lemma 2), there is $\beta > 0$ $(\beta < \alpha_1/2)$ such that $F(\beta) \subset F'$. Here we may assume that the number β is chosen so that $B(\beta) \subset \bigcap_{n=-1}^{1} \gamma^n(B(\alpha_1))$. Put $E = E^{c_0}$ and $E' = E^{c_1} \oplus \cdots \oplus E^{c_k} \oplus E^s \oplus E^u$. For any $v \in \mathbb{R}^r = E \oplus E'$, let $v = (v_1, v_2,$ •

$\cdots, v_r)$ be the representation by components with respect to the fundamental vector of $\mathbb{R}^r = E \oplus E'$. Put $\varepsilon = \beta/8$ and fix any $\delta > 0$ $(\delta < \varepsilon)$. Let $\phi : \mathbb{R}^r \to \mathbb{R}^r$ be the time-one map for the vector field (*) given by

$$(*) \qquad \dot{v}_i = \delta' \chi(v_1) \cdots \chi(v_r) v_i \quad \text{for} \quad 1 \leqq i \leqq r,$$

where $\chi : \mathbb{R} \to \mathbb{R}$ is a function of class C^∞ such that $0 < \chi(t) < 1$ $(\beta/2 < |t| < 2\beta/3)$, $\chi(t) = 1$ $(|t| < \beta/2)$ and $\chi(t) = 0$ $(2\beta/3 \leqq |t|)$, and $\delta' > 0$ is a real number chosen such that $\|\phi(v) - v\| < \delta$ for $v \in \mathbb{R}^r$. Let $\tilde{\phi}$ be a map from $\psi(B(\alpha_1)) \oplus F$ onto itself defined by

$$\tilde{\phi}(x) = \begin{cases} \psi(v) + f & \text{if} \quad f \notin F' \\ \psi(\phi(v)) + f & \text{if} \quad f \in F' \end{cases}$$

for $x = \psi(v) + f \in \psi(B(\alpha_1)) \oplus F$. We shall denote again by $\tilde{\phi}$ the extension on X as $\tilde{\phi}(x) = x$ for $x \notin \psi(B(\alpha_1)) \oplus F$. Define a map $g : X \to X$ by $g(x) = \tilde{\phi} \circ \sigma(x)$ $(x \in X)$. Obviously, $d_1(\sigma, g) < \delta$ and $g \in \mathscr{H}(X)$. Consider $x' = \psi(u)$ where $u = (\beta/4, 0, 0, \cdots, 0) \in E \oplus E' = \mathbb{R}^r$. Then we get

$$d_1(\sigma^n(x'), 0) = d_1(\psi(\gamma^n(u)), 0) = \|\gamma^n(u)\| = \beta/4$$

for all $n \geqq 0$. For any

$$y \in W_\varepsilon(x') = \{z \in X : d_1(z, x') \leqq \varepsilon\} = \psi(B(\varepsilon)) \oplus F(\varepsilon) + x',$$

there are $w \in B(\varepsilon)$ and $f \in F(\varepsilon)$ such that $y = \psi(w + u) + f$. It is clear that $\beta/8 < \|\pi_E(w + u)\| < 3\beta/8$, where $\pi_E : \mathbb{R}^r \to E$ denotes a projection along complementary subspace E'. Hence there is the smallest integer $n_0 \geqq 0$ such that $3\beta/8 < \|(\phi\gamma)^{n_0}(w + u)\| < \alpha_1$ or $d_1(\sigma^{n_0}(f), 0) > 3\beta/8$ $(\sigma^{n_0}(f) \in F)$ holds. Since $\psi_{B(\alpha_1)}$ is an isometry, we can easily obtain that $d_1(g^{n_0}(y), 0) > 3\beta/8$, and so $d_1(\sigma^{n_0}(x'), g^{n_0}(y)) > \beta/8 = \varepsilon$. Therefore (X, σ) is not persistent.

To see that $(2) \to (1)$, we show that if $(\mathbb{R}^r, \gamma)$ is hyperbolic, then (X, σ) is topologically stable and a continuous map $h : X \to X$ is onto. To get the conclusion, it is enough to check that a continuous map h constructed in the proof of the statement $(B) \to (A)$ of [1] (see pp. 133-135 and Correction) is onto. This is sketched as follows (see [1] for details).

There is a 1-to-1 group homomorphism $\psi^* : \mathbb{R}^r/_{\mathrm{Ker}\,\psi} \to \psi(\mathbb{R}^r)$. In [1], $\mathbb{R}^r/_{\mathrm{Ker}\,\psi}$ is denoted by the symbol $V_1 \oplus V_2$. Remark that $\mathrm{Ker}\,\psi \subset \mathbb{Z}^r$ by Lemma 1 (iii). Let $\tilde{d}_0$ denote the metric induced on $V_1 \oplus V_2$ by the metric d_0 of $\mathbb{R}^r$. We note that d_0 is equivalent to the Euclidean metric on $\mathbb{R}^r$ (see [1, p. 123]). Let $\tilde{\gamma} : V_1 \oplus V_2 \to V_1 \oplus V_2$ denote the map induced by γ. Obviously, $\psi^* \circ \tilde{\gamma} = \sigma \circ \psi^*$. Since γ is hyperbolic, $\tilde{\gamma}$ is topologically stable (see [1, pp. 131-132] or [4]). For any $\varepsilon > 0$ (very small), let $\delta > 0$ be the number with the property of topological stability. Take and fix any $f \in \mathscr{H}(X)$ with $d_1(f, \sigma) < \delta$. Then there is a sequence $\{f_n\}_{n=0}^{\infty} \subset \mathscr{H}(X)$ such that $f_n(\psi(\mathbb{R}^r)) = \psi(\mathbb{R}^r)$ $(n \geqq 0)$, $d_1(f_n, \sigma) < \delta$ for n large enough and $f_n \to f$ $(n \to \infty)$. Fix an integer n such that $d_1(f_n, \sigma) < \delta$, and put $\tilde{f}_n(v) = \psi^{*-1} \circ f_n \circ \psi^*(v)$ for $v \in V_1 \oplus V_2$. Then $\tilde{f}_n : V_1 \oplus V_2 \to V_1 \oplus V_2$ is a homeomorphism and $\tilde{d}_0(\tilde{f}_n(v), \tilde{\gamma}(v)) < \delta$ for $v \in V_1 \oplus V_2$. So there is a continuous map $\tilde{h}_n : V_1 \oplus V_2 \to V_1 \oplus V_2$ such that $\tilde{h}_n \circ \tilde{f}_n = \tilde{\gamma} \circ \tilde{h}_n$ and $\tilde{d}_0(\tilde{h}_n(v), v) < \varepsilon$ $(v \in V_1 \oplus V_2)$. Since the natural projection $p : \mathbb{R}^r \to V_1 \oplus V_2$ is a covering projection, there is a lifting $\bar{h}_n : \mathbb{R}^r \to \mathbb{R}^r$ of $\tilde{h}_n$ such that $d_0(\bar{h}_n(v), v) < \varepsilon$ for $v \in \mathbb{R}^r$. Hence by Lemma 3, $\bar{h}_n$ maps $\mathbb{R}^r$ onto itself, and so $\tilde{h}_n(V_1 \oplus V_2) = V_1 \oplus V_2$ (since $\tilde{h}_n \circ p = p \circ \bar{h}_n$). Put $h_n = \psi^* \circ \tilde{h}_n \circ \psi^{*-1}$. Then for an arbitrarily large n, we get that $h_n \circ f_n = \sigma \circ h_n$ on $\psi(\mathbb{R}^r)$, $d_1(h_n(x), x) < \varepsilon$ $(x \in \psi(\mathbb{R}^r))$, $h_n(\psi(\mathbb{R}^r)) = \psi(\mathbb{R}^r)$, and h_n is uniformly continuous (see [1, Correction]). Thus, h_n is extended to a surjective continuous map of X since $\overline{\psi(\mathbb{R}^r)} = X$ by Lemma 1 (ii). We shall denote it by the same symbol. Since $\{h_n\}$ converges uniformly to some continuous map h of X (see [1, Correction]), it follows that $h \circ f = \sigma \circ h$ on X, $d_1(h, id) \leqq \varepsilon$ and $h(X) = X$. The proof is complete.

Acknowledgement.

The authors are pleased to acknowledge many useful and pleasant conversations with Dr. M. Komuro.

References

[1]. N. Aoki, Topological stability of solenoidal automorphisms, Nagoya Math. J. 90 (1983), 119-135 (Correction to : Topological stability of solenoidal automorphisms, ibid., 95 (1984)).

[2]. N. Aoki, The splitting of zero-dimensional automorphisms and applications. (To appear in Collq. Math.)

[3]. J. Lewowicz, Persistence in expansive systems, Ergod. Th. & Dynam. Sys. 3 (1983), 567-578.

[4]. A. Morimoto, The method of the pseudo-orbit tracing property and stability, Tokyo Univ. Seminary notes 39, 1979 (in Japanese).

[5]. M. Sears, Expansive self-homeomorphisms of the Cantor set, Math. Sys. Th. 6 (1972), 129-132.