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$L^2$-Solutions for Nonlinear Schrödinger Equations and Nonlinear Groups

Author(s)

Tsutsumi, Yoshio

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\( L^2 \)-Solutions for Nonlinear Schrödinger Equations and Nonlinear Groups

(Yoshio Tsutsumi)

Faculty of Integrated Arts and Sciences, Hiroshima University, Higashisenda-machi, Naka-ku, Hiroshima 730, Japan

§1. Introduction and main results.

We consider the unique global existence of solutions in a weaker class than the energy space, i.e., \( H^1(\mathbb{R}^n) \) for the Cauchy problem of the nonlinear Schrödinger equation:

\[
(1.1) \quad i \frac{\partial u}{\partial t} = -\Delta u + \lambda |u|^{p-1} u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\
(1.2) \quad u(t_0, x) = u_0(x), \quad x \in \mathbb{R}^n,
\]

where \( t_0 \in \mathbb{R} \) and \( \lambda \in \mathbb{R} \). By \( \alpha(n) \) we denote \( \infty \) if \( n = 1 \) or \( n = 2 \) and \( (n + 2)/(n - 2) \) if \( n \geq 3 \). There are many papers concerning the global existence of solutions for Problem (1.1)-(1.2) (see, e.g., [1]-[2], [4]-[7], [9]-[10] and [13]-[14]). In [1] Baillon, Cazenave and Figueira show that if \( 1 \leq n \leq 3, 1 < p < \alpha(n) \) and \( \lambda > 0 \), Problem (1.1)-(1.2) has a unique global strong solution \( u(t) \in C(\mathbb{R};H^2(\mathbb{R}^n)) \cap C^1(\mathbb{R};L^2(\mathbb{R}^n)) \) for any \( u_0 \in H^2(\mathbb{R}^n) \).

In [2] Ginibre and Velo show that if \( 1 < p < \alpha(n) \) and \( \lambda > 0 \) or if \( 1 < p < 1 + \frac{4}{n} \) and \( \lambda < 0 \), Problem (1.1)-(1.2) has a unique global weak solution \( u(t) \in C(\mathbb{R};H^1(\mathbb{R}^n)) \) for any \( u_0 \in H^1(\mathbb{R}^n) \).
In [6] Strauss shows that if $\lambda > 0$ and $p > 1$, Problem (1.1) - (1.2) has at least one global weak solution $u(t)$ in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n))$ for any $u_0 \in H^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$ (see also [5]). In [10] M. Tsutsumi and N. Hayashi discuss the unique global existence of classical solutions for (1.1)-(1.2) (see also Pecher and von Wahl [4]). In [9] M. Tsutsumi discusses the unique global solution in $\mathcal{D}'(\mathbb{R}^n)$ or in the weighted Sobolev space for (1.1)-(1.2). Recently in [13, 14] N. Hayashi, K. Nakamitsu and M. Tsutsumi have shown that the solution of (1.1)-(1.2) has the smoothing property in some sense. In [13] they also discuss the global existence of solutions of (1.1) - (1.2) for the initial data $u_0 \in L^2(\mathbb{R}^n)$ with $xu_0(x) \in L^2(\mathbb{R}^n)$, when $n = 1$. In almost all of previous papers the solution of (1.1)-(1.2) has been constructed in a space not larger than the energy space, that is, $H^1(\mathbb{R}^n)$, because the proofs in almost all of previous papers are based on the energy inequality.

However, in [7] Strauss constructs the wave operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for the equation (1.1) with $p = 1 + 4/n$ (see [7, Theorem 5]). His results are almost equivalent to the construction in $L^2(\mathbb{R}^n)$ of unique local solutions for (1.1) - (1.2) with $p = 1 + 4/n$. In this paper we prove that when $1 < p < 1 + 4/n$, we can construct the unique global solution of (1.1)-(1.2) for any $u_0$ in $L^2(\mathbb{R}^n)$ (but possibly not in $H^1(\mathbb{R}^n)$). Such a solution is called an "$L^2$-solution". Furthermore, we show that when $1 < p < 1 + 4/n$, the solution operator of the evolution equation (1.1) constitutes a strongly continuous
nonlinear operator group in $L^2(\mathbb{R}^n)$. Our proof is based on the $L^2$-norm conservation law and the dispersive effect of solutions (see, e.g., Lemma 2.2).

We put $U(t) = e^{i\Delta t}$ and $f(z) = \lambda |z|^{p-1}z$ ($z \in \mathbb{C}$). Our main theorem in this paper is the following.

**Theorem 1.1.** Assume that $1 < p < 1 + \frac{4}{n}$. Then, for any $u_0 \in L^2(\mathbb{R}^n)$ and any $t_0 \in \mathbb{R}$ there exists a unique global solution $u(t)$ of (1.1)-(1.2) such that

$$
(1.3) \quad u(t) \in C(\mathbb{R}; L^2(\mathbb{R}^n)) \cap L^r_{loc}(\mathbb{R}; L^{p+1}(\mathbb{R}^n)),
$$

$$
(1.4) \quad u(t) = U(t-t_0) - i \int_{t_0}^{t} U(t-\tau)f(u(\tau)) \, d\tau, \quad t \in \mathbb{R},
$$

$$
(1.5) \quad ||u(t)||_{L^2(\mathbb{R}^n)} = ||u_0||_{L^2(\mathbb{R}^n)}, \quad t \in \mathbb{R},
$$

where $r = \frac{4(p + 1)}{n(p - 1)}$ and the integral in (1.4) is the Bochner integral in $H^{-1}(\mathbb{R}^n)$. Furthermore, let $u_{0j}$, $j = 1, 2, \ldots$, and $u_0$ be such that $u_{0j}, u_0 \in L^2(\mathbb{R}^n)$ and $u_{0j} \rightarrow u_0$ in $L^2(\mathbb{R}^n)$ ($j \rightarrow \infty$). Let $u_j(t)$ and $u(t)$ be the solutions of (1.1) with $u_j(t_0) = u_{0j}$ and $u(t_0) = u_0$, respectively. Then, for each $T > 0$

$$
(1.6) \quad u_j(t) \rightarrow u(t) \text{ in } C([t_0-T, t_0+T]; L^2(\mathbb{R}^n)) \quad (j \rightarrow \infty).
$$

**Remark 1.1.** Theorem 1.1 is almost the same as Theorem 1.1 in [15] except that (1.6) is stronger than (1.6) in [15]. Theorem 1.1 implies the well-posedness in $L^2(\mathbb{R}^n)$ of the Cauchy
problem of the nonlinear Schrödinger equation (1.1) with
\[ 1 < p < 1 + \frac{4}{n}. \]

By Theorem 1.1 we can define the solution operator of the
evolution equation (1.1) as a mapping from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \),
when \( 1 < p < 1 + \frac{4}{n} \). We denote it by \( S(t) \). The following
result is an immediate consequence of Theorem 1.1.

**Corollary 1.2.** Assume that \( 1 < p < 1 + \frac{4}{n} \). Then,
\[ \{ S(t) ; - \infty < t < + \infty \} \]
is a strongly continuous nonlinear
operator group in \( L^2(\mathbb{R}^n) \). That is, \( S(t) \) is a homeomorphism
from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) for each \( t \in \mathbb{R} \), and
\[(1.7) \quad S(t + s) = S(t)S(s), \quad t, s \in \mathbb{R}, \]
\[(1.8) \quad S(0) = I, \]
\[(1.9) \quad S(h)v \rightarrow v \quad \text{in} \; L^2(\mathbb{R}^n) \; (h \rightarrow 0), \; v \in L^2(\mathbb{R}^n), \]
where \( I \) is the identity operator from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \).

Our plan in this paper is as follows. In Section 2 we
summarize several lemmas needed for the proof of Theorem 1.1.
In Section 3 we give a sketch of proof of Theorem 1.1.

We conclude this section with several notations given.
We abbreviate \( L^p(\mathbb{R}^n) \) and \( H^m(\mathbb{R}^n) \) to \( L^p \) and \( H^m \), respectively.
\((\cdot, \cdot)\) denotes the scalar product in \( L^2 \). For a closed interval
I in \( \mathbb{R} \) and a Hilbert space \( H \) we denote the set of all weakly
continuous functions from $I$ to $H$ by $C_w(I;H)$. Let $h(x)$ be an even and positive function in $C_0^\infty(\mathbb{R}^n)$ with $\|h\|_L^1 = 1$. We put $h_j(x) = j^n h(jx)$ for each positive integer $j$. $\ast$ denotes the convolution with respect to spatial variables. In the course of calculations below various constants will be simply denoted by $C$. In particular, $C = C(\ast, \cdots, \ast)$ will denote a constant depending only on the quantities appearing in parentheses.

§2. Lemmas.

In this section we summarize several results needed for the proof of Theorem 1.1.

For $U(t)$ we have the following two lemmas.

**Lemma 2.1.** Let $q$ and $r$ be positive numbers such that $1/q + 1/r = 1$ and $2 \leq q \leq \infty$. For any $t \neq 0$, $U(t)$ is a bounded operator from $L^r$ to $L^q$ satisfying

$$
(2.1) \quad \|U(t)v\|_{L^q} \leq (4\pi |t|)^{n/q - n/2} \|v\|_{L^r}, \quad v \in L^r, \quad t \neq 0,
$$

and for any $t \neq 0$, the map $t \mapsto U(t)$ is strongly continuous. For $q = 2$, $U(t)$ is unitary and strongly continuous for all $t \in \mathbb{R}$.

**Lemma 2.2.** Let $q$ and $r$ be positive numbers such that $1 \leq q - 1 < \alpha(n)$ and $(n/2 - n/q)r = 2$. Then,

$$
(2.2) \quad \|U(\cdot)v\|_{L^r(\mathbb{R};L^q)} \leq C \|v\|_{L^2},
$$

5
where $C = C(n, q)$.

Lemma 2.1 is well known (see, e.g., [2, Lemma 1.2]). For Lemma 2.2, see Strichartz [8, Corollary 1 in §3] and Ginibre and Velo [3, Proposition 7].

Furthermore, we need the following two lemmas.

**Lemma 2.3.** Let $I$ be an open interval in $\mathbb{R}$. Let $1 < q, r < \infty$ and $a, b > 0$. We put

$$
M = \{ v(t) \in L^\infty(I; L^2) \cap L^r(I; L^q); ||v||_{L^\infty(I; L^2)} \leq a, \ ||v||_{L^r(I; L^q)} \leq b \}.
$$

Then $M$ is a closed subset in $L^r(I; L^q)$.

**Lemma 2.4.** Let $T_1$ and $T_2$ be constants with $T_1 < T_2$.
Assume that $v(t) \in C([T_1, T_2]; H^{-1})$ and for some $K > 0$

$$(2.3) \quad ||v(t)||_{L^2} \leq K, \ a.e. \ t \in [T_1, T_2].$$

Then, $v(t) \in C_w([T_1, T_2]; L^2)$ and (2.3) holds for all $t \in [T_1, T_2]$.

Lemmas 2.3 and 2.4 are identical to Lemmas 2.3 and 2.4 in [15], respectively. For the proofs of Lemmas 2.3 and 2.4, see [15, §2].

We conclude this section by giving the following lemma concerning the mollifier $h_j(x)$.
Lemma 2.5. Let $I$ be a bounded closed interval in $\mathbb{R}$. Let $f(t) \in C(I; L^2)$. We put $f_j(t) = (h_j \ast f)(t)$. Then,

\begin{equation}
(2.4) \quad f_j(t) \in \bigcap_{k=1}^{\infty} C(I; H^k), \quad j = 1, 2, \cdots, 
\end{equation}

\begin{equation}
(2.5) \quad \|f_j(t)\|_{H^m} \leq C_{jm} \|f(t)\|_{L^2}, \quad t \in I, \quad j = 1, 2, \cdots,
\end{equation}

for each positive integer $m$,

\begin{equation}
(2.6) \quad f_j(t) \to f(t) \text{ in } C(I; L^2) \quad (j \to \infty),
\end{equation}

where $C_{jm} = C(j, m)$.

**Proof.** (2.4) and (2.5) are clear. We prove only (2.6).

We note that $f(t)$ is uniformly continuous on $I$. Since

\[ \|f_j(t) - f_j(s)\|_{L^2} \leq \|f(t) - f(s)\|_{L^2}, \quad t, s \in I, \]

we conclude that $f_j(t), \ j = 1, 2, \cdots,$ are equi-continuous on $I$. On the other hand, $f_j(t) \to f(t)$ in $L^2 \ (j \to \infty)$ for each $t \in I$. Therefore, we can prove (2.6) by using the same argument as in the proof of the Ascoli-Arzela theorem.

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§3. Sketch of the Proof of Theorem 1.1.

In this section we give a sketch of the proof of Theorem 1.1. By $I_t$ and $I_t$ we denote an open interval $(t_0 - t, t_0 + t)$ and a closed interval $[t_0 - t, t_0 + t]$, respectively, for $t \geq 0$. Let $r = \frac{4(p+1)}{n(p-1)}$ throughout this section.
We have the following result concerning the unique local existence of $L^2$-solutions for (1.1)-(1.2).

**Lemma 3.1.** Assume that $1 < p < 1 + \frac{4}{n}$. Then, for any $t_0 \in \mathbb{R}$ and any $\rho > 0$ there exists a $T = T(p, n, \lambda, \rho) > 0$ such that for any $u_0 \in L^2$ with $\|u_0\|_{L^2} \leq \rho$ Problem (1.1)-(1.2) has a unique local solution $u(t)$:

\[(3.1) \quad u(t) \in C([\bar{T}_T]; L^2) \cap L^r([\bar{T}_T]; L^{p+1}), \]

\[(3.2) \quad u(t) = U(t-t_0)u_0 - i \int_{t_0}^{t} U(t-\tau)f(u(\tau)) \, d\tau, \quad t \in \bar{T}_T, \]

where the integral in (3.2) is the Bochner integral in $H^{-1}$. Furthermore, the solution $u(t)$ satisfies

\[(3.3) \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \bar{T}_T. \]

**Proof.** We only give the outline of the proof of Lemma 3.1. For the details, see [15, §3].

We consider the following integral equation:

\[(3.4) \quad u_j(t) = U(t-t_0)h_j \star u_0 - i \int_{t_0}^{t} U(t-\tau)f(u_j(\tau)) \, d\tau, \quad j = 1, 2, \cdots. \]

From the result of Ginibre and Velo [2, Theorem 3.1] we already know that for each $j$ there exists a unique global solution $u_j(t)$ of (3.4) in $C(\mathbb{R}; H^1)$ such that
\[(3.5) \quad \|u_j(t)\|_{L^2} \leq \|h_j \ast u_0\|_{L^2} \preceq \|u_0\|_{L^2}, \quad t \in \mathbb{R}, \quad j = 1, 2, \ldots,\]

Let \( \rho \) be a positive constant with \( \|u_0\|_{L^2} \leq \rho \). By \( \delta \) we denote the constant appearing in \((2.2)\) with \( q = p+1 \) and \( r = \frac{4(p+1)}{n(p-1)} \).

We note that \( \delta \) depends only on \( n \) and \( p \). We put

\[(3.6) \quad M = \{ v(t) \in L^\infty(I_T;L^2) \cap L^r(I_T;L^{p+1}); \quad \|v\|_{L^\infty(I_T;L^2)} \leq \rho, \quad \|v\|_{L^r(I_T;L^{p+1})} \leq 2\delta \rho \}, \]

where \( T \) is a small positive constant to be determined later.

We note that by Lemma 2.3 \( M \) is closed in \( L^r(I_T;L^{p+1}) \).

We first show that if \( T \) is sufficiently small, then

\[(3.7) \quad u_j(t) \in M \quad \text{for all } j.\]

For \( 0 \leq s \leq T \) we take the \( L^r(I_s;L^{p+1}) \) norm of \((3.4)\) and use \((2.1), (2.2)\) and the generalized Young inequality to obtain

\[(3.8) \quad \|u_j\|_{L^r(I_s;L^{p+1})} \leq \delta \rho + C_0 T^{p/q_1} \|u_j\|_{L^r(I_s;L^{p+1})}^{p/q_1}, \quad 0 \leq s \leq T, \quad j = 1, 2, \ldots,\]

where \( q_1 = \frac{4p}{n+4-4n} \) and \( C_0 = C_0(n, p, \lambda) \). Now we choose \( T > 0 \) so small that there exists a positive number \( y \) satisfying \( C_0 T^{p/q_1} y^p + \delta \rho - y < 0 \) and \( 0 < y \leq 2\delta \rho \). For that purpose, it is sufficient to choose \( T > 0 \) so that

\[(3.9) \quad T < (2C_0(2\delta \rho)^{p-1})^{-q_1/p}.\]

Then we put
(3.10) \( y_0 = \min \{ 2\delta \rho \geq y > 0; \ C_0 T^{\frac{p}{q_1}} y^p + \delta \rho - y = 0 \} \).

If \( T \) is chosen so small that (3.9) holds, then by (3.8) and (3.10) we obtain

(3.11) \( \| u_j \|_{L^r(I_T; L^{P+1})} \leq y_0 \leq 2\delta \rho, \quad j = 1, 2, \ldots \).

(3.5) and (3.11) give us (3.7), if \( T \) is chosen so small that (3.9) holds.

We next consider the estimate of the difference between \( u_j \) and \( u_k \) for any \( j \) and \( k \) with \( j \neq k \). For \( u_j, u_k \in M \) we have

(3.12) \[ \| u_j - u_k \|_{L^r(I_T; L^{P+1})} \leq \delta K(j, k) + C_0 T^{\frac{p}{q_1}} \cdot 2(2\delta \rho)^{P-1} \| u_j - u_k \|_{L^r(I_T; L^{P+1})}, \]

where \( K(j, k) = \| h_j \ast u_0 - h_k \ast u_0 \|_{L^2} \), \( q_1 = \frac{4p}{n+4-np} \) and \( \bar{C}_0 = \bar{C}_0(n, p, \lambda) \). If we choose \( T \) so small in (3.12) that

(3.13) \[ \bar{C}_0 T^{\frac{p}{q_1}} \cdot 2(2\delta \rho)^{P-1} \leq \frac{1}{2}, \]

then we have by (3.12)

(3.14) \[ \| u_j - u_k \|_{L^r(I_T; L^{P+1})} \leq 2\delta K(j, k) \]

for all \( j \) and \( k \). Since \( K(j, k) \to 0 \) (\( j, k \to \infty \)), we obtain by (3.14)

(3.15) \[ \| u_j - u_k \|_{L^r(I_T; L^{P+1})} \to 0 \] (\( j, k \to \infty \)),

if \( T \) is chosen so small that (3.13) holds. In addition we have by (3.15)
(3.16) \[ |(u_j(t) - u_k(t), \psi) | \leq K(j, k) \| \psi \|_{L^2} \]
\[ + CT^2 \| \psi \|_{H^1} \cdot 2(2g_\rho)^{P-1} \| u_j - u_k \|_{L^p(I_T; L^{p+1})} \]
\[ \rightarrow 0 \quad (j, k \rightarrow \infty) \quad \text{uniformly on } \bar{I}_T, \]
for \( \psi \in H^1 \), where \( q_2 = \frac{4+(n+4)p-4p^2}{4(p+1)} > 0 \). (3.16) implies that \( \{u_j(t)\}_{j=1}^\infty \) is the Cauchy sequence in \( C(\bar{I}_T; H^{-1}) \).

Therefore, by (3.7), (3.15), (3.16) and Lemma 2.3 we obtain the solution \( u(t) \) of (1.1)-(1.2) such that

(3.17) \[ u(t) \in L^p(I_T; L^2) \cap L^p(I_T; L^{p+1}) \cap C(\bar{I}_T; H^{-1}), \]

(3.18) \[ u(t) = U(t-t_0)u_0 - i \int_{t_0}^t U(t-\tau) f(u(\tau)) \, d\tau, \quad t \in \bar{I}_T, \]

(3.19) \[ \| u(t) \|_{L^2} \leq \| u_0 \|_{L^2}, \quad \text{a.e. } t \in I_T, \]

(3.20) \[ u_j(t) \to u(t) \text{ in } L^p(I_T; L^{p+1}) \text{ and in } C(\bar{I}_T; H^{-1}) \quad (j \to \infty), \]

where \( T \) is a positive constant determined by (3.9) and (3.13) and the integral in (3.18) is the Bochner integral in \( H^{-1} \).

(3.17), (3.19) and Lemma 2.4 imply that

(3.21) \[ u(t) \in C_w(\bar{I}_T; L^2) \]

and that for all \( t \in \bar{I}_T \) (3.19) holds. The uniqueness of solutions satisfying (3.17-18) follows from the estimate of the type (3.14) and the standard argument.

Thus, for any \( s \in \bar{I}_T \) we can uniquely solve (1.1)-(1.2) in the time interval \([s-T, s+T]\) with the initial time \( t_0 \) and the
initial datum \( u_0 \) replaced by \( s \) and \( u(s) \), respectively, where \( T \) is the same as in the case of the initial time \( t_0 \) and the initial datum \( u_0 \). Therefore, reversing the roles of \( 0 \) and \( t \), we obtain the reverse inequality to (3.19) for all \( t \in \bar{I}_T \), which implies (3.3). (3.3) and (3.21) give us

(3.22) \( u(t) \in C(\bar{I}_T; L^2) \).

This completes the proof of Lemma 3.1.

(Q. E. D.)

We are now in a position to prove Theorem 1.1.

**Proof of Theorem 1.1.** The unique global existence of \( L^2 \)-solutions for (1.1)-(1.2) follows directly from Lemma 3.1, which shows the unique local solvability in \( L^2 \) of (1.1)-(1.2) and the a priori bound of the \( L^2 \)-norm of \( L^2 \)-solutions.

It remains only to prove the continuous dependence of \( L^2 \)-solutions on the initial data. Let \( u_{0j}, j = 1, 2, \ldots \), and \( u_0 \) be such that \( u_{0j} \), \( u_0 \in L^2 \) and \( u_{0j} \to u_0 \) in \( L^2 \) (\( j \to \infty \)). Let \( u_j(t) \) and \( u(t) \) be the global \( L^2 \)-solutions of (1.1) with \( u_j(t_0) = u_{0j} \) and \( u(t_0) = u_0 \), respectively. We put

\[ \rho = \sup \{ \| u_0 \|_{L^2}, \| u_{0j} \|_{L^2}, j = 1, 2, \ldots \} \]

For this \( \rho \), let \( T > 0 \) be defined as in (3.9) and (3.13). Then, by using the same argument as in the proof of Lemma 3.1 we have

(3.23) \( u_j(t) + u(t) \) in \( L^r(\bar{I}_T; L^{p+1}) \) (\( j \to \infty \)),

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\[ (3.24) \quad |(u_j(t) - u(t), g(t))| \leq K \sup_{t \in \overline{I}_T} \|g(t)\|_{H^1} \]
\[ \times (\|u_{0j} - u_0\|_{L^2} + \|u_j - u\|_{L^r(I_T; L^{p+1})}), \quad t \in \overline{I}_T, \quad j = 1, 2, \cdots, \]

for \( g(t) \in C(\overline{I}_T; H^1) \) (see, e.g., (3.15) and (3.16)), where \( K = K(n, p, \lambda, \rho) > 0 \). We evaluate
\[ (3.25) \quad \|u_j(t) - u(t)\|_{L^2}^2 = \langle u_j(t) - u(t), u_j(t) - u(t) \rangle \]
\[ \leq \|u_j(t)\|_{L^2}^2 - \langle u(t), u_j(t) \rangle + \langle u_j(t) - u(t), u(t) \rangle, \quad t \in \overline{I}_T, \quad j = 1, 2, \cdots. \]

We first evaluate the second term at the right hand side of (3.25). Let \( \varepsilon \) be an arbitrary positive constant. We put
\( \hat{u}_k(t) = (h_k \ast u)(t) \) for each positive integer \( k \). By Lemma 2.5 we can choose \( k \) so large that
\[ (3.26) \quad |\langle u_j(t) - u(t), u(t) - \hat{u}_k(t) \rangle| \leq 2\rho \|u(t) - \hat{u}_k(t)\|_{L^2} < \frac{1}{2} \varepsilon, \quad t \in \overline{I}_T. \]

For such a \( k \) we have by (3.23), (3.24) and Lemma 2.5
\[ (3.27) \quad |\langle u_j(t) - u(t), \hat{u}_k(t) \rangle| \leq K \sup_{t \in \overline{I}_T} \|\hat{u}_k(t)\|_{H^1} \]
\[ \times (\|u_{0j} - u_0\|_{L^2} + \|u_j - u\|_{L^r(I_T; L^{p+1})}) < \frac{1}{2} \varepsilon, \quad t \in \overline{I}_T, \]

if \( j \) is sufficiently large. Therefore, we obtain by (3.26) and (3.27)
\begin{equation}
| (u_j(t) - u(t), u(t)) | \leq | (u_j(t) - u(t), \hat{u}_k(t)) | + | (u_j(t) - u(t), u(t) - \hat{u}_k(t)) | < \frac{1}{2} \epsilon + \frac{1}{2} \epsilon = \epsilon, \quad t \in \tilde{T},
\end{equation}

for sufficiently large \( j \). (3.28) implies that

\begin{equation}
| (u_j(t) - u(t), u(t)) | \to 0 \quad (j \to \infty) \quad \text{uniformly on } \tilde{T}.
\end{equation}

We next evaluate the first term at the right hand side of (3.25). Since \( \| u_j(t) \|_{L^2} = \| u_{0j} \|_{L^2} \) and \( \| u(t) \|_{L^2} = \| u_0 \|_{L^2} \) for \( t \in \tilde{T} \), we have by (3.29)

\begin{equation}
2 \left| \| u_j(t) \|_{L^2} - (u(t), u_j(t)) \right| \leq 2 \left| \| u_{0j} \|_{L^2} - \| u_0 \|_{L^2} \right| + \left| (u(t), u_j(t) - u(t)) \right| \\
\to 0 \quad (j \to \infty) \quad \text{uniformly on } \tilde{T}.
\end{equation}

Combining (3.25), (3.29) and (3.30), we obtain

\begin{equation}
 u_j(t) \to u(t) \quad \text{in } C(\tilde{T}, L^2) \quad (j \to \infty).
\end{equation}

On the other hand, the length of \( T \) is determined only by \( n, p, \lambda \) and \( \rho \) (see (3.9) and (3.13)). By the \( L^2 \)-norm conservation law we see that \( \sup \{ \| u(t) \|_{L^2}, \| u_j(t) \|_{L^2}, j = 1, 2, \ldots \} \) is constant for \( t \in \mathbb{R} \). Accordingly, we use the above argument with the initial time \( t_0 \) and the initial data \( u_0, u_{0j}, j = 1, 2, \ldots \), replaced by \( t_0 + T \) and \( u(t_0 + T), u_j(t_0 + T), j = 1, 2, \ldots \), or by \( t_0 - T \) and \( u(t_0 - T), u_j(t_0 - T), j = 1, 2, \ldots \), respectively, to obtain (3.31) with \( \tilde{T} \) replaced by \( \tilde{T}_{2T} \).
Repeating this procedure, we obtain (1.6). This completes the proof of Theorem 1.1.

(Q. E. D.)

REFERENCES


[7] W. A. Strauss, Everywhere defined wave operators, in


