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L^2-Solutions for Nonlinear Schrödinger Equations and Nonlinear Groups

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§1. Introduction and main results.

We consider the unique global existence of solutions in a weaker class than the energy space, i.e., $H^1(\mathbb{R}^n)$ for the Cauchy problem of the nonlinear Schrödinger equation:

$$(1.1) \qquad i\frac{\partial u}{\partial t} = -\Delta u + \lambda|u|^{p-1}u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

$$(1.2) \qquad u(t_0,x) = u_0(x), \quad x \in \mathbb{R}^n,$$

where $t_0 \in \mathbb{R}$ and $\lambda \in \mathbb{R}$. By $\alpha(n)$ we denote ∞ if $n = 1$ or $n = 2$ and $(n + 2)/(n - 2)$ if $n \geqq 3$. There are many papers concerning the global existence of solutions for Problem (1.1)-(1.2) (see, e.g., [1]-[2], [4]-[7], [9]-[10] and [13]-[14]). In [1] Baillon, Cazenave and Figueira show that if $1 \leqq n \leqq 3$, $1 < p < \alpha(n)$ and $\lambda > 0$, Problem (1.1)-(1.2) has a unique global strong solution $u(t) \in C(\mathbb{R};H^2(\mathbb{R}^n)) \cap C^1(\mathbb{R};L^2(\mathbb{R}^n))$ for any $u_0 \in H^2(\mathbb{R}^n)$. In [2] Ginibre and Velo show that if $1 < p < \alpha(n)$ and $\lambda > 0$ or if $1 < p < 1 + \frac{4}{n}$ and $\lambda < 0$, Problem (1.1)-(1.2) has a unique global weak solution $u(t) \in C(\mathbb{R};H^1(\mathbb{R}^n))$ for any $u_0 \in H^1(\mathbb{R}^n)$.

In [6] Strauss shows that if $\lambda > 0$ and $p > 1$, Problem (1.1) -(1.2) has at least one global weak solution $u(t)$ in $L^\infty(\mathbb{R}; H^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n))$ for any $u_0 \in H^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$ (see also [5]). In [10] M. Tsutsumi and N. Hayashi discuss the unique global existence of classical solutions for (1.1)-(1.2) (see also Pecher and von Wahl [4]). In [9] M. Tsutsumi discusses the unique global solution in $\mathcal{S}(\mathbb{R}^n)$ or in the weighted Sobolev space for (1.1)-(1.2). Recently in [13, 14] N. Hayashi, K. Nakamitsu and M. Tsutsumi have shown that the solution of (1.1)-(1.2) has the smoothing property in some sense. In [13] they also discuss the global existence of solutions of (1.1) -(1.2) for the initial data $u_0 \in L^2(\mathbb{R}^n)$ with $xu_0(x) \in L^2(\mathbb{R}^n)$, when $n = 1$. In almost all of previous papers the solution of (1.1)-(1.2) has been constructed in a space not larger than the energy space, that is, $H^1(\mathbb{R}^n)$, because the proofs in almost all of previous papers are based on the energy inequality. However, in [7] Strauss constructs the wave operators from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for the equation (1.1) with $p = 1 + \frac{4}{n}$ (see [7, Theorem 5]). His results are almost equivalent to the construction in $L^2(\mathbb{R}^n)$ of unique local solutions for (1.1) -(1.2) with $p = 1 + \frac{4}{n}$. In this paper we prove that when $1 < p < 1 + \frac{4}{n}$, we can construct the unique global solution of (1.1)-(1.2) for any u_0 in $L^2(\mathbb{R}^n)$ (but possibly not in $H^1(\mathbb{R}^n)$). Such a solution is called an "L^2-solution". Furthermore, we show that when $1 < p < 1 + \frac{4}{n}$, the solution operator of the evolution equation (1.1) constitutes a strongly continuous nonlinear operator group in $L^2(\mathbb{R}^n)$. Our proof is based on the L^2-norm conservation law and the dispersive effect of solutions (see, e.g., Lemma 2.2).

We put $U(t) = e^{i\Delta t}$ and $f(z) = \lambda|z|^{p-1}z$ ($z \in \mathbf{C}$). Our main theorem in this paper is the following.

Theorem 1.1. Assume that $1 < p < 1 + \frac{4}{n}$. Then, for any $u_0 \in L^2(\mathbb{R}^n)$ and any $t_0 \in \mathbb{R}$ there exists a unique global solution $u(t)$ of (1.1)-(1.2) such that

$$(1.3) \qquad u(t) \in C(\mathbb{R}\,; L^2(\mathbb{R}^n)) \cap L^r_{loc}(\mathbb{R}\,; L^{p+1}(\mathbb{R}^n)),$$

$$(1.4) \qquad u(t) = U(t - t_0) - i \int_{t_0}^{t} U(t-\tau) f(u(\tau))\, d\tau, \quad t \in \mathbb{R},$$

$$(1.5) \qquad \|u(t)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)}, \qquad t \in \mathbb{R},$$

where $r = \frac{4(p+1)}{n(p-1)}$ and the integral in (1.4) is the Bochner integral in $H^{-1}(\mathbb{R}^n)$. Furthermore, let u_{0j}, $j = 1,2,\cdots$, and u_0 be such that u_{0j}, $u_0 \in L^2(\mathbb{R}^n)$ and $u_{0j} \to u_0$ in $L^2(\mathbb{R}^n)$ $(j \to \infty)$. Let $u_j(t)$ and $u(t)$ be the solutions of (1.1) with $u_j(t_0) = u_{0j}$ and $u(t_0) = u_0$, respectively. Then, for each $T > 0$

$$(1.6) \qquad u_j(t) \to u(t) \text{ in } C([t_0 - T, t_0 + T]; L^2(\mathbb{R}^n)) \quad (j \to \infty).$$

Remark 1.1. Theorem 1.1 is almost the same as Theoreml.1 in [15] except that (1.6) is stronger than (1.6) in [15]. Theorem 1.1 implies the well-posedness in $L^2(\mathbb{R}^n)$ of the Cauchy problem of the nonlinear Schrödinger equation (1.1) with $1 < p < 1 + \frac{4}{n}$.

By Theorem 1.1 we can define the solution operator of the evolution equation (1.1) as a mapping from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$, when $1 < p < 1 + \frac{4}{n}$. We denote it by $S(t)$. The following result is an immediate consequence of Theorem 1.1.

Corollary 1.2. Assume that $1 < p < 1 + \frac{4}{n}$. Then, $\{ S(t) \; ; \; -\infty < t < +\infty \}$ is a strongly continuous nonlinear operator group in $L^2(\mathbb{R}^n)$. That is, $S(t)$ is a homeomorphism from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ for each $t \in \mathbb{R}$, and

(1.7) $\quad S(t + s) = S(t)S(s), \quad t, s \in \mathbb{R},$

(1.8) $\quad S(0) = I,$

(1.9) $\quad S(h)v \to v$ in $L^2(\mathbb{R}^n)$ $(h \to 0)$, $v \in L^2(\mathbb{R}^n)$,

where I is the identity operator from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

Our plan in this paper is as follows. In Section 2 we summarize several lemmas needed for the proof of Theorem 1.1. In Section 3 we give a sketch of proof of Theorem 1.1.

We conclude this section with several notations given. We abbreviate $L^p(\mathbb{R}^n)$ and $H^m(\mathbb{R}^n)$ to L^p and H^m, respectively. $(\cdot\,,\cdot)$ denotes the scalar product in L^2. For a closed interval I in $\mathbb{R}$ and a Hilbert space H we denote the set of all weakly continuous functions from I to H by $C_w(I;H)$. Let $h(x)$ be an even and positive function in $C_0^\infty(\mathbb{R}^n)$ with $\|h\|_{L^1} = 1$. We put $h_j(x) = j^n h(jx)$ for each positive integer j. $*$ denotes the convolution with respect to spatial variables. In the course of calculations below various constants will be simply denoted by C. In particular, $C = C(*,\cdots,*)$ will denote a constant depending only on the quantities appearing in parentheses.

§2. Lemmas.

In this section we summarize several results needed for the proof of Theorem 1.1.

For $U(t)$ we have the following two lemmas.

Lemma 2.1. Let q and r be positive numbers such that $1/q + 1/r = 1$ and $2 \leqq q \leqq \infty$. For any $t \neq 0$, $U(t)$ is a bounded operator from L^r to L^q satisfying

$$(2.1) \qquad \|U(t)v\|_{L^q} \leqq (4\pi|t|)^{\frac{n}{q}-\frac{n}{2}} \|v\|_{L^r}, \qquad v \in L^r, \quad t \neq 0,$$

and for any $t \neq 0$, the map $t \to U(t)$ is strongly continuous. For $q = 2$, $U(t)$ is unitary and strongly continuous for all $t \in \mathbb{R}$.

Lemma 2.2. Let q and r be positive numbers such that $1 \leqq q - 1 < \alpha(n)$ and $(\frac{n}{2} - \frac{n}{q})r = 2$. Then,

$$(2.2) \qquad \|U(\cdot)v\|_{L^r(\mathbb{R};L^q)} \leqq C\,\|v\|_{L^2},$$

where $C = C(n, q)$.

Lemma 2.1 is well known (see, e.g., [2, Lemma 1.2]). For Lemma 2.2, see Strichartz [8, Corollary 1 in §3] and Ginibre and Velo [3, Proposition 7].

Furthermore, we need the following two lemmas.

Lemma 2.3. Let I be an open interval in $\mathbb{R}$. Let $1 < q, r < \infty$ and $a, b > 0$. We put

$$M = \{ v(t) \in L^{\infty}(I;L^2) \cap L^r(I;L^q);$$

$$\|v\|_{L^{\infty}(I;L^2)} \leqq a, \quad \|v\|_{L^r(I;L^q)} \leqq b \}.$$

Then M is a closed subset in $L^r(I;L^q)$.

Lemma 2.4. Let T_1 and T_2 be constants with $T_1 < T_2$. Assume that $v(t) \in C([T_1, T_2];H^{-1})$ and for some $K > 0$

$$\|v(t)\|_{L^2} \leqq K, \quad \text{a.e. } t \in [T_1, T_2]. \tag{2.3}$$

Then, $v(t) \in C_w([T_1, T_2];L^2)$ and (2.3) holds for all $t \in [T_1, T_2]$.

Lemmas 2.3 and 2.4 are identical to Lemmas 2.3 and 2.4 in [15], respectively. For the proofs of Lemmas 2.3 and 2.4, see [15, §2].

We conclude this section by giving the following lemma concerning the mollifier $h_j(x)$.

<u>Lemma 2.5.</u> Let I be a bounded closed interval in $\mathbb{R}$. Let $f(t) \in C(I;L^2)$. We put $f_j(t) = (h_j * f)(t)$. Then,

$$(2.4) \qquad f_j(t) \in \bigcap_{k=1}^{\infty} C(I;H^k), \quad j = 1,2,\cdots,$$

$$(2.5) \qquad \|f_j(t)\|_{H^m} \leqq C_{jm} \|f(t)\|_{L^2}, \quad t \in I,\ j = 1,2,\cdots,$$

for each positive integer m,

$$(2.6) \qquad f_j(t) \to f(t) \quad \text{in } C(I;L^2) \quad (j \to \infty),$$

where $C_{jm} = C(j, m)$.

<u>Proof.</u> (2.4) and (2.5) are clear. We prove only (2.6). We note that $f(t)$ is uniformly continuous on I. Since

$$\|f_j(t) - f_j(s)\|_{L^2} \leqq \|f(t) - f(s)\|_{L^2}, \quad t, s \in I,$$

we conclude that $f_j(t)$, $j = 1,2,\cdots$, are equi-continuous on I. On the other hand, $f_j(t) \to f(t)$ in L^2 $(j \to \infty)$ for each $t \in I$. Therefore, we can prove (2.6) by using the same argument as in the proof of the Ascoli-Arzela theorem.

(Q. E. D.)

§3. Sketch of the Proof of Theorem 1.1.

In this section we give a sketch of the proof of Theorem 1.1. By I_t and $\bar{I}_t$ we denote an open interval (t_0-t, t_0+t) and a closed interval $[t_0-t, t_0+t]$, respectively, for $t \geqq 0$. Let $r = \frac{4(p+1)}{n(p-1)}$ throughout this section.

We have the following result concerning the unique local existence of L^2-solutions for (1.1)-(1.2).

Lemma 3.1. Assume that $1 < p < 1 + \frac{4}{n}$. Then, for any $t_0 \in \mathbb{R}$ and any $\rho > 0$ there exists a $T = T(p, n, \lambda, \rho) > 0$ such that for any $u_0 \in L^2$ with $\|u_0\|_{L^2} \leqq \rho$ Problem (1.1)-(1.2) has a unique local solution $u(t)$:

$$(3.1) \qquad u(t) \in C(\bar{I}_T; L^2) \cap L^r(I_T; L^{p+1}),$$

$$(3.2) \qquad u(t) = U(t-t_0)u_0 - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))\, d\tau, \quad t \in \bar{I}_T,$$

where the integral in (3.2) is the Bochner integral in H^{-1}. Furthermore, the solution $u(t)$ satisfies

$$(3.3) \qquad \|u(t)\|_{L^2} = \|u_0\|_{L^2}, \quad t \in \bar{I}_T.$$

Proof. We only give the outline of the proof of Lemma 3.1. For the detailes, see [15, §3].

We consider the following integral equation:

$$(3.4) \qquad u_j(t) = U(t-t_0)h_j * u_0 - i\int_{t_0}^{t} U(t-\tau)f(u_j(\tau))\, d\tau, \qquad j = 1,2,\cdots.$$

From the result of Ginibre and Velo [2, Theorem 3.1] we already know that for each j there exists a unique global solution $u_j(t)$ of (3.4) in $C(\mathbb{R}; H^1)$ such that

$$(3.5)\qquad \|u_j(t)\|_{L^2} = \|h_j * u_0\|_{L^2} \leqq \|u_0\|_{L^2}, \quad t \in \mathbb{R}, \quad j = 1,2,\cdots.$$

Let ρ be a positive constant with $\|u_0\|_{L^2} \leqq \rho$. By δ we denote the constant appearing in (2.2) with $q = p+1$ and $r = \frac{4(p+1)}{n(p-1)}$. We note that δ depends only on n and p. We put

$$(3.6)\qquad M = \{ v(t) \in L^\infty(I_T;L^2) \cap L^r(I_T;L^{p+1}); \quad \|v\|_{L^\infty(I_T;L^2)} \leqq \rho,\ \|v\|_{L^r(I_T;L^{p+1})} \leqq 2\delta\rho \},$$

where T is a small positive constant to be determined later. We note that by Lemma 2.3 M is closed in $L^r(I_T;L^{p+1})$.

We first show that if T is sufficiently small, then

$$(3.7)\qquad u_j(t) \in M \qquad \text{for all } j.$$

For $0 \leqq s \leqq T$ we take the $L^r(I_s;L^{p+1})$ norm of (3.4) and use (2.1), (2.2) and the generalized Young inequality to obtain

$$(3.8)\qquad \|u_j\|_{L^r(I_s;L^{p+1})} \leqq \delta\rho + C_0 T^{p/q_1} \|u_j\|^p_{L^r(I_s;L^{p+1})}, \quad 0 \leqq s \leqq T,\ j = 1,2,\cdots,$$

where $q_1 = \frac{4p}{n+4-np}$ and $C_0 = C_0(n, p, \lambda)$. Now we choose $T > 0$ so small that there exists a positive number y satisfying $C_0 T^{p/q_1} y^p + \delta\rho - y < 0$ and $0 < y \leqq 2\delta\rho$. For that purpose, it is sufficient to choose $T > 0$ so that

$$(3.9)\qquad T < (2C_0(2\delta\rho)^{p-1})^{-q_1/p}.$$

Then we put

(3.10) $y_0 = \min\{\, 2\delta\rho \geqq y > 0;\ C_0 T^{p/q_1} y^p + \delta\rho - y = 0 \,\}.$

If T is chosen so small that (3.9) holds, then by (3.8) and (3.10) we obtain

(3.11) $\|u_j\|_{L^r(I_T;L^{p+1})} \leqq y_0 \leqq 2\delta\rho\ , \quad j = 1,2,\cdots.$

(3.5) and (3.11) give us (3.7), if T is chosen so small that (3.9) holds.

We next consider the estimate of the difference between u_j and u_k for any j and k with $j \neq k$. For u_j, $u_k \in M$ we have

(3.12) $$\|u_j - u_k\|_{L^r(I_T;L^{p+1})} \leqq \delta K(j,k) + \bar{C}_0 T^{p/q_1} \cdot 2(2\delta\rho)^{p-1} \|u_j - u_k\|_{L^r(I_T;L^{p+1})}\ ,$$

where $K(j,k) = \|h_j * u_0 - h_k * u_0\|_{L^2}$, $q_1 = \frac{4p}{n+4-np}$ and $\bar{C}_0 = \bar{C}_0(n,\ p,\ \lambda)$. If we choose T so small in (3.12) that

(3.13) $\bar{C}_0 T^{p/q_1} \cdot 2(2\delta\rho)^{p-1} \leqq \frac{1}{2}\ ,$

then we have by (3.12)

(3.14) $\|u_j - u_k\|_{L^r(I_T;L^{p+1})} \leqq 2\delta K(j,k)$

for all j and k. Since $k(j,k) \to 0$ $(j,\ k \to \infty)$, we obtain by (3.14)

(3.15) $\|u_j - u_k\|_{L^r(I_T;L^{p+1})} \to 0 \quad (j,\ k \to \infty),$

if T is chosen so small that (3.13) holds. In addition we have by (3.15)

$$(3.16)\quad |(u_j(t) - u_k(t),\psi)| \leqq K(j,k)\,\|\psi\|_{L^2}$$
$$+ CT^{q_2}\,\|\psi\|_{H^1}\cdot 2(2\delta\rho)^{p-1}\|u_j - u_k\|_{L^r(I_T;L^{p+1})}$$
$$\to 0 \quad (j,\ k \to \infty) \quad \text{uniformly on } \bar{I}_T\ ,$$

for $\psi \in H^1$, where $q_2 = \dfrac{4+(n+4)p-np^2}{4(p+1)} > 0$. (3.16) implies that $\{u_j(t)\}_{j=1}^{\infty}$ is the Cauchy sequence in $C(\bar{I}_T;H^{-1})$.

Therefore, by (3.7), (3.15), (3.16) and Lemma 2.3 we obtain the solution u(t) of (1.1)-(1.2) such that

$$(3.17)\quad u(t) \in L^{\infty}(I_T;L^2)\cap L^r(I_T;L^{p+1})\cap C(\bar{I}_T;H^{-1}),$$

$$(3.18)\quad u(t) = U(t-t_0)u_0 - i\int_{t_0}^{t} U(t-\tau)f(u(\tau))\,d\tau, \quad t \in \bar{I}_T\ ,$$

$$(3.19)\quad \|u(t)\|_{L^2} \leqq \|u_0\|_{L^2}\ , \quad \text{a.e. } t \in I_T\ ,$$

$$(3.20)\quad u_j(t) \to u(t) \text{ in } L^r(I_T;L^{p+1}) \text{ and in } C(\bar{I}_T;H^{-1}) \quad (j \to \infty),$$

where T is a positive constant determined by (3.9) and (3.13) and the integral in (3.18) is the Bochner integral in H^{-1}. (3.17), (3.19) and Lemma 2.4 imply that

$$(3.21)\quad u(t) \in C_w(\bar{I}_T;L^2)$$

and that for all $t \in \bar{I}_T$ (3.19) holds. The uniqueness of solutions satisfying (3.17-18) follows from the estimate of the type (3.14) and the standard argument.

Thus, for any $s \in \bar{I}_T$ we can uniquely solve (1.1)-(1.2) in the time interval [s-T , s+T] with the initial time t_0 and the initial datum u_0 replaced by s and u(s), respectively, where T is the same as in the case of the initial time t_0 and the initial datum u_0. Therefore, reversing the roles of 0 and t, we obtain the reverse inequality to (3.19) for all $t \in \bar{I}_T$, which implies (3.3). (3.3) and (3.21) give us

$$(3.22) \qquad u(t) \in C(\bar{I}_T;L^2).$$

This completes the proof of Lemma 3.1.

(Q. E. D.)

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. The unique global existence of L^2-solutions for (1.1)-(1.2) follows directly from Lemma 3.1, which shows the unique local solvability in L^2 of (1.1)-(1.2) and the a priori bound of the L^2-norm of L^2-solutions.

It remains only to prove the continuous dependence of L^2-solutions on the initial data. Let u_{0j}, $j = 1,2,\cdots$, and u_0 be such that u_{0j}, $u_0 \in L^2$ and $u_{0j} \to u_0$ in L^2 $(j \to \infty)$. Let $u_j(t)$ and $u(t)$ be the global L^2-solutions of (1.1) with $u_j(t_0) = u_{0j}$ and $u(t_0) = u_0$, respectively. We put $\rho = \sup\{\|u_0\|_{L^2}, \|u_{0j}\|_{L^2}, j = 1,2,\cdots\}$. For this ρ, let $T > 0$ be defined as in (3.9) and (3.13). Then, by using the same argument as in the proof of Lemma 3.1 we have

$$(3.23) \qquad u_j(t) \to u(t) \quad \text{in } L^r(I_T;L^{p+1}) \qquad (j \to \infty),$$

$$(3.24)\qquad |(u_j(t) - u(t), g(t))| \leqq K \sup_{t \in \bar{I}_T} \|g(t)\|_{H^1}$$
$$\times (\|u_{0j} - u_0\|_{L^2} + \|u_j - u\|_{L^r(I_T; L^{p+1})}),$$
$$t \in \bar{I}_T, \quad j = 1, 2, \cdots,$$

for $g(t) \in C(\bar{I}_T; H^1)$ (see, e.g., (3.15) and (3.16)), where $K = K(n, p, \lambda, \rho) > 0$. We evaluate

$$(3.25)\qquad \|u_j(t) - u(t)\|^2_{L^2} = (u_j(t) - u(t), u_j(t) - u(t))$$
$$\leqq |\ \|u_j(t)\|^2_{L^2} - (u(t), u_j(t))\ | + |(u_j(t) - u(t), u(t))|,$$
$$t \in \bar{I}_T, \quad j = 1, 2, \cdots.$$

We first evaluate the second term at the right hand side of (3.25). Let ε be an arbitrary positive constant. We put $\tilde{u}_k(t) = (h_k * u)(t)$ for each positive integer k. By Lemma 2.5 we can choose k so large that

$$(3.26)\qquad |(u_j(t) - u(t), u(t) - \tilde{u}_k(t))| \leqq 2\rho \|u(t) - \tilde{u}_k(t)\|_{L^2} < \frac{1}{2}\varepsilon,$$
$$t \in \bar{I}_T.$$

For such a k we have by (3.23), (3.24) and Lemma 2.5

$$(3.27)\qquad |(u_j(t) - u(t), \tilde{u}_k(t))| \leqq K \sup_{t \in \bar{I}_T} \|\tilde{u}_k(t)\|_{H^1}$$
$$\times (\|u_{0j} - u_0\|_{L^2} + \|u_j - u\|_{L^r(I_T; L^{p+1})}) < \frac{1}{2}\varepsilon,$$
$$t \in \bar{I}_T,$$

if j is sufficiently large. Therefore, we obtain by (3.26) and (3.27)

$$(3.28) \quad |(u_j(t) - u(t), u(t))|$$

$$\leqq |(u_j(t)-u(t), \tilde{u}_k(t))| + |(u_j(t)-u(t), u(t)-\tilde{u}_k(t))|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \quad t \in \bar{I}_T,$$

for sufficiently large j. (3.28) implies that

$$(3.29) \quad |(u_j(t)-u(t), u(t))| \to 0 \quad (j \to \infty) \quad \text{uniformly on } \bar{I}_T.$$

We next evaluate the first term at the right hand side of (3.25). Since $\|u_j(t)\|_{L^2} = \|u_{0j}\|_{L^2}$ and $\|u(t)\|_{L^2} = \|u_0\|_{L^2}$ for $t \in \bar{I}_T$, we have by (3.29)

$$(3.30) \quad \left| \|u_j(t)\|_{L^2}^2 - (u(t), u_j(t)) \right|$$

$$\leqq \left| \|u_{0j}\|_{L^2}^2 - \|u_0\|_{L^2}^2 \right| + |(u(t), u_j(t)-u(t))|$$

$$\to 0 \quad (j \to \infty) \quad \text{uniformly on } \bar{I}_T.$$

Combining (3.25), (3.29) and (3.30), we obtain

$$(3.31) \quad u_j(t) \to u(t) \quad \text{in } C(\bar{I}_T; L^2) \quad (j \to \infty).$$

On the other hand, the length of T is determined only by n, p, λ and ρ (see (3.9) and (3.13)). By the L^2-norm conservation law we see that $\sup\{ \|u(t)\|_{L^2}, \|u_j(t)\|_{L^2}, j = 1,2,\cdots\}$ is constant for $t \in \mathbb{R}$. Accordingly, we use the above argument with the initial time t_0 and the initial data u_0, u_{0j}, $j = 1,2,\cdots$, replaced by t_0+T and $u(t_0+T)$, $u_j(t_0+T)$, $j = 1,2,\cdots$, or by t_0-T and $u(t_0-T)$, $u_j(t_0-T)$, $j = 1,2,\cdots$, respectively, to obtain (3.31) with $\bar{I}_T$ replaced by $\bar{I}_{2T}$.

Repeating this procedure, we obtain (1.6). This completes the proof of Theorem 1.1.

(Q. E. D.)

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