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ON SOME ASYMPTOTIC PROBLEMS IN OPTIMAL CONTROL

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Introduction. Let us consider the following boundary value problem

$$(P)\qquad \begin{cases} \mathrm{Max}[\,u-\psi; Au - f\,] = 0 & \text{in } \Omega \\ Bu = 0 & \text{on } \partial\Omega \end{cases}$$

Here f and ψ are given real function defined on a bounded open subset $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$, with boundary $\partial\Omega$ of class C^1. The operators A,B appearing in $(P)_\lambda$ are linear partial differential operators defined, respectively by

$$A = - a_{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j} \qquad x \in \Omega$$

$$B = b_i(x) \frac{\partial}{\partial x_i} \qquad x \in \partial\Omega$$

(we adopt here and in the following the usual convention about summation over repeated indices i,j = 1...N).

We shall assume from now that

(1) $a_{ij} \in C^{0,\gamma}(\Omega) \qquad a_{ij}(x)\, \xi_i \xi_j \geq \alpha\, \xi_i^2 \qquad x \in \Omega$

for some $0<\gamma<1$, $\alpha>0$ and all $\xi = (\xi_1 \dots \xi_N) \in \mathbb{R}^N$,

(2) $b_i \in C^{0,\gamma}(\partial\Omega) \qquad b_i(x) n_i(x) \geq \beta > 0 \qquad x \in \partial\Omega.$

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Here $n_i(x)$, $i = 1 \dots N$, are the components of the unit outward normal vector at $x \in \partial\Omega$.

An equivalent formulation of (P) is to find u such that

$$(\hat{P}) \quad \begin{cases} u \le \psi, \quad Au \le f \qquad (u-\psi)(Au-f) = 0 \quad \text{in } \Omega \\ Bu = 0 \qquad \text{on } \partial\Omega \end{cases}$$

If, for example, $A = -\Delta$, $b_i = n_i$ and $f = 0$ this problem amounts to find the greatest subharmonic function lying below the obstacle ψ and satisfying the Neumann condition on the boundary.

One motivation for the study of (P) comes from stochastic optimal control theory. It can be proved indeed, by a dynamic programming approach, that (P) comprises the Bellman optimality conditions for the optimal undiscounted stopping time problem for a non-degenerate diffusion in Ω with reflecting boundary conditions. We shall not go into details about the stochastic interpretation of problem (P) and refer instead to [1], [2], [3]. Let us only point out in this respect that the relevant unknown in (P) is the contact region $C = \{x \in \Omega \,|\, u(x) = \psi(x)\}$ The first time the reflected diffusion enters C can be shown indeed to be an optimal one (see [1]).

The topics we shall discuss in connection with the unilateral oblique derivative problem (P) are existence and uniqueness of solution in a suitable Sobolev space.

The results presented in the following sections 1 and 2 are contained in a joint paper with M.G. Garroni [5] (see also [4]).

Similar result for first order problems due to the collaboration with J.L. Menaldi will be briefly accounted in § 3 (see [6] for details)

§ 1. Necessary conditions

Let us consider the particular case where $A = -\Delta$, $b_i = n_i$ and assume u is a solution of (P).

Then, as an immediate consequence of the divergence theorem and the boundary condition, one finds that necessarily

(3) $\int_\Omega f dx \geq 0$

Moreover, since it may happen that $u \equiv \psi$, the following condition on ψ necessarily holds

(4) $\frac{\partial \psi}{\partial n} \geq 0$ on $\partial\Omega$.

These simple considerations suggest that similar necessary conditions should be fulfilled in the general case.

The purpose of this section is to identify such conditions, the main difficulty to this end being the lack of differentiability of coefficients which does not allow integration by parts.

To overcome this difficulty, let us consider the linear boundary value problem

(5)
$$\begin{aligned} Av + \lambda v &= f \quad \text{in } \Omega \\ Bv &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

It is proved in [5] that for $\lambda > 0$ and for any $f \in L^q(\Omega)$ $(1 < q < \frac{1}{1-\gamma}, \gamma > \frac{N-1}{N})$, (5) has a unique solution $v_\lambda \in W^{2,q}(\Omega)$ given by

(6) $v_\lambda(x) = \int_0^{+\infty} \int_\Omega e^{-\lambda t} G(x,y,t) f(y) dy dt$.

Here G is the solution of

$$\begin{aligned} &\frac{\partial G}{\partial t} + AG = \delta_{x-y}\ \delta_t && \text{in } \Omega \times]0,T[\\ &G(x,y,0) = 0 && \text{in } \Omega \times \Omega \\ &BG = 0 && \text{in } \partial\Omega \times [0,T]. \end{aligned}$$

The existence of such a Green's function as well as various estimates on G are proved in [8].

Let us denote by J_λ the resolvent operator of problem (5) as defined by (6) and by J^*_λ its adjoint acting on $L^{q'}(\Omega)$, $(\frac{1}{q}+\frac{1}{q'}=1)$.

Theorem 1. Assume $1 < q < \frac{1}{1-\gamma}$, $\gamma > \frac{N-1}{N}$. Then, there exists a unique $m \in L^{q'}(\Omega)$ such that

$$(7)\qquad \begin{aligned} & m = \lambda J^*_\lambda m \quad \text{in } \Omega \\ & m > 0 \quad \text{a.e.} \quad \fint_\Omega m\,dx = 1 \end{aligned}$$

Moreover, there exists $\delta > 0$ such that the following inequality holds

$$(8)\qquad \left\| \int_\Omega G(x,y,t)f(y)dy - \fint_\Omega f m\,dx \right\|_q \leq k\, e^{-\delta t} \| f \|_{L^q}, \quad \forall t \geq 0$$

for some $k \geq 0$ and all $f \in L^q(\Omega)$.

The proof of (7), (8) (see [5] for details) is based on the positivity of G and on general spectral theorems for order preserving compact operators.

Let us point out here that, for more regular f, (8) describes the decay as $t \to +\infty$ of the solution of the initial-boundary value problem for A,B, generalizing well-known results.

Another remark is that m satisfies formally the homogeneous adjoint problem

$$\begin{aligned} & A^*m = 0 && \text{in } \Omega \\ & B^*m = 0 && \text{on } \partial\Omega \\ & m > 0 && \fint_\Omega m\,dx = 1 \end{aligned}$$

and $m \in C^2(\bar{\Omega})$, provided the coefficients a_{ij}, b_i are smooth enough. In the model case considered at the beginning of this section, $m \equiv 1$.

The function m defined by (7) turns out to be the correct one in order to find necessary conditions for the solvability of (P). Suppose, for simplicity, that $\psi \equiv 0$. Then we have

Lemma 1. Assume $f \in L^q(\Omega)$ $(1 < q < \frac{1}{1-\gamma},\ \gamma > \frac{N-1}{N})$. If $u \in W^{2,q}(\Omega)$ is a solution of (P), then necessarily

(9) $$⨍_\Omega fm\,dx \geq 0$$

Outline of the proof. Let u be a solution of (P). Then, the following inequality holds

(10) $$-f^- \leq Au \leq f \quad \text{in } \Omega$$

(see [5] for the proof and [7] for general discussion of Lewy–Stampacchia type inequalities).

Hence

$$Au = g \quad \text{in } \Omega$$
$$Bu = 0 \quad \text{on } \partial\Omega$$

for some $g \in L^q(\Omega)$, $-f^- \leq g \leq f$ in Ω.

Equivalently, u satisfies

(11) $$u = J_\lambda(g + \lambda u) \qquad \text{for any } \lambda > 0.$$

Multiply both sides of (11) by m and integrate over Ω to find that

$$\int_\Omega um\,dx = \int_\Omega gJ^*_\lambda m\,dx + \lambda\int_\Omega uJ^*_\lambda m\,dx = \frac{1}{\lambda}\int_\Omega gm\,dx + \int_\Omega um\,dx$$

that is

$$\int_\Omega gm\,dx = 0$$

Since $f \geq g$, this yields (9).

§ 2. Existence and uniqueness of solutions of (P)

The main result of this section is that the necessary condition (9) turns out to be also a sufficient one for the solvability of (P).

The following theorem can be regarded as a Fredholm alternative type result for the nonliner boundary value problem (P).

We assume again for simplicity $\psi \equiv 0$ (the general case can be easily reduced to this by changing the unknown).

Theorem 2. Assume $f \in L^q(\Omega)$ $\left(1 < q < \frac{1}{1-\gamma},\ \gamma > \frac{N-1}{N}\right)$. Then,

(a) If $\int_\Omega fm\,dx > 0$ then (P) has unique solution $u \in W^{2,q}(\Omega)$

(b) If $\int_\Omega fm\,dx = 0$ then there exists a maximum function u (in the almost everywhere partial ordering) such that

$$Au = f \,,\quad u \leq 0 \quad \text{in } \Omega$$
$$Bu = 0 \quad \text{on } \partial\Omega$$

Outline of the proof.

The first step is to consider for $\lambda > 0$ the following problem

$$(P)_\lambda \qquad \begin{aligned} &\operatorname{Max}[u_\lambda;\ Au_\lambda + \lambda u_\lambda - f] = 0 \quad \text{in } \Omega \\ &Bu_\lambda = 0 \quad \text{on } \partial\Omega \end{aligned}$$

Problem $(P)_\lambda$ has a unique solution $u_\lambda \in W^2_q(\Omega)$ and

(12) $$u_\lambda = \operatorname{Sup}\{ v \in W^2_q(\Omega) : v \leq 0;\ Av + \lambda v \leq f \text{ in } \Omega,\ Bv \leq 0 \text{ on } \partial\Omega\}$$

(13) $\quad -f^- \leq Au_\lambda + \lambda u_\lambda \leq f,$

(see [5]). As a consequence of (13), u_λ satisfies

$$\begin{aligned} Au_\lambda + \lambda u_\lambda &= g_\lambda - f^- \quad \text{in } \Omega \\ Bu_\lambda &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

for some $g_\lambda \in L^q(\Omega)$, $0 \leq g_\lambda \leq f^+$. Hence,

(14) $\quad u_\lambda = J_\lambda(g_\lambda - f^-).$

Multiply both sides of (14) by λm and integrate over Ω to obtain taking (7) into account, that

$$\lambda ⨍_\Omega u_\lambda m dx = ⨍_\Omega \lambda J_\lambda(g_\lambda - f^-) m dx = ⨍_\Omega (g_\lambda - f^-) m dx.$$

This yields

(15) $\quad -C_1 \| f \|_q \leq \lambda ⨍_\Omega u_\lambda m dx \leq 0,$

for some constant C_1 independent of λ.

Let us consider then the function w_λ defined by

$$w_\lambda = u_\lambda - ⨍_\Omega u_\lambda m dx ,$$

and observe that w_λ satisfies

(16) $$\begin{aligned} Aw_\lambda + \lambda w_\lambda &= g_\lambda - f^- - \lambda C_\lambda \quad \text{in } \Omega \\ Bw_\lambda &= 0 \quad \text{on } \partial\Omega \\ ⨍_\Omega w_\lambda m dx &= 0, \end{aligned}$$

where $C_\lambda = ⨍_\Omega u_\lambda m dx$. The same argument as above shows that $h_\lambda = g_\lambda - f^- - \lambda C_\lambda$ satisfies

$$\int_\Omega h_\lambda m dx = 0$$

From (8) it follows that

$$(17)\quad \| w_\lambda \|^q_{L^q} = \| J_\lambda(h_\lambda) \|^q_{L^q} \le \int_0^{+\infty} e^{-\lambda t} \| \int_\Omega G(x,y,t) h_\lambda(y) dy \|^q_{L^q} dt \le$$

$$\le k \| h_\lambda \|^q_{L^q} \int_0^{+\infty} e^{-q(\lambda+\delta)} dt \le \frac{k^q \| h_\lambda \|^q_{L^q}}{q\,\delta}.$$

Thanks to (15), $\| h_\lambda \|_{L^q}$ is uniformly bounded and therefore (17) yields

$$\| w_\lambda \|_{L^q} \le C, \quad C \text{ independent of } \lambda.$$

This and the equation (16) give finally

$$(18)\qquad \| w_\lambda \|_{W^{2,q}} \le C, \qquad C \text{ independent of } \lambda.$$

Let us proceed now to the proof of statement (a), sending the interested reader to [5] for case (b).

The idea is to show that if $\int_\Omega fmdx > 0$, then C_λ is uniformly bounded. This, together with estimate (18) will imply the uniform boundedness of $\| u_\lambda \|_{W^{2,q}}$.

Let

$$\lim_{\lambda \to 0} \lambda C_\lambda = L \le 0$$

$$\lim_{\lambda \to 0} w_\lambda = w \quad \text{in } W^{2,q}(\Omega) - \text{weak}$$

(these limits exist thanks to (15) and (18)) and assume by contradiction that

$$C_\lambda \to -\infty \qquad \text{as} \quad \lambda \to 0.$$

Then, it is easy to check that for sufficiently small λ, w_λ satisfies

$$Aw_\lambda + \lambda w_\lambda = f - \lambda C_\lambda \quad \text{in } \Omega$$
$$Bw_\lambda = 0 \quad \text{on } \partial\Omega$$

Letting $\lambda \to 0$ in the above we obtain

$$Aw = f - L \quad \text{in } \Omega$$
$$Bw = 0 \quad \text{on } \partial\Omega$$

This implies

$$⨍_\Omega fmdx = L \leq 0,$$

which contradicts the assumption $⨍_\Omega fmdx > 0$.

Hence C_λ and, consequently, $\| u_\lambda \|_{W^{2,q}}$ is bounded. At this point it is not difficult to show that some subsequential $W^{2,q}$ - weak limit u of u_λ solves (P).

To prove the uniqueness, let $\tilde{u}$ be another solution of (P). It is immediate that $\tilde{u}$ satisfies

$$\tilde{u} \leq 0 \quad A\tilde{u} + \lambda\tilde{u} \leq f \quad \text{in } \Omega$$
$$B\tilde{u} = 0 \quad \text{on } \partial\Omega$$

for any $\lambda > 0$. This, together with (12), yields

$$\tilde{u} \leq u_\lambda$$

and, taking the limit as $\lambda \to 0$,

(19) $$\tilde{u} \leq u.$$

On the other hand, from (P) it follows that

$$A(u - \tilde{u})(u - \tilde{u}) \leq 0 \quad \text{in} \quad \Omega$$

Taking (19) into account this gives

$$A(u - \tilde{u}) = g \quad \text{in } \Omega$$
$$B(u - \tilde{u}) = 0 \quad \text{on } \partial\Omega .$$

for some $g \leq 0$. But then, necessarily $\int_\Omega gmdx = 0$, so that $g = 0$. Then

$$u - \tilde{u} = \lambda J_\lambda (u - \tilde{u}) \quad \text{for any } \lambda > 0$$

Multiplying by m and integrating on Ω we find

$$\int_\Omega (u - \tilde{u})mdx = \lambda \int_\Omega (u - \tilde{u})mdx.$$

Hence, $\int_\Omega (u - \tilde{u})mdx = 0$ and, by (19), $u - \tilde{u} = 0$.

Remark 1. Theorem 2 extends previous results of A. Bensoussan - J.L. Lions [9], M. Robin [3].

§ 3. Some remarks on the deterministic optimal stopping time problem

Let us denote by $y_x(\cdot)$ the solution of the ordinary differential equation

$$\frac{dy}{dt} = g(y(t)), \quad t > 0$$

$$y(0) = x \in \mathbb{R}^N$$

Here $g = (g_1 \dots g_N) : \mathbb{R}^N \to \mathbb{R}^N$ is assumed to satisfy the following conditions

$$(20) \qquad \begin{aligned} &|g(x) - g(x')| \le L|x-x'| \ ; \quad |g(x)| \le M \\ &(g(x) - g(x'))\cdot(x - x') \le -\gamma|x - x'|^2 \end{aligned}$$

for all x,x' in $\mathbb{R}^N$ and some positive constants L,M,γ.

It is well known that (20) imply that g has a unique stationary point x_0 and that

$$|y_x(t) - y_{x'}(t)| \le e^{-\gamma t}|x - x'| \qquad \forall t > 0.$$

The optimal stopping time problem is to find, for any initial position x, a time $\theta^*_x \ge 0$ such that

$$J(x,\theta^*_x) = \operatorname*{Inf}_{\theta \ge 0} J(x,\theta).$$

Here above

$$J(x,\theta) = \int_0^\theta f(y_x(t))e^{-\lambda t}\, dt + \psi(y_x(\theta))\, e^{-\lambda\theta}$$

with $\lambda > 0$, f and ψ are given functions such that

$$(21) \qquad \begin{aligned} &|f(x) - f(x')| \le C|x - x'| \qquad |f(x)| \le D \\ &|\psi(x) - (x')| \le C|x - x'| \qquad |\psi(x)| \le D \end{aligned}$$

for all $x,x' \in \mathbb{R}^N$ and positive constants C,D.

It is proved in [6] that, under the assumptions made, the value function V_λ of the optimal stopping problem, defined by

$$V_\lambda(x) = \operatorname*{Inf}_{\theta \ge 0} J(x,\theta), \quad \lambda > 0$$

is the unique bounded Lipschitz continuous solution of the Bellman equation

$$(\tilde{P})_\lambda \qquad \mathrm{Max}\,[V-\psi ;- \sum_{i=1}^{N} g_i \frac{\partial V}{\partial x_i} + \lambda V - f\,] = 0 \quad \text{in } \mathbb{R}^N,$$

in the viscosity sense (see [2]). Moreover,

$$|V_\lambda(x)| \leq C, \qquad |V_\lambda(x) - V_\lambda(x')| \leq C|x - x'|$$

with C independent on $\lambda > 0$.

The behaviour of V_λ as $\lambda \to 0$ is described by the following theorem. Let us recall that we denote by x_o the unique point in $\mathbb{R}^N$ such that $g(x_o) = 0$.

Theorem 3. Let us assume (20), (21). If $f(x_o) \geq 0$, then $V_\lambda \to V$ locally uniformly in $\mathbb{R}^N$ and V is a viscosity solution of

$$\mathrm{Max}\,[V - \psi;\, - \sum_{i=1}^{N} g_i \frac{\partial V}{\partial x_i} - f\,] = 0 \quad \text{in } \mathbb{R}^N$$

On the other hand, if $f(x_o) < 0$, then V_λ does not converge but $V_\lambda - V_\lambda(x_o) \to \tilde{V}$ locally uniformly in $\mathbb{R}^N$ and $\tilde{V}$ is a viscosity solution of

$$- \sum_{i=1}^{N} g_i \frac{\partial V}{\partial x_i} = f - f(x_o).$$

We refer to [6] for the proof and to [10] for further results in this direction.

Final Remark

The asymptotic behaviour as $\lambda \to 0^+$ of the solutions of problems $(P)_\lambda$ and $(\tilde{P})_\lambda$ is formally identical and in both cases it depends on the average value of f with respect to some measure.

For problem $(P)_\lambda$ this is the measure $d\mu = m dx$ with m given by

Lemma 1, while for $(\tilde{P})_\lambda$ the same role is played by the Dirac measure δ_{x_0} concentrated at the unique stationary point of g.

Both these measures can be interpreted as invariant measures for the semigroups having $- a_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$ and $- g_i \frac{\partial}{\partial x_i}$, respectively, as generators on suitable defined domains.

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