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On the 2-local structure of groups of characteristic 2 type

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1. A trichotomy theorem.

By definition, a group is of characteristic 2 type if it has even order and every 2-local subgroup L satisfies the condition $C_L(O_2(L)) \le O_2(L)$. For finite groups G of characteristic 2 type and $S \in Syl_2(G)$, we shall denote by $M(S)$ or $M(S, G)$ the set of all maximal 2-local subgroups of G containing S. A familiar example of groups of characteristic 2 type is a simple group of Lie type G^* defined over a finite field of characteristic 2. Suppose G^* has rank ℓ and take $S^* \in Syl_2(G)$. It can be shown that $M(S^*)$ is equal to the set of all maximal parabolic subgroups (that is, parabolic subgroups of rank $\ell - 1$) containing the Borel subgroup $B^* = N_{G^*}(S^*)$, which is one of the reasons why the set $M(S)$ played an important role in the recently finished program to classify the finite simple groups of characteristic 2 type.

More recent investigation to revise the classification focuses attention on rank one parabolic subgroups of G^* rather than maximal parabolic subgroups. To be more specific, we define the (2-generated) core of each rank one parabolic subgroup P of G^* to be the subgroup $O^{2'}(P)$ generated by all 2 elements of P, and define the corresponding objects in an arbitrary finite group G as follows.

Definition. If a subgroup Y of a group X is contained in a unique maximal subgroup of X, we say that X is Y-irreducible. For finite groups G and $S \in Syl_2(G)$, we denote by $C'(S)$ or $C'(S, G)$ the set of all S-irreducible subgroups of G, and we denote by $C(S)$ or $C(S, G)$ the set of all elements X of $C'(S)$ such that $O_2(X) \neq 1$.

It can be shown that if $X \in C(S^*, G^*)$ then either $X \leq B^*$ or X is the core of some rank one parabolic subgroup containing B^*. Thus, we observe that one of the following holds in G^*.

(1^*) $|M(S^*)| = 1$.

(2^*) There exist elements X_i $(i = 1, 2)$ of $C(S^*)$ such that $O_2(\langle X_1, X_2 \rangle) = 1$.

(3^*) There exist elements X_i $(i = 1, 2)$ of $C(S^*)$ such that $O_2(\langle X_1, X_2 \rangle) \neq 1$ and, whenever $\langle X_1, X_2 \rangle \leq M \in M(S^*)$, $M/O_2(M)$ has a quasisimple component L whose central factor group $L/Z(L)$ is isomorphic to a simple Lie type group of characteristic 2 and rank at least 2.

Indeed, (1^*) holds if and only if $\ell = 1$ and (2^*) holds if and only if $\ell = 2$. When $\ell \geq 3$, take two adjacent nodes of the Dynkin diagram for G^* and let X_i $(i = 1, 2)$ be the cores of the associated rank one parabolic subgroups containing B^*. Then the X_i satisfy the conditions of (3^*). Here and elsewhere, however, we consider the groups $Sp_4(2)' \cong A_6$, $G_2(2)' \cong PSU_3(3)$, and ${}^2F_4(2)'$ to be Lie type groups of characteristic 2 and rank 2, although they do not possess BN-pairs at characteristic 2. We note that simple Lie type groups of characteristic 2 and rank

1 are the groups $SL_2(2^m)$, $PSU_3(2^m)$, and $Sz(2^{2m-1})$ $(m \geq 2)$, and that they are sometimes called Bender groups.

The purpose of this paper is to show that an analogous trichotomy holds true for an arbitray group of characteristic 2 type, which is presumably the first nontrivial remark made on the relationship between $M(S)$ and $C(S)$. In order to state our result, we need the following:

Definition. For finite groups G and $S \in Syl_2(G)$, we denote by $L(S)$ or $L(S, G)$ the set of all subgroups Y of G that contain S and satisfy $O_2(Y) \neq 1$.

We say that a finite group G is almost simple if the generalized Fitting subgroup $F^*(G)$ is a nonabelian simple group. For almost simple groups G and $S \in Syl_2(G)$, we denote by $C''(S)$ or $C''(S, G)$ the set of all elements X of $C'(S)$ such that $N_X(S \cap F^*(G)) \neq X$. We note that $C''(S)$ is nonempty (see 2.1) and define

$$D''(S) = D''(S, G) = \bigcap O_2(X) \quad (X \in C''(S)).$$

We say that the almost simple group G is large if $D''(S, G) \neq 1$ for $S \in Syl_2(G)$.

Our main result may now be stated.

Theorem. Let G be a group of characteristic 2 type and take $S \in Syl_2(G)$. Then one of the following holds.

(1) $|M(S)| = 1$.

(2) There exist elements X_i $(i = 1, 2)$ of $C(S)$ such that $O_2(\langle X_1, X_2 \rangle) = 1$.

(3) There exist elements X_i $(i = 1, 2)$ of $C(S)$ such that $O_2(\langle X_1, X_2 \rangle) \neq 1$ and, whenever $\langle X_1, X_2 \rangle \leq Y \in L(S)$,

$Y/O_2(Y)$ has a quasisimple component $K/O_2(Y)$ such that the almost simpe group

$$A_S(K) = N_S(K)K/C_{N_S(K)K}(K/O_{2,2'}(K))$$

is large.

For the sake of brevity, we shall abuse the terminology and call $K/O_2(Y)$ as in Case (3) above a <u>large quasisimple component of</u> $Y/O_2(Y)$ (with respect ot S).

The Theorem leaves some open problems. First, in Case (3) the group $A = A_S(K)$ is a large almost simple group and $A = TF^*(A)$ for $T \in Syl_2(A)$. Therefore, we would pose the following:

Problem 1. Find all large almost simple groups A such that $A = TF^*(A)$ for $T \in Syl_2(A)$.

Of course, since we aim to apply the Theorem to revisions of the classification, we may assume that A is a "known" group (or a K-group). It is reported that Kantor and Liebeck-Saxl obtained a complete list of maximal subgroups of odd index of almost simple groups, which could presumably be used to enumerate the groups in Problem 1.

If an almost simple group A satisfies the condition $A \in C'(T, A)$ for $T \in Syl_2(A)$, then $A \in C''(T, A)$ and it follows that A is not large. Therefore, the groups in Problem 1 must be contained in the list of almost simple groups A such that $A \notin C'(T, A)$ and $A = TF^*(A)$ for $T \in Syl_2(A)$. Aschbacher has worked for such a list [1, Th. 2] (in fact, Aschbacher considers almost simple groups A such that $A \in C'(T, A)$). We see that if $F^*(A)$ is a Bender group and $A = TF^*(A)$ for $T \in Syl_2(A)$

then $A \in C'(T, A)$ and so A is not large. This shows that the large quasisimple component of $Y/O_2(Y)$ in Case (3) is not a central extension of any Bender groups, and thus Case (3) fits in well with Case (3^*) occuring in the Lie type groups G^*.

There are certain easy subcases of Problem 1. For instance, if $F^*(A)$ is of Lie type and characteristic 2, then A is large if (and only if) $A \notin C'(T, A)$. To see this, take an arbitrary $X \in C''(T, A)$. If $A \notin C'(T, A)$, then $X \neq A$, so $X \cap F^*(A) \neq F^*(A)$, and $X \cap F^*(A)$ is contained in a maximal parabolic subgroup of $F^*(A)$ by a result of Tits (see (1.6) of [2]). Hence, it follows that $C_X(O_2(X)) \leq O_2(X)$ and, consequently, $Z(T) \leq O_2(X)$. Thus, $Z(T) \leq D''(T, A)$ and A is large. This argument applies also to the case $F^*(A) \cong Sp_4(2)'$, $G_2(2)'$, or ${}^2F_4(2)'$ provided we suitably define maximal parabolic subgroups of these simple groups.

Also, if $A_n \leq A \leq \Sigma_n$, then A is large only if n is even. If n is odd, then T ($\in Syl_2(A)$) fixes precisely one letter in $\Omega = \{1, 2, \ldots, n\}$. Assume that T fixes 1 and take an arbitrary T-orbit Δ ($\neq \{1\}$) on Ω. Then T is contained in the direct product $Y = \Sigma_{\{1\} \cup \Delta} \times U$ of the symmetric group $\Sigma_{\{1\} \cup \Delta}$ on $\{1\} \cup \Delta$ and a Sylow 2-subgroup U of the symmetric group on $\Omega - \{1\} \cup \Delta$. The intersection $X = Y \cap A$ is contained in $C''(T, A)$ and $O_2(X) \leq U$. Thus, $D''(T, A) = 1$ and A is not large.

Condition (3) says that an element of $M(S)$ is large in some sense. Therefore, the Theorem probably could be used to analyze characteristic 2 type groups with small 2-local subgroups. As an extreme case, consider the case that every 2-local subgroup of G is solvable. Then Case (3) is ruled out and Case

(1) can easily be handled, so only Case (2) remains to be investigated. In particular, the Theorem contains Theorem A of [3] as a special case, because if the X_i in Case (2) are solvable then each X_i is a $\{2, p_i\}$-group for some odd prime p_i. The Theorem presumably could be used to analyze other small characteristic 2 type groups such as quasithin groups, although Case (3) is not automatically ruled out in such groups.

In addition to finding applications of the Theorem to small characteristic 2 type groups, we would pose to investigate each of the three cases (1), (2), (3) for general groups of characteristic 2 type. Foote and others have already investigated Case (1) in the broader context of the theory of blocks [4, pp. 37 - 42] (see also an article of Stellmacher [4, pp. 123 - 125]), but a much simpler direct approach is desirable.

Goldschmidt [5] determined the structure of groups X_i $(i = 1, 2)$ having a common 2-subgroup S of index 3 such that no nonidentity subgroup of S is normal both in X_1 and in X_2, and we have ever since been asking in what direction Goldschmidt's theorem should be extended. Case (2) of the Theorem suggests one possible extension: that is, we would pose the following:

Problem 2. Determine the structure of groups X_i $(i = 1, 2)$ that satisfy the following conditions:

(a) X_i $(i = 1, 2)$ have a common 2-subgroup S;

(b) $|X_i : S|$ is odd and X_i is S-irreducible $(i = 1, 2)$;

(c) no nonidentity subgroup of S is normal both in X_1 and in X_2;

(d) $C_{X_i}(O_2(X_i)) \leq O_2(X_i)$ $(i = 1, 2)$.

Without being aware of the Theorem, several people including the author have already considered various special cases of Problem 2 (e.g. Stellmacher [6] and Gomi-Tanaka [7]). Their results are far from giving a satisfactory answer to Problem 2 because very strong additional assumptions are made, but some of them have already had applications to simple groups of characteristic 2 type (for instance, see [3]).

Case (3) reminds us of Mason's problems concerning parabolic type subgroups [4, pp. 155 - 157]. No essential progress has been made on them to the knowledge of the author, and they will deserve more attention.

The Theorem is proved by consideration of certain graphs which are implicit in the following definition.

Definition. Let G be a finite group and $S \in Syl_2(G)$. If $C(S)$ is nonempty, we define

$$D(S) = D(S, G) = \bigcap O_2(X) \quad (X \in C(S)).$$

We denote by $E(S)$ or $E(S, G)$ the set of all unordered pairs (X_1, X_2) of elements of $C(S)$ such that either

(1) $O_2(\langle X_1, X_2 \rangle) = 1$ or

(2) $O_2(\langle X_1, X_2 \rangle) \neq 1$ and, whenever $\langle X_1, X_2 \rangle \leq Y \in L(S)$, we have $D(S, Y) \neq O_2(Y)$.

In the Lie type groups G^*, a pair (X_1, X_2) of elements of $C(S^*)$ is contained in $E(S^*)$ if and only if each X_i is the core of some rank one parabolic subgroup P_i containing B^* and P_1 is joined to P_2 in the Dynkin diagram for G^* (we identify the P_i with the associated nodes). Thus, if we form a graph with vertex set $C(S^*)$ and edge set $E(S^*)$, then this graph is, essentially, equal to the "Dynkin graph" obtained from the Dynkin diagram by reducing double or triple bonds to single bonds. Our result below shows that the set $E(S)$ is in general nonempty.

1.1. Let $X_1 \in C(S)$ and assume $X_1 \not\leq N_G(D(S))$. Then $(X_1, X_2) \in E(S)$ for some $X_2 \in C(S)$.

Proof. Suppose false, and take an arbitrary element X of $C(S)$. Then $(X_1, X) \notin E(S)$ and so there exists an element Y of $L(S)$ such that $\langle X_1, X \rangle \leq Y$ and $D(S, Y) = O_2(Y)$. Therefore, there exists a subset, C_X, of $C(S)$ such that $X \in C_X$ and X_1 normalizes the intersection

$$D_X = \bigcap O_2(Z) \quad (Z \in C_X).$$

Clearly, $D(S) \leq D_X$ for each $X \in C(S)$ and hence $D(S)$ is contained in the intersection

$$D = \bigcap D_X \quad (X \in C(S)),$$

while since $O_2(X) \geq D_X$ for each $X \in C(S)$, we have $D(S) \geq D$. Therefore, $D(S) = D$ and it follows that $X_1 \leq N_G(D(S))$. This is a contradiction.

Our next result gives information on $E(S)$.

1.2. Let $(X_1, X_2) \in E(S)$ and assume $\langle X_1, X_2 \rangle \leq Y \in L(S)$. Then $Y/O_2(Y)$ has a large quasisimple component.

A proof of this result will be given in Section 2. We conclude this section by deriving our theorem from 1.1 and 1.2. Let G be a group of characteristic 2 type, $S \in Syl_2(G)$, and assume that $|M(S)| > 1$. Note that if $S \neq M \in M(S)$ then $M = \langle C(S, M) \rangle$ (see 2.1). Since $M(S) \neq \{S\}$, $C(S)$ is not the empty set and the group $D(S)$ is defined. Moreover, since G is of characteristic 2 type, we have $C_X(O_2(X)) \leq O_2(X)$ for all $X \in C(S)$ and so $Z(S) \leq D(S)$. Hence $D(S)$ is not the identity and $N_G(D(S))$ is a 2-local subgroup. Since $M(S) \neq \{N_G(D(S))\}$, there exists an element X_1 of $C(S)$ such that $X_1 \not\leq N_G(D(S))$. By 1.1, $(X_1, X_2) \in E(S)$ for some $X_2 \in C(S)$. If $O_2(\langle X_1, X_2 \rangle) = 1$, we are done. If $O_2(\langle X_1, X_2 \rangle) \neq 1$ and $\langle X_1, X_2 \rangle \leq Y \in L(S)$, then $Y/O_2(Y)$ has a large quasisimple component by 1.2. This completes the proof of the Theorem.

As we have seen, the restriction that G be of characteristic 2 type can be relaxed to require only that $C_X(O_2(X)) \leq O_2(X)$ for all $X \in C(S)$, which is satisfied, for instance, if every $M \in M(S)$ satisfies the condition $C_M(O_2(M)) \leq O_2(M)$. This is in line with Gorenstein's approach to revising the classification.

2. 2-Irreducible subgroups and quasisimple components.

We shall devote this section to technical details for the proof of 1.2. In addition to the notation defined in Section 1, we shall use the following notation. By $D'(S, G)$, we denote the intersection

$$\cap O_2(X) \quad (X \in C'(S, G))$$

provided $C'(S, G)$ is not empty. For groups X and its proper subgroups Y, we denote by $N(Y, X)$ the set of all maximal subgroups of X that contain Y. Thus, $C'(S, G)$ consists of all subgroups X of G with $S < X$ and $|N(S, X)| = 1$. We use Bender's notation $F^*(G)$ and $E(G)$ for the generalized Fitting subgroup and the maximal semisimple normal subgroup of a finite group G ($F^*(G)$ has already appeared in Section 1; $F^*(G) = F(G)E(G)$ and $F(G)$ is the Fitting subgroup). The well known property

$$C_G(F^*(G)) \leq F^*(G)$$

is of fundamental importance in this section. Further important properties of $F^*(G)$ and $E(G)$ may be found in [8] and [9]. Now, we begin the proof of 1.2.

2.1. Let G be a finite group, $S \in Syl_2(G)$, and assume $S \neq G$. Then $G = \langle C'(S, G) \rangle$.

Proof. If $G \in C'(S, G)$, there is nothing to prove, so we assume that there exist two distinct elements M_i of $N(S, G)$ $(i = 1, 2)$. Then $S < M_i < G$ and, arguing by induction on $|G : S|$, we have $M_i = \langle C'(S, M_i) \rangle$, so $G = \langle M_1, M_2 \rangle = \langle C'(S, G) \rangle$.

2.2. Let G be a finite group, $S \in Syl_2(G)$, and N a normal subgroup of G. If $Y/N \in C'(SN/N, G/N)$, then there exists an element X of $C'(S, G)$ such that $Y = XN$.

Proof. Let $N(SN/N, Y/N) = \{M/N\}$. By 2.1, there exists an element X of $C'(S, Y)$ such that $X \not\leq M$. As $SN \leq XN \not\leq M$, we must have $XN = Y$.

2.3. Let G be a finite group and $S \in Syl_2(G)$. Then the following hold.

(1) If $S < H \leq G$, then $D'(S, G) \leq D'(S, H)$.

(2) If N is a normal subgroup with $SN \neq G$ and $*$ denotes the natural homomorphism of G onto G/N, then $D'(S, G)^* \leq D'(S^*, G^*)$.

Proof. Since $C'(S, G) \geq C'(S, H)$, (1) follows. To prove (2), let $Y \in C'(S^*, G^*)$. Then $Y = X^*$ for some $X \in C'(S, G)$ by 2.2, and clearly $O_2(X)^* \leq O_2(Y)$. So $D'(S, G)^* \leq O_2(Y)$ for all $Y \in C'(S^*, G^*)$ and (2) follows.

2.4. Let G be a finite group with $O_2(G) = 1$ and assume that $D'(S, G) \neq 1$ for $S \in Syl_2(G)$. Then the following hold.

(1) $E(G) \neq 1$.

(2) If $*$ denotes the natural homomorphism of G onto $G/O(G)$, then $[E(G)^*, D'(S^*, S^*E(G)^*)] \neq 1$.

Proof. First, our assumption and 2.3 show that $D'(S, H) \neq 1$ whenever $S < H \leq G$. To prove (1), we assume that $O(G) \neq 1$ because $O_2(G) = 1$. Then for each $X \in C'(S, SO(G))$, we have $O^2(X) \leq O(G)$, so $[O_2(X), O^2(X)] = 1$, and hence $X = SO^2(X) \leq N_G(D'(S, SO(G)))$. Thus, $D'(S, SO(G)) \leq O_2(SO(G)) \leq C_G(O(G))$ by 2.1 and, our assumption yields that $C_G(O(G)) \not\leq O(G)$. Since $O_2(G) = 1$, we conclude that $E(G) \neq 1$.

Define $H = SE(G)O(G)$. Then $SO(G) \neq H$ and $D'(S, H)^* \leq$

$D'(S^*, H^*)$ by 2.3, so it suffices to prove $[E(G)^*, D'(S, H)^*] \neq 1$. Suppose this is false. Then $[E(G), D'(S, H)] = 1$ by the three-subgroup lemma. Also, if $O(G) \neq 1$, then $D'(S, H) \leq D'(S, SO(G))$ by 2.3 and $D'(S, SO(G)) \leq C_G(O(G))$ as shown before. Therefore, $[E(G)O(G), D'(S, H)] = 1$ and, since $O_2(G) = 1$, it follows that $D'(S, H) = 1$, contrary to our assumption.

2.5. Let G be a finite group such that $G = SE(G)$ for $S \in Syl_2(G)$. If $[E(G), D'(S, G)] \neq 1$, then there exists a quasisimple component L of $E(G)$ such that $[\langle L^S \rangle, D'(S, S\langle L^S \rangle)] \neq 1$.

Proof. For each quasisimple component L of $E(G)$, we have $D'(S, G) \leq D'(S, S\langle L^S \rangle)$ by 2.3. Hence the assertion follows.

The next three lemmas deal with the following situation.

2.6 Hypothesis. G is a finite group, L is a subgroup of even order with $G = S\langle L^S \rangle$ for $S \in Syl_2(G)$, $\langle L^S \rangle = L_1 \times L_2 \times \cdots \times L_n$, and $L^S = \{L_1, L_2, \cdots, L_n\}$.

Under this hypothesis, set

$$N = \langle L^S \rangle, \qquad H = N_S(L)L,$$

and for each subgroup K of H that contains $N_S(L)$, define

$$K^u = S\langle (K \cap L)^S \rangle.$$

Also, assume

$$L = L_1$$

and, for each subgroup M of G that contains S, define

$$M^d = N_S(L)(L \cap (M \cap N)L_2 \cdots L_n).$$

Finally, take elements s_i of S $(i = 1, 2, \cdots, n)$ so that

$$L^{s_i} = L_i.$$

2.7. Under Hypothesis 2.6, the following hold.

(1) M^d and K^u are subgroups with $N_S(L) \leq M^d \leq H$, $S \leq K^u$, and $K \leq K^u$.

(2) $K^u \cap N = (K \cap L)^{s_1} \times \cdots \times (K \cap L)^{s_n}$ with $(K \cap L)^S = \{(K \cap L)^{s_i} \mid 1 \leq i \leq n\}$ and $N_S(K \cap L) = N_S(L)$.

(3) $(K^u)^d = K$ and $(M^d)^u \geq M$.

(4) If $N_S(L) \leq K < H$, then $K^u \neq G$.

Proof. As $N_S(L)$ normalizes $M \cap N$ and permutes the L_i $(2 \leq i \leq n)$, M^d is a subgroup with $N_S(L) \leq M^d \leq H$. By the definition, K^u is a subgroup containing S and, as $K = N_S(L)(K \cap L)$ we have $K \leq K^u$ and $N_S(L) \leq N_S(K \cap L)$. On the other hand, $1 \neq S \cap L = N_S(L) \cap L \leq K \cap L$ and $L \cap L_i = 1$ for $i \geq 2$, so $N_S(K \cap L) \leq N_S(L)$. Thus, $N_S(K \cap L) = N_S(L)$ and, consequently, $(K \cap L)^S = \{(K \cap L)^{s_i} \mid 1 \leq i \leq n\}$. Now,

$$\begin{aligned} S \cap N &= (S \cap L_1)\cdots(S \cap L_n) \\ &= (S \cap L)^{s_1}\cdots(S \cap L)^{s_n} \\ &\leq \langle (K \cap L)^S \rangle, \end{aligned}$$

so as $K^u \cap N = (S \cap N)\langle (K \cap L)^S \rangle$,

$$K^u \cap N = (K \cap L)^{s_1} \times \cdots \times (K \cap L)^{s_n}.$$

Hence

$$L \cap (K^u \cap N)L_2 \cdots L_n = K \cap L$$

and so

$$(K^u)^d = N_S(L)(K \cap L) = K.$$

Also, if $K \neq H$, then $K \cap L \neq L$, $K^u \cap N \neq N$, and so $K^u \neq G$. Let $x = x_1 \cdots x_n \in M \cap N$ with $x_i \in L_i$ $(1 \leq i \leq n)$. Then

$$x_i \in L_i \cap (M \cap N)L_1 \cdots L_{i-1}L_{i+1} \cdots L_n$$

$$= (L \cap (M \cap N)L_2 \cdots L_n)^{s_i}$$
$$\leq (M^d \cap L)^{s_i}.$$

Therefore, $M \cap N \leq \langle (M^d \cap L)^S \rangle$ and $M = S(M \cap N) \leq (M^d)^u$.

2.8. Under Hypothesis 2.6, if $L = \langle (S \cap L)^L \rangle$, then the mappings d and u induce bijections between the sets $N(S, G)$ and $N(N_S(L), H)$, each being the inverse mapping of the other.

Proof. Let $M \in N(S, G)$. If $M^d = H$, then $L \leq (M \cap N)L_2 \cdots L_n$, so $L = \langle (S \cap L)^L \rangle \leq \langle (S \cap L)^{M \cap N} \rangle \leq M$, and hence $G = S \langle L^S \rangle = M$, a contradiction. Thus, $M^d \neq H$ and we can take $K \in N(M^d, H)$. Since $M \leq M^{du} \leq K^u \neq G$ by 2.7, we have $M = M^{du}$ and $M^{du} = K^u$. By (3) of 2.7, the latter equality yields that $M^d = K$ and thus $M^d \in N(N_S(L), H)$. Conversely, let $K \in N(N_S(L), H)$. Recall that $K^u \neq G$ and take $M \in N(K^u, G)$. Then $K = K^{ud} \leq M^d < H$, so $K = M^d$ and $K^u = M^{du} = M$. Thus, $K^u \in N(S, G)$. We have shown that d and u induce mappings between $N(S, G)$ and $N(N_S(L), H)$ and that they each are the inverse mapping of the other. This completes the proof.

2.9. Under Hypothesis 2.6, if $[N, D'(S, G)] \neq 1$ then the intersection

$$\cap\, O_2(K) \quad (K \in C'(N_S(L), H) \text{ and } N_K(S \cap L) \neq K)$$

does not centralize L.

Proof. Let $K \in C'(N_S(L), H)$ and assume $N_K(S \cap L) \neq K$. Define $M = K^u$. Then $M = S(M \cap N)$, $M \cap N = (K \cap L)^{s_1} \times \cdots \times (K \cap L)^{s_n}$, and $(K \cap L)^S = \{(K \cap L)^{s_i} \mid 1 \leq i \leq n\}$ by 2.7. Thus, Hypothesis 2.6 is satisfied by M and $K \cap L$ in places of G and L, respectively. Moreover, since K is $N_S(L)$-irreducible and $N_K(S \cap L) \neq K$, it follows that $K = N_S(L) \langle (S \cap L)^K \rangle$, and hence

$K \cap L = \langle (S \cap L)^K \rangle = \langle (S \cap K \cap L)^{K \cap L} \rangle$. Thus, we can apply 2.8 to M and $K \cap L$. As $N_S(K \cap L)(K \cap L) = K$ by 2.7, 2.8 shows that there is a bijection between $N(S, M)$ and $N(N_S(L), K)$. Therefore, $M \in C'(S, G)$. Now,

$$[K \cap L, O_2(M)] \leq [M \cap N, O_2(M)] \leq O_2(M \cap N).$$

Hence

$$\langle (K \cap L)^{O_2(M)} \rangle \leq (K \cap L)O_2(M \cap N)$$

and so

$$\langle O^2(K \cap L)^{O_2(M)} \rangle \leq K \cap L \leq L.$$

Now, $N_K(S \cap L) \neq K$ and so $O^2(K \cap L) \neq 1$. Thus, $O_2(M) \leq N_S(L) \leq K$ and, since $K \leq M$ by 2.7, we conclude that $O_2(M) \leq O_2(K)$. Therefore, $D'(S, G) \leq O_2(K)$. Since K is arbitrary and $N = \langle L^S \rangle$, the assertion follows.

2.10. Let G be a finite group with $O_2(G) = 1$ and assume that $D'(S, G) \neq 1$ for $S \in Syl_2(G)$. Then G has a quasisimple component L such that, for $A = N_S(L)L/C_{N_S(L)L}(L/Z(L))$ and $T \in Syl_2(A)$, the intersection

$$\cap O_2(X) \quad (X \in C'(T, A) \text{ and } N_X(T \cap F^*(A)) \neq X)$$

is not the identity: that is, A is a large almost simple group.

Proof. Let $*$ denote the natural homomorphism of G onto $G/O(G)$. Then

$$[E(G)^*, D'(S^*, S^*E(G)^*)] \neq 1$$

by 2.4, and so by 2.5 applied to $S^*E(G)^*$, there exists a quasisimple component L of G such that

$$[\langle (L^*)^{S^*} \rangle, D'(S^*, S^* \langle (L^*)^{S^*} \rangle)] \neq 1.$$

Now, define $H = N_S(L)L$ and observe that $N_{S^*}(L^*)L^* = H^*$

because $N_G(LO(G)) = N_G(L)$. Then, by 2.9 applied to $S^* \langle (L^*)^{S^*} \rangle$, we have that the intersection

$$\cap O_2(X) \quad (X \in C'(N_{S^*}(L^*), H^*) \text{ and } N_X(S^* \cap L^*) \neq X)$$

does not centralize L^*. Since $C_{H^*}(L^*) = O_2(H^*)$, we conclude that for $B = H^*/C_{H^*}(L^*)$ and $T \in Syl_2(B)$, the intersection

$$\cap O_2(X) \quad (X \in C'(T, B) \text{ and } N_X(T \cap F^*(B)) \neq X)$$

is not the identity. Now, $A \cong B$ and thus we have proved 2.10.

Now, let G be a finite group and $S \in Syl_2(G)$. If $(X_1, X_2) \in E(S, G)$ and $\langle X_1, X_2 \rangle \leq Y \in L(S, G)$, then we can apply 2.10 to $Y/O_2(Y)$ and easily find out a large quasisimple component of $Y/O_2(Y)$. Thus, we have proved 1.2.

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