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# CHARACTERIZATIONS OF NORMAL APPROXIMATE SPECTRA

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1. Introduction. In [4], one of the authors introduced a normal spectrum of an operator in a C*-algebra **A**. An operator $T$ is a normal topological divisor of zero if there is a sequence $\{A_n\}$ of operators in **A** such as $\|A_n\| = 1$,

$$\|TA_n\| \longrightarrow 0 \quad \text{and} \quad \|T^*A_n\| \longrightarrow 0.$$

The set $\sigma_n(T)$ of all $z$'s such that $T - z$ is a normal topological divisor of zero is called here the normal spectrum of $T$.

Such a kind of normality of spectra of operators is introduced by several authors [2,3,5] independently about ten years ago, cf. also [1]. A scalar $z$ is called a normal approximate propervalue of $T$ if one of the following conditions is satisfied:

(i) There is a sequence $\{x_n\}$ of unit vectors such as

$$\|(T - z)x_n\| \longrightarrow 0 \quad \text{and} \quad \|(T - z)^*x_n\| \longrightarrow 0.$$

(ii) There is no $s > 0$ such as

$$(T - z)^*(T - z) + (T - z)(T - z)^* \geq s.$$

(iii) There is a character $\varphi$ of the C*-algebra $C^*(T)$ generated by $T$ and the identity such as $z = \varphi(T)$.

All normal approximate propervalues of $T$ form a compact set $\pi_n(T)$, which is called the normal approximate spectrum of $T$ by [1]. By (ii) and (iii), it is clear that $\pi_n(T)$ is purely algebraic, which is determined within $C^*(T)$. So we have the following problems: (1) Is the normal spectrum purely algebraic ? (2) Are there any relations between $\sigma_n(T)$ and $\pi_n(T)$ ? (They are not discussed in [4].)

In this note, we shall give a solution to the above problems as follows: They are just the same (considering a C*-algebra acts faithfully on a Hilbert space). As an application, one can give a C*-algebraic proof to the reciprocity stated above in (iii).

2. Normal spectra. Now we shall give a C*-algebraic characterization of the normal approximate spectrum.

Theorem. <u>The normal spectrum is nothing but the normal approximate spectrum</u>: $\sigma_n(T) = \pi_n(T)$.

<u>Proof</u>. First note that $0 \in \sigma_n(T)$ if and only if there is a sequence $\{A_n\}$ of positive operators in $\mathbf{A}$ such that $\|A_n\| = 1$,

$$TA_n \longrightarrow 0 \quad \text{and} \quad T^*A_n \longrightarrow 0.$$

Suppose that $0 \in \pi_n(T)$. Since $T^*T + TT^*$ is not invertible by (ii), there is a sequence $\{A_n\}$ of positive operators in $\mathbf{A}$ such that $\|A_n\| = 1$,

$$(T^*T + TT^*)A_n \longrightarrow 0.$$

Since $A_n(T^*T + TT^*)A_n \longrightarrow 0$, we have

$$A_nT^*TA_n \longrightarrow 0 \quad \text{and} \quad A_nTT^*A_n \longrightarrow 0,$$

or equivalently

$$TA_n \longrightarrow 0 \quad \text{and} \quad T^*A_n \longrightarrow 0.$$

Conversely, assume that $0 \notin \pi_n(T)$, i.e., $T^*T + TT^* \geq s$ for some $s > 0$. For any $B \geq 0$ with $\|B\| = 1$, since

$$BT^*TB + BTT^*B \geq sB^2,$$

it follows that

$$\|TB\|^2 + \|T^*B\|^2 \geq s,$$

which implies that $0 \notin \sigma_n(T)$.

3. Applications. In this section, we shall give another proofs to the following characterizations of normal approximate propervalues.

Corollary 1. *For $T$ in a unital $C^*$-algebra $\mathbf{A}$, a scalar $z$ belongs to $\pi_n(T)$ if and only if the left ideal generated by $T$ and $T^*$ is proper in $\mathbf{A}$, i.e.,*

$$\mathbf{A}(T - z) + \mathbf{A}(T - z)^* \neq \mathbf{A}.$$

*Proof.* Suppose that $0 \in \pi_n(T) = \sigma_n(T)$. Then there is a sequence $\{B_n\}$ in $\mathbf{A}$ such $\|B_n\| = 1$, $TB_n \longrightarrow 0$ and $T^*B_n \longrightarrow 0$. If there exist $A$ and $B$ in $\mathbf{A}$ such that $AT + BT^* = 1$, then

$$1 = \|B_n\| = \|ATB_n + BT^*B_n\| \leq \|A\|\,\|TB_n\| + \|B\|\,\|T^*B_n\| \longrightarrow 0.$$

This is a contradiction.

Conversely, if $0 \notin \pi_n(T)$, then $T^*T + TT^*$ is invertible. Since $(AT^*)T + (AT)T^* = 1$ for some $A$ in $\mathbf{A}$, it follows that $\mathbf{A}T + \mathbf{A}T^* = \mathbf{A}$.

Finally we shall give a simple proof to the following reciprocity;

Corollary 2. *For $T$ in a unital $C^*$-algebra, a scalar $z$ belongs to $\pi_n(T)$ if and only if there is a character $\varphi$ on $C^*(T)$ such as $\varphi(T) = z$.*

<u>Proof</u>. If $0 \in \pi_n(T)$, then there exists $\{B_n\}$ in $C^*(T)$ such that $TB_n \longrightarrow 0$, $T^*B_n \longrightarrow 0$ and $\|B_n\| = 1$. Since $\|B_n\| = 1$, there is a state $f_n$ on $C^*(T)$ such that $f_n(B_n{}^*B_n) = 1$. So we define a state $\varphi$ by the composition of a Banach limit Lim on $l^\infty$;

$$\varphi(A) = \mathrm{Lim}\, f_n(B_n{}^*AB_n) \qquad \text{for } A \in C^*(T).$$

Since $\varphi(p(T, T^*)) = p(0, 0)$ for any non-commutative polynomial p on T and $T^*$, a state $\varphi$ is a character with $\varphi(T) = 0$.

Conversely, if there is a character $\varphi$ such that $\varphi(T) = 0$, then $\varphi(C^*(T)T + C^*(T)T^*) = 0$. Therefore we have $C^*(T)T + C^*(T)T^* \neq C^*(T)$, which implies that $0 \in \pi_n(T)$ by Corollary 1.

Remark. In the final part of the above, Corollary 1 is not necessary, cf. [1; I, Theorem 1]: If $\varphi(T) = 0$ for some character $\varphi$, then

$$\varphi(T^*T + TT^*) = 0.$$

Since $\varphi(1) = 1$, it is impossible that there is $s > 0$ such as $T^*T + TT^* \geqq s$.

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