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On disjoint ordered pairs of operators

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We obtain sufficient conditions for the nonexistence of non-zero intertwining operators between two nonnormal operators. We say that such pair of operators is disjoint.

1. Let $B(X)$ be the algebra of all bounded linear operators on a complex Banach space X. For $A \in B(X)$ and for a closed set $\sigma \subset \mathbb{C}$, let

$$X_A(\sigma) = \{x \in X \ ; \ (zI-A)f(z) \equiv x \text{ for some analytic function } f : \mathbb{C}\setminus\sigma \to X\}$$

and for an arbitrary $\sigma \subset \mathbb{C}$, let

$$X_A(\sigma) = \cup\{X_A(\tau) \ ; \ \tau \subset \sigma \text{ and } \tau \text{ is closed}\}.$$

The set $X_A(\sigma)$ is called the spectral manifold of A. It is known that $X_A(\sigma)$ is an invariant linear manifold of A and that if $\sigma_1 \subset \sigma_2$ then $X_A(\sigma_1) \subset X_A(\sigma_2)$. And it is clear that $X_A(\sigma) = X_A(\sigma \cap \sigma(A))$, $X_A(\sigma(A)) = X$, $X_A(\emptyset) = \{0\}$ and that $X_A(\sigma) \subset \bigcap_{z \notin \sigma}(A-zI)X$ for any closed set $\sigma \subset \mathbb{C}$.

Clancey [1] proved the following

<u>Proposition.</u> Let T on H be a hyponormal operator (i. e., $T^*T \geq TT^*$) and $\sigma \subset \mathbb{C}$ be a closed set, then $X_T(\sigma) = \bigcap_{z \notin \sigma}(T-zI)H$ and, in particular, $\bigcap_{z \in \sigma(T)} (T-zI)H = \bigcap_{z \in \mathbb{C}}(T-zI)H = X_T(\emptyset) = \{0\}$ where $\sigma(T)$ denotes the spectrum of T.

<u>Corollary 1.</u> If T on H is a subnormal operator with the minimal normal extension N on K, then $\bigcap_{z \in \sigma(N)} (T-zI)H = \{0\}$.

The following theorem is a slite modification of [6].

<u>Theorem 1.</u> For $\varphi \in H^\infty$, let T_φ be the analytic Toeplitz operator on H^2 defined by the relation $(T_\varphi f)(z) = \varphi(z)f(z)$. Then $\bigcap_{z \in \delta}(T_\varphi - \varphi(z)I)H^2 = \{0\}$ if δ is an infinite set having a limit point inside $\{z \in \mathbb{C} \ ; \ |z| = 1\}$.

2. For $A \in B(X)$ and $B \in B(Y)$, we shall say that the ordered pair (A, B) is disjoint if the only bounded linear operator C mapping X into Y and satisfying the equation CA = BC (i. e., C intertwines A and B) is zero.

<u>Lemma 1.</u> If CA = BC for $C \in B(X,Y)$, then $CX_A(\sigma) \subset X_B(\sigma)$ for

an arbitrary set $\sigma \subset \mathbb{C}$. In particular $CX \subset X_B(\sigma(A))$.

Then, we have only to seek such $\sigma \subset \mathbb{C}$ as $X_A(\sigma)^\sim = X$ and $X_B(\sigma) = \{0\}$, in particular, we may prove $X_B(\sigma(A)) = \{0\}$ in order to show that the pair (A,B) is disjoint.

The following theorem is well known. But we give here a simple proof.

Theorem 2. [7] If $\sigma(A) \cap \sigma(B) = \emptyset$, then the pair (A, B) is disjoint.

Proof. $X_B(\sigma(A)) = X_B(\sigma(A) \cap \sigma(B)) = X_B(\emptyset) = \{0\}$.

Theorem 3. Let T be a subnormal operator on H with the minimal normal extension N on K. If $\sigma(A) \cap \sigma(N) = \emptyset$, then the pair (A, T) is disjoint.

Proof. By the assumption, there is an open set D such that $\sigma(N) \subset D$ and $\sigma(A) \cap D = \emptyset$. Then $X_T(\sigma(A)) \subset X_T(\mathbb{C} \setminus D) = \bigcap_{z \in D} (T-zI)H \subset \bigcap_{z \in \sigma(N)} (T-zI)H = \{0\}$ by Proposition and by Corollary 1.

Theorem 4. Let T_φ be an analytic Toeplitz operator on H^2. If $\sigma(T_\varphi) \not\subset \sigma(A)$, then the pair (A, T_φ) is disjoint.

Proof. It is known that $\sigma(T_\varphi)$ is the closure of $\{\varphi(z); |z|<1\}$. Let $\tau = \{\varphi(z); |z| < 1\} \cap [\mathbb{C} \setminus \sigma(A)]$, then τ is either a non-empty open set or a singleton, depending on whether $\{\varphi(z); |z|<1\}$ is an open set or a singleton (that is, whether φ is non-constant or constant). In either case, $\delta = \varphi^{-1}(\tau) \cap \{z \varepsilon \mathbb{C} ; |z|<1\}$ is a non-empty open subset in $\{z\varepsilon\mathbb{C} ; |z|<1\}$ and hence, by the assumptio there is an open set D such that $\tau \subset D$ and that $\sigma(A) \cap D = \emptyset$. Then $X_{T_\varphi}(\sigma(A)) \subset X_{T_\varphi}(\mathbb{C} \setminus D) = \bigcap_{z\varepsilon D} (T_\varphi - zI)H^2$ by Proposition $\subset \bigcap_{z\varepsilon\tau} (T_\varphi - zI)H^2 = \bigcap_{z\varepsilon\delta} (T_\varphi - \varphi(z)I)H^2 = \{0\}$ by Theorem 1.

Corollary 2. [4] Let T_φ, T_ψ be two analytic Toeplitz operators on H^2. If $\{\psi(z) ; |z|<1\} \not\subset \sigma(T_\varphi)$, then the pair (T_φ, T_ψ) is disjoint.

Let A and B are bounded linear operators on two Hilbert spaces H and K respectively.

Lemma 2. [2] Let CA = BC for C ε $B(H, K)$. If C has dense range and if B is hyponormal, then $\sigma(B) \subset \sigma(A)$.

If $CA = BC$ for $C \in B(H, K)$ implies that $CA^* = B^*C$, then $\ker[C]^\perp$ and $\mathrm{range}[C]^\sim$ are reducing subspaces for A and B respectively and it is easily seen that $A|\ker[C]^\perp$ and $B|\mathrm{range}[C]^\sim$ are normal and hence $\sigma(A|\ker[C]^\perp) = \sigma(B|\mathrm{range}[C]^\sim)$ by Lemma 2.

$A \in \mathbf{B}(H)$ is dominant if there is a number M_λ for each $\lambda \in \mathbb{C}$ such that $\|(A-\lambda I)^* x\| \leqq M_\lambda \|(A-\lambda I)x\|$ for all $x \in H$. If there is a constant M such that $M_\lambda \leqq M$ for all $\lambda \in \mathbb{C}$, A is called M-hyponormal and if $M = 1$, A is hyponormal.

<u>Theorem 5.</u> Let $A^* \in B(H)$ be M-hyponormal and let $B \in \mathbf{B}(K)$ be dominant. If $\sigma(A^{(n)}) \cap \sigma(B^{(n)}) = \emptyset$, then the pair (A, B) is disjoint, where $A^{(n)}$ denotes the normal part of A.

<u>Proof.</u> By [8], $CA = BC$ for $C \in \mathbf{B}(H, K)$ implies that $CA^* = B^*C$ and hence, by the arguments after Lemma 2, we have

$$\sigma(B|\mathrm{range}[C]^\sim) = \sigma(A|\ker[C]^\perp) \cap \sigma(B|\mathrm{range}[C]^\sim) \subset \sigma(A^{(n)}) \cap \sigma(B^{(n)}) = \emptyset$$

and $C = 0$.

$A \in \mathbf{B}(H)$ is paranormal if $\|Ax\|^2 \leqq \|A^2x\| \|x\|$ for all $x \in H$.

If $A^* \in \mathbf{B}(H)$ be an isometry and if $B \in \mathbf{B}(K)$ be a paranormal contraction (i. e., $\|B\| \leqq 1$), then, by [5], it is easily seen that $CA = BC$ for $C \in B(H, K)$ implies $CA^* = B^*C$. And then $A|\ker[C]^\perp$ and $B|\mathrm{range}[C]^\sim$ are unitary because $\sigma(B|\mathrm{range}[C]^\sim) = \sigma(A|\ker[C]^\perp) \subset \sigma(A^{(n)}) = \sigma(A^{(u)})$ by the arguments after Lemma 2, where $A^{(u)}$ denotes the unitary part of A. And hence we have

<u>Theorem 6.</u> Let $A^* \in B(H)$ be an isometry and let $B \in B(K)$ be a paranormal contraction. If $\sigma(A^{(u)}) \cap \sigma(B^{(u)}) = \emptyset$, then the pair (A, B) is disjoint.

<u>Proof.</u> $\sigma(B|\mathrm{range}[C]^\sim) = \sigma(A|\ker[C]^\perp) \cap \sigma(B|\mathrm{range}[C]^\sim)$ $\subset \sigma(A^{(u)}) \cap \sigma(B^{(u)}) = \emptyset$ and $C = 0$.

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