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PERTURBATION BOUNDS FOR EIGENVALUES : A SURVEY

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This talk is addressed to the following problem, which besides being of theoretical interest is of considerable practical importance : if we know how close two matrices are then how far apart can their eigenvalues be? Various aspects of this problem have been studied by pure mathematicians, physicists and numerical analysts. We will give a survey of results on global variational inequalities for eigenvalues.

Definition: Let A, B be two nxn complex matrices. Enumerate their eigenvalues with multiplicity as Eig A = $\{\alpha_1,\ldots,\alpha_n\}$ and Eig B = $\{\beta_1,\ldots,\beta_n\}$ respectively. The <u>optimal matching distance</u> between the eigenvalues of A and the eigenvalues of B is defined as

$$d(\text{Eig } A, \text{Eig } B) = \min_{\sigma} \max_{1\le i\le n} |\alpha_i - \beta_{\sigma(i)}|$$

where σ runs over all permutations of n indices.

Notice that if d(Eig A, Eig B) = δ then Eig B can be obtained from Eig A by moving each point by a distance at most δ. Further, δ is the smallest number with this property.

Let $||A||$ denote the operator bound norm of A.

Problem I: If $||A-B||$ is known what can be said about d(Eig A, Eig B)?

Some prominent results on this are summarised below.

Theorem 1: (H.Weyl, 1912) If A, B are Hermitian then

$$d(\text{Eig } A, \text{Eig } B) \le ||A-B||$$

Theorem 2: (P.Lax, 1958) If all real linear combination of A and B have only real eigenvalues then

$$d(\text{Eig } A, \text{Eig } B) \leq ||A-B||$$

Theorem 3: (R.Bhatia and C.Davis, 1984) If A, B are unitary then

$$d(\text{Eig } A, \text{Eig } B) \leq ||A-B||.$$

Problem II: It is a long standing conjecture that the inequality $d(\text{Eig } A, \text{Eig } B) \leq ||A-B||$ holds for normal matrices A and B. This remains an open problem. Theorems 1 and 3 deal with the most important special subclasses of normal matrices. In addition to these results, the above conjecture has been proved for the following special cases:

(i) when A is Hermitian and B is skew-Hermitian (V.S.Sunder, 1982)

(ii) when A,B are constant multiples of unitaries (R.Bhatia and J.Holbrook, 1985)

(iii) when A,B and A-B are all normal (R.Bhatia, 1982).

Something weaker than the conjecture is the content of

Theorem 4: (R.Bhatia, C.Davis and A.McIntosh, 1983) There exists a universal constant c such that for all normal matrices A,B

$$d(\text{Eig } A, \text{Eig } B) \leq c\ ||A-B||.$$

Remark: It has been shown by P.Koosis *et al* that the constant c in the above theorem is bounded by 2.91.

When A,B are arbitrary nxn matrices the inequalities are not as simple. Several results on this problem have been obtained by A.Ostrowski (1957), P.Henrici (1962), R.Bhatia and K.Mukherjea (1979), R.Bhatia and S.Friedland(1981) and L.Elsner (1982). A sample result is

Theorem 5: If A,B are nxn complex matrices then

$$d(\mathrm{Eig}\ A, \mathrm{Eig}\ B) \leq n^{1+1/n}(2M)^{1-1/n}||A-B||^{1/n}$$

where $M = \max(||A||, ||B||)$.

Remark: The exponent $1/n$ of $||A-B||$ in the above inequality has long been known to be essential. Dimensional considerations then show that the factor $M^{1-1/n}$ is also essential. Some minor improvements of the other factors in the above inequality are known. But it is yet an open problem to find the best constants in the above inequality.

As Theorems 1-4 above show that the result of Theorem 5 can be improved when A,B lie in some special classes of matrices. It would be interesting to find more large and natural classes of matrices for which the result of Theorem 5 can be improved.

We now consider some different metrics on the space of matrices. We say that $|||\cdot|||$ defines a unitarily-invariant norm on matrices if for all A and for all unitary U and V we have

$$|||A||| = |||UA\ V|||.$$

Examples of such norms are given by the operator norm $||A||$, the Hilbert-Schmidt norm or the Frobenius norm defined by

$$||A||_F = \Big(\sum_{i,j} |a_{ij}|^2\Big)^{1/2} = (\mathrm{tr}\ A^*A)^{1/2},$$

where tr denotes the trace of a matrix and a_{ij} are the entries of the nxn matrix A. More examples are given by the Schatten p norms, $1 \leq p \leq \infty$. (See Schatten's book quoted in the references).

These norms are of theoretical as well as practical interest (e.g. a numerical analyst may prefer to work with a more easily computable norm).

Given a unitarily invariant norm $|||.|||$ define a distance between Eig A and Eig B as

$$|||(\text{Eig } A, \text{ Eig } B)||| = \min_{\sigma} |||\text{diag}(\alpha_1 - \beta_{\sigma(1)}, \ldots, \alpha_n - \beta_{\sigma(n)})|||,$$

where σ runs over all permutations and $\text{diag}(x_1, \ldots, x_n)$ denotes the diagonal matrix with diagonal entries $x_1, \ldots, x_n$.

Note that in our earlier notation

$$d(\text{Eig } A, \text{ Eig } B) = ||(\text{Eig } A, \text{ Eig } B)||$$

where $||.||$ is the operator norm.

Problem III. If $|||A-B|||$ is known for some unitarily-invariant norm, what can be said about $|||(\text{Eig } A, \text{ Eig } B)|||$?

The following theorem, which is a significant generalisation of Theorem 1, follows from a theorem of Lidskii (1950) and seems to have been first noted by Mirsky (1960).

Theorem 6. If A,B are Hermitian then for every unitarily-invariant norm

$$|||(\text{Eig } A, \text{ Eig } B)||| \leq |||A-B|||.$$

Remark: It has been shown by R.Bhatia (1982) that the above result holds, more generally, when A,B and A-B are normal. However, it fails in the case of arbitrary normal matrices. It fails even when A,B are unitary matrices.

For unitary matrices the best result is given by

Theorem 7. (R.Bhatia, C.Davis, A.McIntosh, 1983) If A,B are unitary matrices then for every unitarily-invariant norm

$$|||(\text{Eig } A, \text{ Eig } B)||| \leq \frac{\pi}{2} |||A-B|||.$$

For arbitrary normal matrices the following result holds

Theorem 8. (A.J.Hoffman and H.W.Wielandt, 1953). If A,B are normal matrices then

$$||(\text{Eig } A, \text{ Eig } B)||_F \leq ||A-B||_F$$

where $||A||_F$ denotes the Frobenius norm of A.

Recently T.Ando and R.Bhatia have obtained some results for some special normal matrices and for the family of Schatten p norms.

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