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On automorphic cuspidal representations of U(2,2)

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Introduction

In this paper, we study the hypercuspidality of automorphic cuspidal representations of U(2,2).

The hypercuspidality in the case of the symplectic group was introduced by I. I. Piatetski-Shapiro [5]. When $G = GSp_4$, for a given cusp form f on G_A, f is called "hypercuspidal" if the Whittaker function corresponding to f vanishes. Let $L_0^2(G_A)$ be the space of cusp forms on G_A. We denote by $L_{0,1}^2(G_A)$ the orthogonal complement of the space of all hypercuspidal forms in $L_0^2(G_A)$. Then any irreducible cuspidal representation in $L_{0,1}^2$ has a unique non-trivial Whittaker model. Thus, the multiplicity one theorem holds for $L_{0,1}^2$.

Analogously, we define the hypercuspidality in the case of U(2,2) by vanishing of some Whittaker functions occuring in the Fourier expansion of a cusp form. More precisely, for a cusp form f on U(2,2), we consider a Fourier expansion of f with respect to the center of the maximal unipotent subgroup of the Borel subgroup. Then we obtain two Whittaker functions W_f and V_f occuring in the Fourier expansion, where W_f is an ordinary Whittaker function and V_f is defined in §1. We note that in the case of Sp_4, the function V_f did not appear in the smilar Fourier expansion of a cusp form f. In terms of these functions, we say f is "U-cuspidal" (resp. "N-cuspidal") if W_f (resp. V_f) vanishes. Moreover, if both function W_f and function V_f vanish, f is called "hypercuspidal".

Next, using the dual reductive pair, we investigate cuspidal representations obtained from the Weil-lifting of those of U(1,1) or U(2,1). Roughly speaking, we have the following:

(1) Cuspidal representations obtained from the Weil-lifting of those of U(1,1) are U-cuspidal.

(2) Let τ be a cuspidal representation of U(2,1). Let $\Theta(\tau,\psi)$ be a cuspidal representation obtained from the Weil-lifting of τ. Then,

(a) if τ is non-hypercuspidal in a sense of [1], then $\Theta(\tau,\psi)$ is N-cuspidal, and

(b) if τ is hypercuspidal in a sense of [1], then $\Theta(\tau,\psi)$ is hypercuspidal.

The details of proof will be given in my Master thesis at Tôhoku University.

Notation

Let F be a global field whose characteristic is different from 2 and let $\mathbb{A}_F$ be the adele ring of F. Let E be a quadratic extension of F, and denote its Galois involution by $x \to \bar{x}$. We fix once and for all an element i in E such that $\bar{i} = -i$ and a non trivial character ψ of $\mathbb{A}_F/F$.

1. Fourier expansions and the hypercuspidality

In this section, we give a definition of the hypercuspidality for cusp forms on U(2,2).

Let V be a 4-dimensional vector space over E with basis $\{e_1, e_2, e_3, e_4\}$, and $(\ ,\ )_V$ the skew-hermitian form on V which is represented by the matrix $\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$ with respect to $\{e_1, e_2, e_3, e_4\}$. Let

$$G_F = \{\ g \in GL_4(E)\ |\ g\begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}\ \}$$

and

$$H_F = \{\ h \in GL_2(E)\ |\ h\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} {}^t\bar{h} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \}.$$

Let B_F be the Borel subgroup of G_F such that its maximal torus is

$$T_F = \{ \begin{pmatrix} a & & & \\ & b & & \\ & & \bar{a}^{-1} & \\ & & & \bar{b}^{-1} \end{pmatrix} |\ a,b \text{ in } E^*\ \},$$

and its unipotent radical is

$$U_F = \{ \begin{pmatrix} 1 & a & x-\bar{a}b & b \\ 0 & 1 & \bar{b}-\bar{a}y & y \\ & & 1 & 0 \\ & 0 & -\bar{a} & 1 \end{pmatrix}\ |\ a,b \text{ in } E,\ x,y \text{ in } F\ \}.$$

Let P_F be the parabolic subgroup stabilizing the isotropic line Ee Then P_F is the product $L_F N_F$ of the Levi subgroup

$$L_F = \{ \begin{pmatrix} a' & & & \\ & a & & b \\ & & \bar{a}'^{-1} & \\ & c & & d \end{pmatrix}\ |\ a' \text{ in } E^*,\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ in } H_F\ \},$$

and the unipotent radical

$$N_F = \left\{ \begin{pmatrix} 1 & a & x-\bar{a}b & b \\ 0 & 1 & \bar{b} & 0 \\ & & 1 & 0 \\ & 0 & -\bar{a} & 1 \end{pmatrix} \middle| \; a,b \text{ in } E,\ x \text{ in } F \right\}.$$

Let Z_F be the center of U_F:

$$Z_F = \left\{ \begin{pmatrix} I_2 & \begin{matrix} x & 0 \\ 0 & 0 \end{matrix} \\ 0 & I_2 \end{pmatrix} \middle| \; x \text{ in } F \right\}.$$

For each ξ, ζ in E and t in F, we define characters $\psi_{(\xi,t)}$, $\psi_{(\xi,\zeta)}$ and ψ_t of $U_F\backslash U_{\mathbb{A}}$, $N_F\backslash N_{\mathbb{A}}$ and $Z_F\backslash Z_{\mathbb{A}}$, respectively, by

$$\psi_{(\xi,t)}\left(\begin{pmatrix} 1 & a & x-\bar{a}b & b \\ 0 & 1 & \bar{b}-\bar{a}y & y \\ & & 1 & 0 \\ & 0 & -\bar{a} & 1 \end{pmatrix}\right) = \psi(Tr_{E/F}(\xi a) + ty),$$

$$\psi_{(\xi,\zeta)}\left(\begin{pmatrix} 1 & a & x-\bar{a}b & b \\ 0 & 1 & \bar{b} & 0 \\ & & 1 & 0 \\ & 0 & -\bar{a} & 1 \end{pmatrix}\right) = \psi(Tr_{E/F}(\xi a + \zeta b))$$

and

$$\psi_t\left(\begin{pmatrix} I_2 & \begin{matrix} x & 0 \\ 0 & 0 \end{matrix} \\ 0 & I_2 \end{pmatrix}\right) = \psi(tx).$$

Further we put $E^1 = \{ a \in E^* \mid a\bar{a} = 1\}$ and $\mathbb{A}_E^1 = \{ a \in \mathbb{A}_E^* \mid a\bar{a} = 1\}$. Then the center $C(G_{\mathbb{A}})$ of $G_{\mathbb{A}}$ is isomorphic to $\mathbb{A}_E^1$. For a character χ of $E^1\backslash\mathbb{A}_E^1$, let $\mathcal{A}_0(G_{\mathbb{A}})_\chi$ denote the space consisting of cusp forms on $G_{\mathbb{A}}$ which transform according to χ under $C(G_{\mathbb{A}})$. For each cusp form f on $G_{\mathbb{A}}$, we define three Whittaker functions corresponding to f by

$$W_f^{\psi_{(\xi,t)}}(g) = \int_{U_F\backslash U_{\mathbb{A}}} \overline{\psi_{(\xi,t)}(u)} f(ug)du,$$

$$V_f^{\psi(\xi,\zeta)}(g) = \int_{N_F\backslash N_{\mathbb{A}}} \overline{\psi_{(\xi,\zeta)}(n)} f(ng)dn$$

and

$$J_f^{\psi_t}(g) = \int_{Z_F\backslash Z_{\mathbb{A}}} \overline{\psi_t(z)} f(zg)dz.$$

First, for a cusp form f on $G_{\mathbb{A}}$, we consider a Fourier expansion of f along Z. Fix g in $G_{\mathbb{A}}$. As a function on the compact abelian group $Z_F\backslash Z_{\mathbb{A}}$, f(zg) has a Fourier expansion of the form

$$f(g) = \int_{Z_F\backslash Z_{\mathbb{A}}} f(zg)dz + \sum_{t\in F^*} J_f^{t}(g).$$

Let [F*] (resp. [E*]) be a complete set of representatives of $N_{E/F}(E^*)$ (resp. E^1) in $F^*/N_{E/F}(E^*)$ (resp. E^*/E^1). Then by the analogy to [4] Lemma 6.2, we obtain the following:

<u>Proposition 1.</u> <u>For each cusp form f on</u> $G_{\mathbb{A}}$, <u>one has</u>

$$f(g) = \sum_{t\in[F^*]} \Big\{ \sum_{\gamma\in R_F\backslash L_F} W_f^{\psi(1,t)}(\gamma g) + \sum_{\gamma\in L(1,ti)\backslash L_F} V_f^{\psi(1,ti)}(\gamma g) +$$

$$+ \sum_{a\in[E^*]} J_f^{\psi_t}\left(\begin{pmatrix} a & & & \\ & 1 & & \\ & & \bar{a}^{-1} & \\ & & & 1 \end{pmatrix} g\right)\Big\},$$

<u>where</u>

$$R_F = \left\{ \begin{pmatrix} a & & & \\ & a & & ab \\ & & a & \\ & & & a \end{pmatrix} \;\middle|\; a \text{ in } E^1,\ b \text{ in } F \right\}$$

<u>and</u>

$$L(1,ti) = \left\{ \begin{pmatrix} a' & & & \\ & a & & (a'-d)t^{-1}i^{-1} \\ & & \bar{a}'^{-1} & \\ & (a'-d)ti & & d \end{pmatrix} \in L_F \right\}.$$

Now let

$$W(\psi) = \{ (W_f^{\psi}(1,t))_{t\in[F^*]} \mid f \in \mathcal{A}_0(G_{\mathbb{A}})_\chi \}$$

and

$$V(\psi) = \{ (V_f^{\psi}(1,ti))_{t\in[F^*]} \mid f \in \mathcal{A}_0(G_{\mathbb{A}})_\chi \}.$$

We define a linear map D from $\mathcal{A}_0(G_{\mathbb{A}})_\chi$ to $W(\psi)\oplus V(\psi)$ by

$$D(f) = ((W_f^{\psi}(1,t))_t, (V_f^{\psi}(1,ti))_t).$$

In terms of this linear map, we give the following

Definition. Let f be a cusp form on $G_{\mathbb{A}}$. We say f is N-*cuspidal* (resp. U-*cuspidal*) if f is contained in $D^{-1}(W(\psi))$ (resp, $D^{-1}(V(\psi))$). Further we say f is *hypercuspidal* if f is contained in Ker(D).

We can show that these spaces are invariant by the action of the Hecke algebra of $G_{\mathbb{A}}$ and independent of a choice of a character ψ and a representative set $[F^*]$.

2. Lifting from U(1,1) to U(2,2)

In this section, we consider the Weil-lifting $\Theta(\tau,\psi)$ of an irreducible automorphic cuspidal representation τ of $H_{\mathbb{A}}$ to $G_{\mathbb{A}}$, and investigate the cuspidality of $\Theta(\tau,\psi)$.

Let W be a 2-dimensional vector space over E, $(\ ,\ )_W$ the skew-hermitian form on W which is represented by the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ with respect to a suitable basis. We consider the symplectic space $X_F = (V\otimes W)_F$ obtained by taking the imaginary part of the hermitian form $(\ ,\ )_W \cdot (\ ,\ )_V$. Thus X_F is a 16-dimensional space over F, and we have a dual reductive pair $(H,G) \subset Sp_{16}(F)$.

In the same manner as in [1], §6 and §8, we choose and fix one Weil-representation ω_ψ of $G_{\mathbb{A}}H_{\mathbb{A}}$. Let $X_F = X_1 \oplus X_2$ be a complete poralization of X_F and $S(X_{1,\mathbb{A}})$ the Schwarz - Bruhat space on $X_{1,\mathbb{A}}$.

Now suppose (τ, V_τ) is an automorphic cuspidal representation of $H_{\mathbb{A}}$ in the space of cusp forms on $H_{\mathbb{A}}$. For each ϕ in V_τ and Φ in $S(X_{1,\mathbb{A}})$, we put

$$\Theta_\psi^\Phi(g,h) = \sum_{v \in X_{1,F}} \omega_\psi(gh)\Phi(v) \qquad (h \in H_A,\ g \in G_A),$$

$$f_\phi^\Phi(g) = \int_{H_F \backslash H_A} \Theta_\psi^\Phi(g,h)\phi(h)dh.$$

We call the representation of G_A realized on

$$\Theta(\tau,\psi) = \{ f_\phi^\Phi \mid \phi \text{ in } V_\tau,\ \Phi \text{ in } S(X_{1,A})\}$$

the "Weil-lifting" of τ.

We define an embedding $H_A \hookrightarrow Sp_8(\mathbb{A}_F)$ by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} \alpha(a) & 0 & 0 & \beta(b) \\ 0 & \alpha(a) & -\beta(b) & 0 \\ 0 & \gamma(c) & \delta(d) & 0 \\ -\gamma(c) & 0 & 0 & \delta(d) \end{pmatrix},$$

where for any x in E

$$\alpha(x) = \begin{pmatrix} Re(x) & Im(x) \\ -N_{E/F}(i)Im(x) & Re(x) \end{pmatrix}, \quad \beta(x) = \begin{pmatrix} Im(x) & -Re(x) \\ Re(x) & N_{E/F}(i)Im(x) \end{pmatrix},$$

$$\gamma(x) = \begin{pmatrix} N_{E/F}(i)Im(x) & -Re(x) \\ Re(x) & Im(x) \end{pmatrix}, \quad \delta(x) = \begin{pmatrix} Re(x) & N_{E/F}(i)Im(x) \\ -Im(x) & Re(x) \end{pmatrix}.$$

According to this embedding, the Weil-representation ω_ψ° of $Sp_8(\mathbb{A}_F)$ can be restricted to SU(1,1). Furthermore, in the same manner as in [1], it can be extended to an ordinary representation ω_ψ° of $H_{\mathbb{A}}$. This extension is determined only up to twisting by a character of $\mathbb{A}_E^1$ composed with the determinant. Therefore we choose one such extension ω_ψ° in accordance with the choice of the ordinary representation ω_ψ of $H_{\mathbb{A}}G_{\mathbb{A}}$. Then the Weil-representation ω_ψ° can be realized on the Schwarz - Bruhat space $S(W_{\mathbb{A}})$ of $W_{\mathbb{A}}$. Hence, for each $\Phi \in S(W_{\mathbb{A}})$, we put

$$\Theta_\Phi(h) = \sum_{w \in W_F} \omega_\psi^\circ(h)\Phi(w)$$

and denote by $\Theta(\psi,\chi^{-1})$ the space consisting of theta-series Θ_Φ which transform according to χ^{-1} under the center of $H_{\mathbb{A}}$.

Theorem 2. Let (τ, V_τ) be an irreducible cuspidal representation of H_A in $\mathcal{A}_0(H_A)_\chi$.

(1) If τ is non-trivial, then $\Theta(\tau,\psi)$ is also non-trivial.

(2) $\Theta(\tau,\psi)$ is cuspidal if and only if τ is orthogonal to $\Theta(\psi,\chi^{-1})$.

(3) If $\Theta(\tau,\psi)$ is cuspidal and non-trivial, then it is U-cuspidal, but not hypercuspidal.

3. Lifting from U(2,1) to U(2,2)

We use the similar argument as in §2.

Let W be a 3-dimensional vector space over E with a basis $\{w_{-1}, w_0, w_1\}$ and $(\ ,\ )_W$ the hermitian form which is represented by the matrix

$$\begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

with respect to $\{w_{-1}, w_0, w_1\}$. Let G° be the corresponding unitary group, and N° the maximal unipotent subgroup of G°:

$$N_F^\circ = \left\{ \begin{pmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,z \text{ in } E,\ z + \bar{z} = -a\bar{a} \right\}.$$

In the same manner as in §2, we have a dual reductive pair $(G,G^\circ)\subset Sp_{24}(F)$. Further, for an irreducible cuspidal representation (τ,V_τ) of $G^\circ_{\mathbb{A}}$, we denote by $\Theta(\tau,\psi)$ the Weil-lifting of it.

For the general theory of cusp forms on $G^\circ_{\mathbb{A}}$, we refer to [1]. We define a character ψ_o of $N^\circ_F\backslash N^\circ_{\mathbb{A}}$ by

$$\psi_o\left(\begin{pmatrix} 1 & a & z \\ 0 & 1 & -\bar{a} \\ 0 & 0 & 1 \end{pmatrix}\right) = \psi(Tr_{E/F}(a)).$$

For $\phi \in L^2_0(G^\circ_F\backslash G^\circ_{\mathbb{A}})$, we put

$$W_\phi^{\psi_o}(g) = \int_{N^\circ_F\backslash N^\circ_{\mathbb{A}}} \overline{\psi_o(n)}\phi(ng)dn.$$

Also we put

$$L^2_{0,0}(G^\circ_F\backslash G^\circ_{\mathbb{A}}) = \{\ \phi \in L^2_0(G^\circ_F\backslash G^\circ_{\mathbb{A}})\ \mid\ W_\phi^{\psi_o} \equiv 0\ \},$$

$$L^2_{0,1}(G^\circ_F\backslash G^\circ_{\mathbb{A}}) = \text{the orthocomplement of } L^2_{0,0} \text{ in } L^2_0.$$

These spaces are invariant under $G^\circ_{\mathbb{A}}$ and independent of ψ. Clearly, we have an orthogonal decomposition

$$L^2_0(G^\circ_F\backslash G^\circ_{\mathbb{A}}) = L^2_{0,0}(G^\circ_F\backslash G^\circ_{\mathbb{A}}) \oplus L^2_{0,1}(G^\circ_F\backslash G^\circ_{\mathbb{A}}).$$

We know from [1] that the multiplicity one theorem holds for $L^2_{0,1}$.

Now for each x in F^*, we take a vector w_x in W such that $(w_x,w_x)_W = x$, and let $G^\circ_{x,F}$ be the stabilizer of w_x in G°_F. Then we obtain a following

Proposition 3. $\Theta(\tau,\psi)$ *is cuspidal if and only if*

$$\int_{G^\circ_{x,F}\backslash G^\circ_{x,\mathbb{A}}} \phi(gh)dg = 0$$

for any x *in* F^*, ϕ *in* V_τ *and* h *in* $G^\circ_{\mathbb{A}}$.

In paticular, if we take $w_x = \frac{1}{2}w_{-1} + xw_1$, then for any x in F*

$$G^{\circ}_{x,F} \supset \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a \text{ in } E^1 \right\}.$$

Thus if V_τ satisfies the condition

$$(\#) \quad \int_{E^1\backslash \mathbb{A}^1_E} \phi\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} g\right) da = 0 \quad \text{for any } \phi \text{ in } V_\tau \text{ and } g \text{ in } G^{\circ}_{\mathbb{A}},$$

then $\Theta(\tau,\psi)$ is cuspidal.

<u>Theorem 4</u>. (1) <u>Suppose</u> $(\tau,V_\tau) \subset L^2_{0,1}(G^{\circ}_F\backslash G^{\circ}_{\mathbb{A}})$. <u>If τ is non-trivial</u>, <u>then</u> $\Theta(\tau,\psi)$ <u>is also non-trivial</u>. <u>Moreover</u>, <u>if</u> V_τ <u>satisfies the condition (#)</u>, <u>then</u> $\Theta(\tau,\psi)$ <u>is N-cuspidal</u>, <u>but not hypercuspidal</u>.
(2) <u>Suppose</u> $(\tau,V_\tau) \subset L^2_{0,0}(G^{\circ}_F\backslash G^{\circ}_{\mathbb{A}})$. <u>If</u> V_τ <u>satisfies the condition (#)</u>, <u>then</u> $\Theta(\tau,\psi)$ <u>is hypercuspidal</u>.

In the proof, we take a complete poralization of $X_F = (V\otimes W)_F$ by $X_F = X_1 \oplus X_2$, where $X_1 = e_1\otimes W + e_2\otimes W$ and $X_2 = e_3\otimes W + e_4\otimes W$. Under this decomposition of X_F, we can give explicitly the action of the Weil-representation ω_ψ of $G_{\mathbb{A}}G^{\circ}_{\mathbb{A}}$ to Schwarz - Bruhat space $S(X_{1,\mathbb{A}}) \simeq S(W_{\mathbb{A}} \oplus W_{\mathbb{A}})$. In the case (1), we put $f = f^{\Phi}_{\phi} \in \Theta(\tau,\psi)$, where $\Phi \in S(W_{\mathbb{A}} \oplus W_{\mathbb{A}})$ and $\phi \in V_\tau$. Then by computing W^{ψ}_f directly, we have

$$W_f^{\psi(1,\frac{1}{2}t)} \equiv 0 \quad \text{for } 1 \neq t \in [F^*]$$

$$W_f^{\psi(1,\frac{1}{2})}(g) = \int_{Z^{\circ}_{\mathbb{A}}\backslash G^{\circ}_{\mathbb{A}}} \omega_\psi(gh)\Phi(w_1,w_0)W_{\phi}^{\psi_0}(h)dh,$$

where Z° is the center of N°. In paticular, the latter formula defines the "local Weil-lifting" of a non-degenerate admissible representation of $G^{\circ}_{F_v}$ to G_{F_v}.

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