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Author(s)	Nogura, Tsugunori
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Problems for convergence properties

愛媛大(理学部) 野倉嗣紀 (Tsugunori Nogura)

§. 1. Definitions. All spaces are assumed to be  $T_3$  and  $T_1$  topological spaces. A space  $X$  is said to be bi-sequential if, whenever  $\mathcal{F}$  is a filter in  $X$  with a cluster point  $x$ , then there exists a countable filter base  $\mathcal{A}$  in  $X$  which converges to  $x$  and whose elements intersect all elements of  $\mathcal{F}$ . If the definition of bi-sequential space is modified by restricting  $\mathcal{F}$  to be a countable filter base, the resulting concept is said to be strongly Fréchet. A space  $X$  is said to be Fréchet if  $x \in \bar{A}$  for  $A \subset X$ , then there exists a sequence in  $A$  converging to the point  $x$ .

Let  $X$  be a space. A collection  $\mathcal{A}$  of convergent sequences of  $X$  is said to be a sheaf in  $X$  if all member of  $\mathcal{A}$  converge to the same point of  $X$ , which is said to be the vertex of the sheaf  $\mathcal{A}$ . In this paper all sheaves are assumed to be countable. We

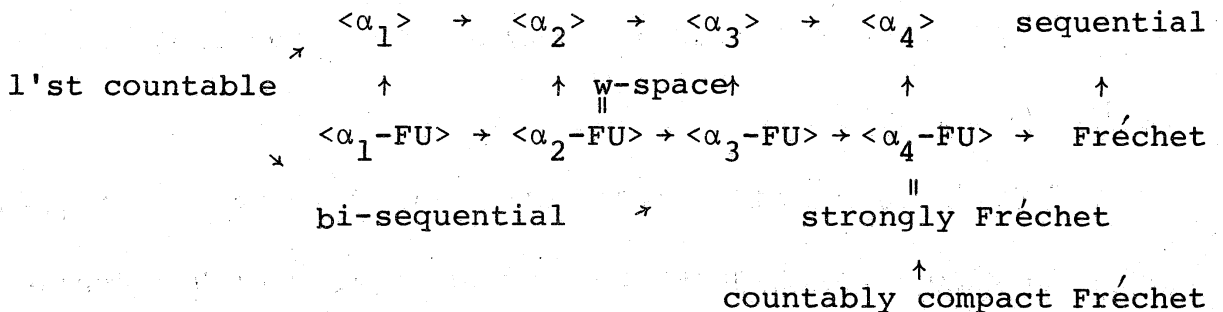
consider the following four properties of  $X$  which were introduced by Arhangel'skii [1,2].

Let  $\mathcal{A}$  be a sheaf in  $X$  with the vertex  $x \in X$ . Then there exists a sequence  $B$  converging to  $x$  such that:

- ( $\alpha_1$ )  $|A - B| < \aleph_0$  for  $A \in \mathcal{A}$ ,
- ( $\alpha_2$ )  $|A \cap B| = \aleph_0$  for  $A \in \mathcal{A}$ ,
- ( $\alpha_3$ )  $|\{A \in \mathcal{A} : |A \cap B| = \aleph_0\}| = \aleph_0$ ,
- ( $\alpha_4$ )  $|\{A \in \mathcal{A} : A \cap B \neq \emptyset\}| = \aleph_0$ .

The class of spaces satisfying the property ( $\alpha_i$ ) for every sheaf  $\mathcal{A}$  and vertex  $x \in X$  is denoted by  $\langle \alpha_i \rangle$  for  $i = 1, 2, 3, 4$ . We denote by  $\langle \alpha_i \text{-FU} \rangle$  the intersection of the class of Fréchet spaces and the class  $\langle \alpha_i \rangle$  for  $i = 1, 2, 3, 4$ .

The following diagram shows the relationship between the above spaces and other spaces.



§. 2. Classification problems.

Problem 2-1. Is there a "naive" countable  $\langle \alpha_1 \text{-FU} \rangle$ -space which is not first countable?

Problem 2-2. Is there a "naive"  $\langle \alpha_2 \rangle$ -space which is not an  $\langle \alpha_1 \rangle$ -space?

Remark. Olson's example [see 5, Introduction] is an  $\langle \alpha_2 \text{-FU} \rangle$ -space, and not first countable, so in every model of set theory it solves either Problem 1 or 2. If we omit "countable" in Problem 1, then we get an example. In fact  $\Sigma$ -product of more than countable number of first countable spaces is such a space [2, 6.16].

Problem 2-3. Is there a "naive"  $\langle \alpha_3 \rangle$ -space which is not an  $\langle \alpha_2 \rangle$ -space?

Remark. If we assume (CH), then there exists an  $\langle \alpha_3 \rangle$ -space which is not an  $\langle \alpha_2 \rangle$ -space.

§. 3. Product problems for Fréchet spaces.

Let  $P$  be a class of spaces. Let  $\mathcal{F}(P) = \{X: X \times Y \text{ is Fréchet for any } Y \in P\}$ .

We use the following notations:

$C$  = the class of compact Fréchet spaces,

$CC$  = the class of countably compact Fréchet spaces,

$B$  = the class of bi-sequential spaces,

$S$  = the class of strongly Fréchet spaces

Problem 3-1. Is  $\mathcal{F}(C) = \mathcal{F}(CC)$  ?

Problem 3-2. Give inner characterizations of classes  $\mathcal{F}(C)$ ,  $\mathcal{F}(CC)$  and  $\mathcal{F}(S)$ .

Remark.  $\langle \alpha_3\text{-FU} \rangle \subset \mathcal{F}(CC) \subset \mathcal{F}(C) \subset S$  and  $B \subset \mathcal{F}(S) \subset \mathcal{F}(CC)$ .

Note that  $\mathcal{F}(B) = S$ . If we assume (CH), then  $\langle \alpha_3\text{-FU} \rangle \subsetneq \mathcal{F}(CC)$  and  $\mathcal{F}(S) \subsetneq \mathcal{F}(CC)$ .

Problem 3-3. Is there a "naive" example of  $\mathcal{F}(CC)$ -space which is not bi-sequential?

Problem 3-4. Is there a "naive" example of  $\mathcal{F}(CC)$ -space which is not an  $\langle \alpha_3\text{-FU} \rangle$ -space?

Problem 3-4. Is  $B = \mathcal{F}(S)$ ?

§. 4. Miscellaneous problems.

Problem 4-1(Arhangelskii). Is  $t(X^2) = t(X)$  for each countably compact regular space  $X$ ?

Problem 4-2(Arhangelskii). Give an inner characterization of subsequential spaces (A space is said to be subsequential if it can be embedded in a sequential space.).

Problem 4-3(Gerlits-Nagy). Let  $X$  and  $Y$  be  $G$ -spaces.

Is  $t(X \times Y) \leq \aleph_1$ ? A space  $X$  is said to be  $G$ -space if each countable subspace is first countable and  $X$  has countable

tightness.

A space is said to be quasi-prime if  $X$  is embedded as a closed subset of  $\prod_{i \in \mathbb{N}} Y_i$ , then there exists an  $n \in \mathbb{N}$  such that  $X$  is embedded in  $\prod_{i=1}^n Y_i$ , where  $Y_i$ ,  $i \in \mathbb{N}$  are arbitrary spaces. For example  $\mathbb{N} \cup \{p\}$  ( $p \in \mathbb{N}^*$ ) is quasi-prime [3].

Problem 4-3. Let  $Q$  be the space of rationals. Is  $\beta Q$  quasi-prime?

Problem 4-4. Is  $Q$  quasi-prime?

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