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A sufficient condition for a bipartite graph to have a k-factor.

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In this paper, we consider only finite undirected simple graphs. A graph denoted by $(X,Y;E)$ is a bipartite graph with partite sets X and Y and edge set $E \subset X \times Y$. If A is a subset of vertices, $N(A)$ denotes the set of vertices adjacent to one of the vertices of A. For two disjoint subsets of vertices A and B, $e(A,B)$ denotes the number of the edges joining A and B. A vertex x is often identified with $\{x\}$. So $e(x,B)$ means $e(\{x\},B)$ and $N(x)$ means $N(\{x\})$. The other notations may be found in [1].

A k-regular spanning subgraph is called a *k-factor*. In a bipartite graph $(X,Y;E)$, *a complete k-matching from X to Y* is defined as a spanning subgraph such that the degree of each vertex of X is k, and the degree of each vertex of Y is at most k. We abbreviate the complete 1-matching from X to Y as *a complete matching from X to Y.*

Theorem A (Hall[2]). *A bipartite graph $(X,Y;E)$ has a complete matching from X to Y if and only if $|N(S)| \geq |S|$ holds for all $S \subset X$.*

The next theorem, first proved by Ore and Ryser, gives a necessary and sufficient condition for a bipartite graph $(X,Y;E)$ to have a complete k-matching from X to Y. Now, for $S \subset X$ and $T \subset Y$, we define

$$\delta(S,T) := e(S,Y-T) + k\,|T| - k\,|S|.$$

Theorem B (Ore, Ryser[4]). *A bipartite graph $(X,Y;E)$ has a complete k-matching from X to Y if and only if $\delta(S,T) \geq 0$ holds for all $S \subset X$ and all $T \subset Y$.*

In this paper, we give a sufficient condition for the existence of a complete k-matching in a bipartite graph, which is an extension of Hall's theorem (Theorem A). Katerinis proved the following theorem.

Theorem C (Katerinis[3]). *If a bipartite graph $(X,Y;E)$ satisfies (C.1) and (C.2), then $(X,Y;E)$ has a 2-factor.*

(C.1) $|X| = |Y| \geq 2$.

(C.2) *For all $M \subset X$,*

$$|N(M)| \geq \frac{3}{2}|M| \qquad \text{if } |M| < \left\lfloor \frac{2}{3}|Y| \right\rfloor,$$

$$|N(M)| = |Y| \qquad \text{if } |M| \geq \left\lfloor \frac{2}{3}|Y| \right\rfloor.$$

As a generalization of Theorem C, we give our main result in this paper.

Theorem 1. *Suppose $k \geq 2$. If a bipartite graph $(X,Y;E)$ satisfies (1.1), (1.2) and (1.3), then $(X,Y;E)$ has a complete k-matching from X to Y.*

(1.1) $|X| \leq |Y|$, $|Y| \geq k$.

(1.2) *For every $M \subset X$ satisfying $|M| < \left\lfloor (k-1+\frac{1}{k})^{-1}|Y| \right\rfloor$, $|N(M)| \geq (k-1+\frac{1}{k})|M|$ holds.*

(1.3) *For every $M \subset X$ satisfying $|M| \geq \left\lfloor (k-1+\frac{1}{k})^{-1}|Y| \right\rfloor$, $|N(M)| = |Y|$ holds.*

In case of $|X| = |Y|$, a complete k-matching from X to Y is equivalent to a

k-factor. Therefore, Theorem 1 also gives a sufficient condition on the existence of a k-factor. Hence, in case of $k=2$, Theorem 1 implies Theorem C. Moreover, if we apply Theorem 1 to the case of $k=1$, then we have the non-trivial implication of Hall's theorem.

The next theorem is slightly stronger than Theorem 1. Hence, we prove Theorem 2 instead of Theorem 1.

Theorem 2. *Suppose $k \geq 2$. If a bipartite graph $(X,Y;E)$ satisfies (2.1), (2.2) and (2.3), then $(X,Y;E)$ has a complete k matching from X to Y.*

(2.1) $|X| \leq |Y|$, $|Y| \geq k$.

(2.2) *For every $M \subset Y$ satisfying $|M| < \left\lceil (k-1+\frac{1}{k})^{-1} |Y| \right\rceil$ and $|M| \equiv 1$ (modulo k), $|N(M)| \geq (k-1+\frac{1}{k}) |M|$.*

(2.3) *For every $M \subset Y$ satisfying $|M| \geq \left\lceil (k-1+\frac{1}{k})^{-1} |Y| \right\rceil$, $|N(M)| = |Y|$ holds.*

Before proving Theorem 2, we state the following lemma.

Lemma 3. *Let $k \geq 2$ be an integer, and $G = (X,Y;E)$ be a bipartite graph satisfying $|X| \leq |Y|$ and $|Y| \geq k$. Suppose there exist $S \subset X$ and $T \subset Y$ such that $\delta(S,T) < 0$. If we choose such S and T so that $S \cup (Y-T)$ is minimal, then (3.1), (3.2), (3.3) and (3.4) hold.*

(3.1) *For any vertex x of S, $e(x, Y-T) \leq k-1$ holds. Therefore $e(S, Y-T) \leq (k-1)|S|$ holds.*

(3.2) *For any vertex y of $Y-T$, $e(S,y) \leq k-1$ holds.*

(3.3) $|N(S)| < (k-1+\frac{1}{k}) |S|$.

(3.4) *There exists a subset M of S such that $|M| \equiv 1$ (modulo k) and $|N(M)| < \left\lceil (k-1+\frac{1}{k}) |M| \right\rceil$ holds.*

Proof. Suppose there exists a vertex x of S such that $e(x, Y-T) \geq k$. Let $S' := S - \{x\}$. Then

$$\begin{aligned}\delta(S',T) &= k\,|T| + e(S',Y-T) - k\,|S'| \\ &\leq k\,|T| + e(S,Y-T) - k - k\,|S| + k \\ &= \delta(S,T) < 0.\end{aligned}$$

This contradicts the minimality of $S \cup (Y-T)$. Thus we obtain (3.1).

Similarly, suppose there exists $y \in Y-T$ such that $e(S,y) \geq k$, and let $T' := T \cup \{y\}$. Then $\delta(S,T') \leq \delta(S,T) < 0$, contradicting the minimality of $S \cup (Y-T)$, and (3.2) follows.

Since G is a bipartite graph, $|N(S)| \leq |T| + e(S,Y-T)$. By the fact that $\delta(S,T) < 0$ and (3.1),

$$\begin{aligned}|N(S)| &\leq |T| + e(S,Y-T) \\ &< |S| + (1-\frac{1}{k})e(S,Y-T) \\ &\leq |S| + (1-\frac{1}{k})(k-1)|S| \\ &= (k-1+\frac{1}{k})|S|.\end{aligned}$$

Thus (3.3) is obtained.

If $|S| \equiv 1$ (*modulo* k), then immediately (3.4) holds. By the fact that $\delta(S,T) < 0$, $S \neq \phi$. Hence let $|S| \equiv 1 + r$ (*modulo* k) where $1 \leq r \leq k-1$ and R be a subset of S such that $e(R, Y-T)$ is maximum over $|R| = r$ and $R \subset S$. Let $d := \min\{e(x,Y-T); x \in R\}$ and $M := S - R$. Then we have

$$\begin{aligned}&e(x,Y-T) \geq d \quad \textit{for all } x \in R \\ &e(x',Y-T) \leq d \quad \textit{for all } x' \in M\end{aligned}$$

and by (3.1), $d \leq k-1$. On the other hand, $|N(M)| \leq |T| + e(M,Y-T)$. Therefore,

$$\begin{aligned}|N(M)| &\leq |T| + e(M,Y-T) \\ &< |S| - \frac{1}{k}e(S,Y-T) + e(M,Y-T) \\ &= (1-\frac{1}{k})e(M,Y-T) - \frac{1}{k}e(R,Y-T) + |S|\end{aligned}$$

$$\le (1-\frac{1}{k})d\,|M| - \frac{1}{k}d\,|R| + |S|$$
$$= \frac{d}{k}\{(k-1)|M| - |R|\} + |S|$$
$$\le \frac{(k-1)^2}{k}|M| - \frac{k-1}{k}|R| + |M| + |R|$$
$$= (k-1+\frac{1}{k})|M| + \frac{1}{k}|R|$$
$$\le \left\lfloor (k-1+\frac{1}{k})|M| \right\rfloor$$

and (3.4) follows. ▪

Proof of Theorem 2. We assume that $(X,Y;E)$ has no complete k-matching from X to Y. By Theorem B, there exist $S\subset X$ and $T\subset Y$ satisfying $\delta(S,T) < 0$. We may assume that the situations of (3.1) – (3.4) occur. By the assumption (2.2) and (2.3), we have $N(M) = Y$. Since $M\subset S$, also we have $N(S) = Y$.

Let $z := \min\{e(S,y); y\in Y-T\}$. Note that $Y-T\neq\phi$, by the assumption that $\delta(S,T) < 0$. Hence z is well-defined, and $1 \le z \le k-1$, by (3.2) and the fact that $|N(S)|=Y$. The neighborhood of S, that is Y, is at most $|T| + \frac{1}{z}e(S,Y-T)$. Hence by (3.1) and the fact that $\delta(S,T) < 0$,

$$|Y| = |N(S)| \le |T| + \frac{1}{z}e(S,Y-T)$$
$$< |S| + (\frac{1}{z}-\frac{1}{k})e(S,Y-T)$$
$$\le |S| + (\frac{1}{z}-\frac{1}{k})(k-1)|S|$$
$$= \frac{k^2-k+z}{kz}|S|. \qquad (1)$$

Let y_0 be a vertex of $Y-T$ such that $e(S,y_0) = z$, and let $S_0 := S - N(y_0)$. Since $N(S_0)\subset Y - \{y_0\}$, S_0 cannot satisfy the conclusion of (2.3). Hence

$$|S_0| = |S| - z \le \frac{k}{k^2-k+1}|Y| - 1. \qquad (2)$$

By (1) and (2),

$$k(k-1)(z-1)|S| < z(z-1)(k^2-k+1)$$
$$|S| < (k-1+\frac{1}{k})\cdot\frac{z}{k-1}$$

$$\leq k-1+\frac{1}{k}.$$

Thus $|S|<k$ and therefore $|M|=1$. But (3.4) and the fact that $N(M)=Y$ imply that $|Y|=|N(M)|<\left\lceil k-1+\frac{1}{k}\right\rceil=k$. This contradicts the assumption that $|Y|\geq k$. ▪

Theorem 2 is in some sense best possible. The graph $(X,Y;E)$ defined as the following shows that the condition $|N(M)|\geq(k-1+\frac{1}{k})|M|$ of (2.2) cannot be replaced by $|N(M)|\geq\left\lceil(k-1+\frac{1}{k})|M|\right\rceil-1$ (see Fig. 1).

$X:=A\cup A'$
  where $A=\{a_1,\cdots,a_{km+1}\}$
  $A'=\{a_{km+2},\cdots,a_n\}$ $n\geq(k^2+k+1)m+2k-1.$
$Y:=B\cup C\cup D$
  where $B=\{b_{ij}\mid 1\leq i\leq km+1,\ 1\leq j\leq k-1\}$
  $C=\{c_1,\cdots,c_m\}$
  $D=\{d_1,\cdots,d_l\}$ $(k-1)(km+1)+m+l\geq n.$
$E=\{a_ib_{ij}\mid 1\leq i\leq km+1,\ 1\leq j\leq k-1\}\cup(A\times C)\cup(A'\times Y).$

Moreover, in this graph all but one M ($=A$) satisfy (2.2).

Besides, the conditions (1.2) and (1.3) of Theorem 1 cannot be unified to the condition:

$$|N(M)|\geq\min\{|Y|,(k-1+\frac{1}{k})|M|\}\quad \textit{for all } M\subset X. \tag{1.4}$$

The graph in Fig. 2 satisfies (1.1) and (1.4) but has no complete k-matching from X to Y.

But the graphs which satisfy (1.1) and (1.4) and have no complete k-matching from X to Y have a similar induced subgraph. Finally, we prove the next theorem.

Theorem 4. *Suppose $k\geq 2$. And also suppose that $G=(X,Y;E)$ is a bipartite graph such that $|X|\leq|Y|$ and $|Y|\geq k$. If G satisfies (1.4) and G has no*

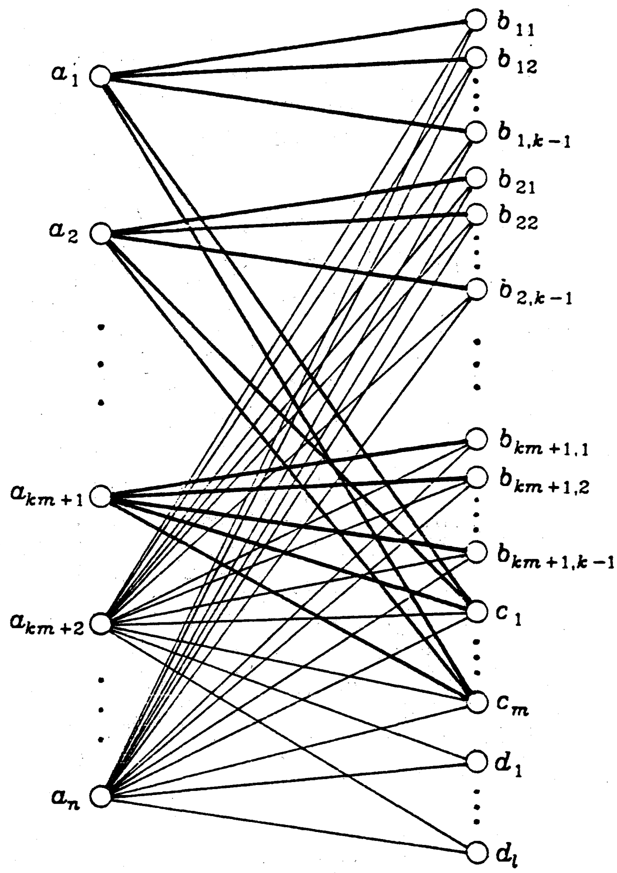

Fig. 1.

complete k-matching from X to Y, then there exist $S \subset X$ *and* $T \subset Y$ *such that (4.1), (4.2) and (4.3) hold.*

(4.1) $|S| = k|T| + 1.$

(4.2) $e(S,y) = 1$ *for all* $y \in Y-T$.

(4.3) $e(x, Y-T) = k-1$ *for all* $x \in S$.

Proof. Since G has no complete k-matching from X to Y, we may assume that we have (3.1) – (3.4). Therefore we have $N(S) = Y$. Let $z := \min\{e(S,y); y \in Y-T\}$. Since $Y-T \neq \phi$, z is well-defined, and $1 \leq z \leq k-1$. Now, we have

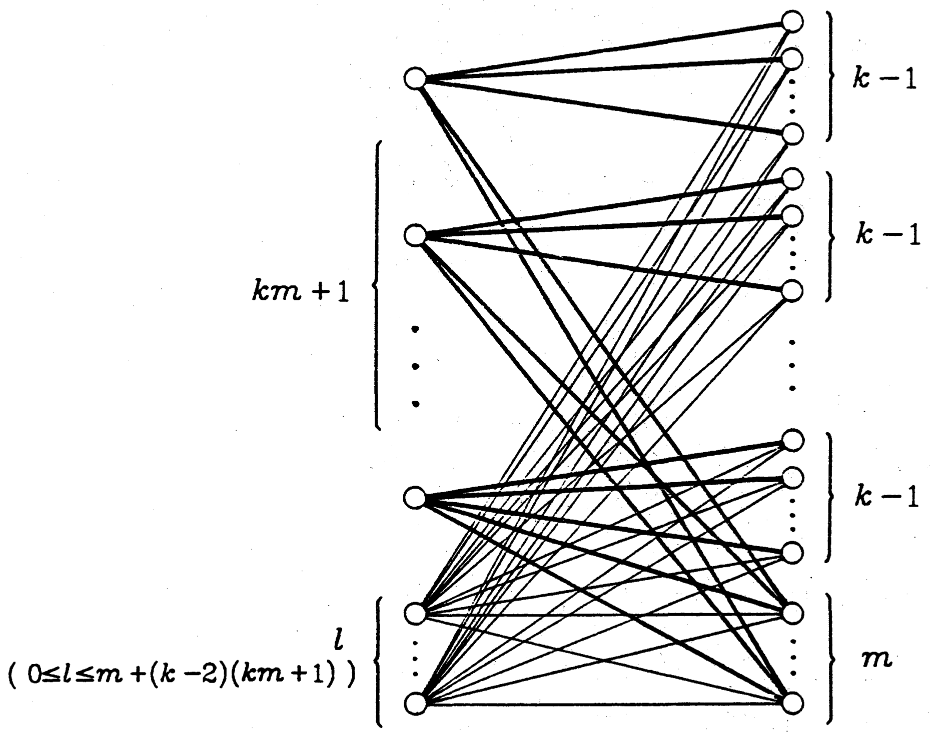

Fig. 2.

$$|Y| = |N(S)| \le |T| + \frac{1}{z} e(S, Y-T)$$
$$< \frac{k^2-k+z}{kz} |S|.$$

Let $y_0 \in Y-T$ such that $e(S, y_0) = z$, and let $S_0 := S - N(y_0)$. Since $N(S_0) \subset Y - \{y_0\}$, we have

$$|Y| - 1 \ge |N(S_0)| \ge (k-1+\frac{1}{k})|S_0| = (k-1+\frac{1}{k})(|S|-z).$$

By the above two inequalities, we have

$$\frac{k^2-k+z}{kz}|S| > |Y| \ge (k-1+\frac{1}{k})(|S|-z) + 1,$$
$$(k^2-k+z)|S| > z(k^2-k+1)(|S|-z) + kz,$$
$$z^2(k^2-k+1) - kz > (z-1)\, k\, (k-1)|S|,$$

Since $|Y| \ge k$, $|S| \ge |M| \ge k+1$. Hence

$$z^2(k^2-k+1) - kz > (z-1)k(k-1)(k+1),$$
$$z^2(k^2-k+1) - zk^3 + k(k^2-1) > 0.$$

We claim that the only situation that $z = 1$ makes this inequality true. Suppose $z \geq 2$, and let $f_k(z) := z^2(k^2-k+1) - zk^3 + k(k^2-1)$. Then, since $k \geq z+1 \geq 3$,

$$\begin{aligned} f_k(2) &= 4(k^2-k+1) - 2k^3 + k(k^2-1) \\ &= -k^3 + 4k^2 - 5k + 4 \\ &= -k(k-2)^2 - k + 4 < 0, \end{aligned}$$

and

$$\begin{aligned} f_k(k-1) &= (k-1)^2(k^2-k+1) - (k-1)k^3 + k(k^2-1) \\ &= (k-1)\{1 - (k-1)(k-2)\} < 0. \end{aligned}$$

Hence, $2 \leq z \leq k-1$ implies $f_k(z) < 0$, which is a contradiction. Thus the claim follows.

Define

$$U := \{u \in Y-T; e(S,u) = 1\},$$
$$W := \{w \in Y-T; e(S,w) \geq 2\} = Y-T-U.$$

Since $z=1$, $U \neq \phi$. We choose $u \in U$ arbitrarily, and let x_u be the only neighborhood of u in S. Now, define α, β and γ as the following non-negative integers (especially, note that $\gamma \geq 1$).

$$\alpha := \sum_{w \in W} (e(S,w) - 1),$$
$$\beta := \sum_{x \in S} (k - 1 - e(x,Y-T)),$$
$$\gamma := e(x_u, U).$$

By these definitions, we have

$$|Y| = |N(S)| = |T| + e(S,Y-T) - \alpha, \quad (3)$$
$$e(S,Y-T) = (k-1)|S| - \beta, \quad (4)$$
$$|Y| - \gamma \geq |N(S-\{x_u\})| \geq (k-1+\frac{1}{k})(|S|-1). \quad (5)$$

By the definitions of α and β,

$$\alpha \geq e(x_u, W), \quad (6)$$
$$\beta \geq k - 1 - e(x_u, Y-T). \quad (7)$$

Thus we have

$$\alpha + \beta + \gamma \geq k-1. \tag{8}$$

By (3) and the fact that $\delta(S,T) < 0$,

$$|Y| < |S| + (1-\frac{1}{k})e(S,Y-T) - \alpha.$$

And by (4),

$$|Y| < (k-1+\frac{1}{k})|S| - \alpha - (1-\frac{1}{k})\beta. \tag{9}$$

Thus with (5), we have

$$\alpha + \gamma + (k-1)(\alpha+\beta+\gamma) < k^2 - k + 1.$$

If $\alpha + \beta + \gamma \geq k$, we have $\alpha + \gamma < 1$. This contradicts the fact that $\gamma \geq 1$. Therefore in (8), hence also in (6) and (7), the equality holds.

From the equality of (7), we have $e(x, Y-T) = k-1$ for all $x \in S - \{x_u\}$. From the equality of (6), for all $w \in W$, $e(S,w) = 2$ and $x_u \in N(w) \cap S$, and hence $\alpha = |W|$.

First we claim that $W = \phi$. In case of $k = 2$, $W = \phi$ is an immediate consequence of (3.2). Thus it suffices to show the claim in case of $k \geq 3$. Assume $W \neq \phi$ and let $w_0 \in W$. Since $|S| \geq k+1 \geq 4$ and $e(S,w_0) = 2$, there exists $x_0 \in S - N(w_0)$. We note that $x_0 \neq x_u$. Because of the fact that $e(x_0, Y-T) = k-1 > \alpha = |W|$, there exists $v \in U \cap N(x_0)$. Especially $v \neq u$, and the only neighborhood of v in S, say x_v, is x_0. Hence the similar arguments lead us to the fact that $x_v \in N(w) \cap S$ for all $w \in W$. But $x_v = x_0 \not\in S \cap N(w_0)$. This is a contradiction. Thus we have the claim, and therefore (4.2) holds.

Next we prove (4.3). It suffices to show that $e(x_u, Y-T) = k-1$. Since $|S| \geq k+1 \geq 3$, we can take $x' \in S - \{x_u\}$. From the above arguments, we can regard x' as the only neighborhood in S of some vertex $v \in U = Y-T$. Then by the similar arguments, $e(x, Y-T) = k-1$ holds for all $x \in S - \{x'\}$. Especially $e(x_u, Y-T) = k-1$ holds.

The results given above show that $\alpha = \beta = 0$ and $\gamma = k-1$. Hence by (5) and (9),

$$(k-1+\frac{1}{k})|S| - \frac{1}{k} \leq |Y| < (k-1+\frac{1}{k})|S|,$$

and so

$$|Y| = (k-1)|S| + \frac{|S|-1}{k}$$

This implies (4.1), for we have $|Y-T| = (k-1)|S|$ from (4.2) and (4.3). And we complete the proof. ▪

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