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Regular subgraphs of a regular graph (※)

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1. Introduction

We consider a finite graph, which may have multiple edges but has no loops. A graph without multiple edges is called a simple graph. We denote the vertex set and the edge set of a graph G by V(G) and E(G), respectively. For a vertex v of a subgraph H, we write $d_H(v)$ for the degree of v in H. A graph G is called an r-regular graph if $d_G(x)=r$ for all $x\in V(G)$, a spanning r-regular subgraph is called an r-factor. Notation and definitions not given in this paper can be found in [1].

We shall consider the following problem. For given integers k and r, $0<k<r$, does an r-regular graph contain a k-regular subgraph ? Let us begin with some known results on this problem. The first theorem is well-known as Petersen's 2-factorization theorem.

Theorem A.(Petersen [9]) Every 2r-regular graph has a 2k-factor for every integer k, $0<k<r$.

Theorem B.(Taskinov[10]) Every (2r+1)-regular graph has a (2k+1)-regular subgraph for every integer k, $0<k<r$.

Taskinov [10] also proved that every 4-regular simple graph contains a 3-regular subgraph, which was conjectured by C. Berge. The next conjecture given by Chvatal et al [3] is stil open.

Conjecture Every 4-regular graph of even order contains a 3-regular subgraph.

In this paper we shall prove the following two theorems.

(※) 文献[10]のタイトルと一見同じであるが違う点に注意されたい。

Theorem 1. Let r be an odd integer and k be an even integer such that $2 \le k \le 2r/3$. Then every r-regular graph contains a k-regular subgraph.

Theorem 2. Let r be an even integer and k be an odd integer such that $1 \le k \le r/2$. Then every r-regular graph of even order contains a k-regular subgraph. Moreover, every r-regular graph of odd order with edge-connectivity two contains a k-regular subgraph.

Note that for every integer r, $r \ge 6$, there exists an r-regular simple graph which has no (r-1)-regular subgraphs ([7],[10]). Furthermore, for every odd integer $r>0$, there exists a 2r-regular graph of odd order that has no (r+2)-regular subgraphs. For example, such a graph is obtained from the complete graph K_3 of order 3 by replacing each edge of K_3 by r-multiple edges.

2. Proofs of Theorems

We begin by introduing some definitions. Let G be a graph, and g and f be integer-valued functions defined on V(G) such that $g(x) \le f(x)$ for all $x \in V(G)$. A spanning subgraph F of G is called a (g,f)-factor if $g(x) \le d_F(x) \le f(x)$ for all $x \in V(G)$. A (g,f)-factor satisfying $g(x)=f(x)$ for all $x \in V(G)$ is briefly called an f-factor. For a vertex subset X of G, we write G-X for the subgraph of G induced by $V(G) \setminus X$. For an edge subset Y of G, G-Y denotes the subgraph of G obtained from G by deleting all the edges in Y. For two disjoint subsets S and T of V(G), we denote by $e_G(S,T)$ the number of edges of G joining S and T. For a set $\{a,b,\dots,d\}$ of integers, a graph G is called an $\{a,b,\dots,d\}$-graph if $d_G(x) \in \{a,b,\dots,d\}$ for all $x \in V(G)$. A spanning $\{a,b,\dots,d\}$-subgraph is called an $\{a,b,\dots,d\}$-factor.

Lemma 1.(Lovasz [8]) Let G be a graph, and g and f be integer-valued functions defined on V(G) satisfying $g(x) \le f(x)$ for all $x \in V(G)$. Then G has a (g,f)-factor if and only if

$$\delta(S,T) = \sum_{t\in T}(d_G(t)-g(t)) + \sum_{s\in S} f(s) - e_G(S,T) - h(S,T) \geq 0$$

for all disjoint subsets S and T of V(G), where h(S,T) denotes the number of components C of G-(S∪T) such that g(x)=f(x) for all x∈V(G), and

$$\sum_{x\in V(C)} f(x) + e_G(T,V(C)) \equiv 1 \pmod 2.$$

Lemma 2.([5],[6]) Let G be an n-edge-connected graph (n≥1), f be an integer-valued function defined on V(G), θ be a real number such that 0≤θ≤1, and A and B be disjoint subsets of E(G). If (1), (2) and one of {(3a),(3b)} hold, then G has an f-factor F such that F⊃A and F∩B=ϕ.

(1) $\sum_{x\in V(G)} f(x) \equiv 0 \pmod 2$.

(2) $\varepsilon = \sum_{x\in V(G)} |f(x)-\theta d_G(x)| + 2\theta|A| + 2(1-\theta)|B| < 2$.

(3a) $n\theta\geq 1$ and $n(1-\theta)\geq 1$

(3b) $\{f(x) \mid x\in V(G)\}$ consists of even numbers and $m(1-\theta)\geq 1$, where $m \in\{n,n+1\}$ and $m\equiv 1 \pmod 2$.

Lemma 3. [2] Let k be an even integer and r be an odd integer such that $2\leq k\leq 2r/3$. Then every 2-edge-connected r-regular graph has a k-factor.

Note that this lemma can be obtained from Lemma 2 with A=B=ϕ, m=3, θ =k/r and f(x)=k for all x∈V(G).

Proof of Theorem 1. Let G be a connected r-regular graph. By Lemma 3, we may assume G has bridges. Then G has a block H which is a 2-edge-connected {r-1,r}-subgraph having exactly one vertex of degree r-1. Put θ=k/r, A=B=ϕ and n=2, and define a function f on V(H) by f(x)=k for all x∈V(H). We can show that H, θ and f satify conditions (1),(2) and (3b) of Lemma 2 since

$$\varepsilon = |k-(k/r)(r-1)| < 2 \quad \text{and} \quad m(1-\theta) = 3(1-k/r) \geq 1.$$

Therefore H has an f-factor, which is a desired k-regular subgraph of G.

<u>Lemma 4.</u> [4] Let k be an odd integer and r be an even integer such that $1\le r/4\le k\le 3r/4$. Then every 4-edge-connected r-regular graph of even order has a k-factor.

Note that the above lemma can be proved by Lemma 2 with $A=B=\phi$, $n=4$, $\theta=k/r$ and $f(x)=k$ for all $x\in V(G)$.

<u>Lemma 5.</u> Let k be an odd integer and r be an even integer such that $1\le r/4\le k\le 3r/4$. Let H be a 3-edge-connected graph of even order. If H is an $\{r-1,r\}$-graph having exactly two vertices of degree $r-1$, or an $\{r-2,r\}$-graph having exactly one vertex of degree $r-2$, then H has a k-factor.

<u>Proof</u> Suppose that H is a 3-edge-connected $\{r-1,r\}$-graph with exactly two vertices u and v of degree $r-1$. We construct an r-regular graph H^* from H by adding a new edge uv. Then H^* is a 4-edge-connected r-regular graph of even order. Put $A=\phi$, $B=\{uv\}$, $\theta=k/r$ and $n=4$, and define a function f by $f(x)=k$ for all $x\in V(H^*)$. Then (1),(2) and (3a) of Lemma 2 hold since $\varepsilon=2(1-\theta)|B|=2(1-k/r)<2$, $n\theta=4k/r\ge 1$ and $n(1-\theta)=4(1-k/r)\ge 1$. Hence H^* has a k-factor F which does not contain the edge uv. Therefore, F is a desired k-factor of H. If H is a $\{r-2,r\}$-graph with one vertex of degree $r-2$, then H is 4-edge-connected, and so we can prove the theorem by Lemma 2.

<u>Lemma 6.</u> Let k be an odd integer greater than 1, and let H be a 3-edge-connected graph of odd order. If H is $\{2k-1,2k\}$-graph having exactly two vertices of degree $2k-1$, or a $\{2k-2,2k\}$-graph having exactly one vertex of degree $2k-2$, then H has a $\{k-1,k\}$-factor F such that $d_F(w)=k-1$ and $d_F(x)=k$ for all $x\in V(G)\backslash w$ for any given vertex w of degree $2k-1$ or $2k-2$.

<u>Proof</u> Set $\theta=1/2$, $n=3$ and define a function f on $V(G)$ by $f(w)=k-1$ and $f(x)=k$ for all $x\in V(G)\backslash w$. Then (1),(2) and (3a) of Lemma 2 hold. Hence H has an f-factor, which is a $\{k-1,k\}$-factor with the required property.

Lemma 7. Let k be an odd integer greater than or equal to three, and G be a connected {2,2k}-graph with at least one vertex of degree 2. Then G has a {0,1,k}-factor F such that

$$d_F(x) = \begin{cases} 0 \text{ or } 1 & \text{if } d_G(x)=2 \\ k & \text{if } d_G(x)=2k. \end{cases}$$

Proof Define two functions g and f on V(G) as

$$g(x) = \begin{cases} 0 & \text{if } d_G(x)=2 \\ k & \text{otherwise,} \end{cases} \quad \text{and} \quad f(x) = \begin{cases} 1 & \text{if } d_G(x)=2 \\ k & \text{otherwise.} \end{cases}$$

We shall prove that G has a (g,f)-factor, which is a desired factor. Suppose that G has no (g,f)-factor. By Lemma 1, there exist disjoint subsets S and T of V(G) such that

$$\delta(S,T) = \sum_{t\in T}(d_G(t)-g(t)) + \sum_{s\in S} f(s) - e_G(S,T) - h(S,T) < 0$$

Among all the pairs (S',T') of disjoint subsets of V(G) such that $\delta(S',T')$ is minimum, choose a pair (S,T) so that $|S\cup T|$ is minimum. Since G has a vertex of degree 2, $\delta(\phi,\phi)=-h(\phi,\phi)=0$. Hence $S\cup T\neq\phi$. We first show that T does not contain any vertex of degree 2. Suppose that T contains a vertex u of degree 2. Let $e_1=uw_1$ and $e_2=uw_2$ be the edges of G incident with u, where we allow $w_1=w_2$. We can easily obtain

$$\delta(S,T\backslash u) = \delta(S,T) - 2 + e_G(S,u) + h(S,T) - h(S,T\backslash u)$$

We consider three cases. If $w_1,w_2\in S$, then $e_G(S,u)=2$ and $h(S,T)=h(S,T\backslash u)$, and so $\delta(S,T\ u)=\delta(S,T)$, which contradicts the choice of S and T. If $w_1\in S$ and $w_2\notin S$, then $e_G(S,u)=1$ and $h(S,T)-h(S,T\ u)\leq 1$, and so $\delta(S,T\backslash u)\leq\delta(S,T)$, a contradiction. If $w_1,w_2\notin S$, tehn $e_G(S,u)=0$ and $h(S,T\ u)- h(S,T\backslash u)\leq 2$, and so $\delta(S,T\backslash u)\leq\delta(S,T)$, a contradiction. Therefore, H has no vertex of degree 2.

Let $C_1,C_2,\ldots,C_m$ be the components of $G-(S\cup T)$ which satisfy the conditions on $h(S,T)$ in Lemma 1, in particular, $m=h(S,T)$. Since $d_G(x)=2k$ for all $x\in V(C_i)$ and

$$\sum_{x\in V(C_i)} d_G(x) = 2|E(C_i)| + e_G(S\cup T,V(C_i)),$$

we have $e_G(S\cup T,V(C_i))\equiv 0 \pmod 2$. Hence

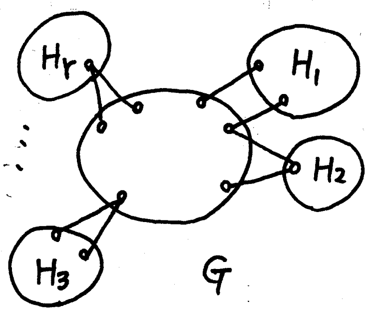
$$e_G(S\cup T,V(C_i)) \geq 2 \quad \text{for all } i,\ 1\leq i\leq m. \qquad (1)$$

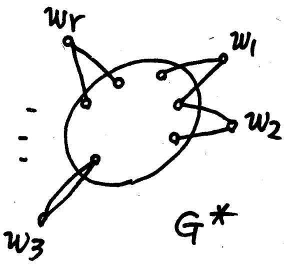
It follows that

$$\begin{aligned}\delta(S,T) &= \frac{1}{2}\sum_{t\in T} d_G(t) + \frac{1}{2}\sum_{s\in S} d_G(s) - e_G(S,T) - h(S,T)\\ &\geq \frac{1}{2}\{e_G(T,S) + \sum_{i=1}^{m} e_G(T,V(C_i))\} + \frac{1}{2}\{ e_G(S,T) + \sum_{i=1}^{m} e_G(S,V(C_i))\}\\ &\qquad - e_G(S,T) - m\\ &= \sum_{i=1}^{m}\{ \frac{1}{2} e_G(S\cup T,V(C_i)) - 1\} \geq 0 \qquad (\text{by } (1)),\end{aligned}$$

which is a required contradiction. Consequently, G has a (g,f)-factor.

Proof of Theorem 2. Let k be an odd integer greater than or equal to three, and let G be a connected 2k-regular graph. If G is 4-edge-connected and of even order, then G has a k-factor by Lemma 4. Hence we may assume that the edge-connectivity of G is two. Note that the order of G may be odd. Choose a cut $\{e,f\}$ of G $(e,f\in E(G))$ so that a component of $G-\{e,f\}$ is minimal. If $\{c,d\}$ is a cut of H, then $\{c,d\}$ is not a cut of G and $\{e,c,d\}$ is a cut of G, which contradicts the fact that G is eulerian. Thus H is 3-edge-connected. If H is of even order, then H contains a k-factor by Lemma 5. Hence we may assume that H is of odd order. Let $\{e_1,f_1\}$ $\{e_2,f_2\}$, ... ,$\{e_r,f_r\}$ be the cuts of G such that a component of H_i of $G-\{e_i,f_i\}$ is a 3-edge-connected. We obtain a $\{2,2k\}$-graph G from G by constracting each H_i to one vertex w_i (see Figure). By Lemma 6, G has a $\{0,1,k\}$-factor F such that $d_F(w_i)=0$ or 1, and $d_F(x)=k$ for all $x\in V(G)\backslash\{w_1,\ldots,w_r\}$. Without loss of generality, we may assume that $d_F(w_i)=1$





for each i, $1 \le i \le t$, and $d_F(w_j)=0$ for each j, $t<j\le r$. Let $u_1w_1, \ldots, u_tw_t$ be edges of F, and let $u_1v_1, \ldots, u_tv_t$ be edges of G such that $v_i \in V(H_i)$ for every i, $1\le i\le t$. For each i, $1\le i\le t$, we can take a $\{k-1,k\}$-factor $L(i)$ of H_i such that $d_{L(i)}(v_i)=k-1$ and $d_{L(i)}(x)=k$ for all $x\in V(H_i)\setminus v_i$ by Lemma 6. Consequently, we obtain a k-regular subgraph $F\cup L(1)\cup \ldots \cup L(t)$. The theorem follows from the above result and Theorems A and B.

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A list of results related to this paper can be found in the following survey article.
J. Akiyama and M. Kano, Factors and factorizations of graphs —— A survey, J. of Graph Theory 9 (1985) 1 - 42

On factors of regular graphs, the reader should refer to
M. Kano, Factors of regular graphs, J. Combinatorial Theory (B),to apper.