CLASSIFICATION OF SEMI-REGULAR GROUP DIVISIBLE DESIGNS

WITH $\lambda_2 = \lambda_1 + 1$ *

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Group divisible (GD) designs with parameters $v = mn, b, r, k$, $\lambda_1, \lambda_2$ satisfying $\lambda_2 = \lambda_1 + 1$ have strong statistical significance in terms of optimality. In this paper, we attempt to classify semi-regular GD designs satisfying $\lambda_2 = \lambda_1 + 1$ by expressing all the parameters in terms of at most four integral parameters. As special cases, available series of semi-regular GD designs can be derived.

1. Introduction

The largest, simplest and perhaps most important class of 2-associate partially balanced incomplete block designs is known as group divisible (GD). A GD design is an arrangement of $v$ (= mn) treatments in $b$ blocks such that each block contains $k$ (< v) distinct treatments; each treatment is replicated $r$ times; and the treatments can be divided into $m$ groups of $n$ (≥ 2) treatments each, any two treatments occurring together in $\lambda_1$ blocks if they belong to the same group, and in $\lambda_2$ blocks if they belong to different groups. For the usual incidence matrix $N$ of the GD design, $NN'$ has eigenvalues $r - \lambda_1$ (= $\theta_1$, say) and $rk - \lambda_2 v$ (= $\theta_2$, say) other than $rk$, with the respective multiplicities $m(n - 1)$ and $m - 1$.

* Supported in part by Grants 59540043(C) and 60530014(C), Japan.
Depending on values of the eigenvalues, GD designs are classified into three subtypes: (a) singular if \( \theta_1 = 0 \); (b) semi-regular (SR) if \( \theta_1 > 0 \) and \( \theta_2 = 0 \); (c) regular if \( \theta_1 > 0 \) and \( \theta_2 > 0 \).

From a well-known relation \( r(k - 1) = (n - 1)\lambda_1 + n(m - 1)\lambda_2 \), it holds that \( \theta_1 - \theta_2 = n(\lambda_2 - \lambda_1) \). Hence, if \( |\theta_1 - \theta_2| = 1 \), then any GD design does not exist. Furthermore, if \( |\theta_1 - \theta_2| \) is a prime, \( p \), say, then \( n = p \) and \( |\lambda_2 - \lambda_1| = 1 \). Note that in a singular GD design \( \lambda_1 > \lambda_2 \); in an SRGD design \( \lambda_2 > \lambda_1 \). From a point of view of statistical optimality, it is known (cf. Takeuchi [4]) that a GD design with \( \lambda_2 = \lambda_1 + 1 \) is A- and E-optimal. In the above sense, a restriction "\( \lambda_2 = \lambda_1 + 1 \)" has a special meaning on existence and optimality. We shall here consider GD designs satisfying \( |\lambda_1 - \lambda_2| = 1 \) and attempt to classify them in a closed form. The case of SRGD designs, in particular, will be considered in detail.

2. Singular and regular designs

In a singular GD design, it is known (cf. Bose and Connor [1]) that the existence of a balanced incomplete block (BIB) design with parameters \( v^*, b^*, r^*, k^*, \lambda^* \) is equivalent to the existence of a singular GD design with parameters \( v = nv^*, b = b^*, r = r^*, k = nk^*, \lambda_1 = r^*, \lambda_2 = \lambda^* \) for every \( n \). Hence a singular GD design satisfying \( \lambda_1 = \lambda_2 + 1 \) is only of the form as \( v = mn, b = m, r = m - 1, k = (m - 1)n, \lambda_1 = m - 1, \lambda_2 = m - 2 \), which can always be constructed from a trivial BIB design with parameters \( v^* = b^* = m, r^* = k^* = m - 1, \lambda^* = m - 2 \).

In a regular GD design, though there are possibilities of \( \lambda_1 - \lambda_2 = \pm 1 \), Mukerjee, Kageyama and Bhagwandas [2] characterized a regular GD design satisfying \( rk - \lambda_2 v = 1 \) and \( \lambda_2 = \lambda_1 + 1 \) as a symmetrical design whose parameters are expressed in terms of only two integral parameters. It seems to be difficult to characterize a regular GD design satisfying
\( \lambda_1 - \lambda_2 = \pm 1 \) without further restrictions on parameters.

3. Characterization of SRGD designs

The following observations will be helpful in the sequel. Consider the equation

\[
px - qy = w, \tag{1a}
\]

where \( p \) and \( q \) are relatively prime positive integers and \( w \) is a non-negative integer. Given \( p,q \) and \( w \), it is easily seen that (1a) has positive integral-valued solutions \((x,y)\). Furthermore, if \((x_1, y_1)\) and \((x_2, y_2)\) are any two distinct positive integral-valued solutions of (1a), then either \( x_1 < x_2, y_1 < y_2 \) or \( x_1 > x_2, y_1 > y_2 \). Hence there exists a solution, say \((x^*, y^*)\) of (1a), depending on \( p,q \) and \( w \), such that if \((\bar{x}, \bar{y})\) be any other solution then \( x^* < \bar{x}, y^* < \bar{y} \). The solution \((x^*, y^*)\) will be called the minimal solution of (1a). It may be seen that every positive integral-valued solution of (1a) is of the form

\[
(x^* + tq, y^* + tp) \quad (t = 0,1,2,\ldots).
\]

In particular, the minimal solution of

\[
px - qy = 1 \tag{1b}
\]

will be denoted by \((x_0, y_0)\), where, of course, \( x_0 = x_0(p,q) \) and \( y_0 = y_0(p,q) \) are functions of \( p \) and \( q \). Also, with \( x_0 \) defined as above, the minimal solution of

\[
px - qy = x_0 \tag{1c}
\]

will be denoted by \((g_0, h_0)\), where \( g_0 = g_0(p,q) \) and \( h_0 = h_0(p,q) \) are functions of \( p \) and \( q \). Since \( p \) and \( q \) are relatively prime, one has

\[
\{(qj + 1) \mod p: j = 1,2,\ldots,p\} = \{0,1,\ldots,p-1\}
\]

and hence

\[
y_0 \leq p. \tag{2}
\]

It may further be seen that \( y_0 \) and \( p \) are relatively prime.
Consider now an SRGD design with parameters \( v = mn, b, r, k, \lambda_1, \lambda_2 \), where
\[
\begin{align*}
\text{rk} - \lambda_2 v &= 0, \\
\lambda_2 &= \lambda_1 + 1.
\end{align*}
\]  
(3)  
(4)

The relation (3), together with \( r(k - 1) = (n - 1)\lambda_1 + n(m - 1)\lambda_2 \), implies
\[
r = n + \lambda_1.
\]  
(5)

Since for an SRGD design \( k \) must be an integral multiple of \( m \) (cf. Raghavarao [3]), let
\[
k = cm,
\]  
(6)

where \( c \) is a positive integer and by (3)-(6),
\[
c = n(\lambda_1 + 1)/(n + \lambda_1) = (\lambda_1 + 1) - (\lambda_1 + 1)\lambda_1/(n + \lambda_1).
\]  
(7)

Also, by (5)-(7),
\[
b = vr/k = (n + \lambda_1)^2/(\lambda_1 + 1).
\]  
(8)

Clearly, \( n \) and \( \lambda_1 \) are such that both \( b \) and \( c \) are positive integers.

Defining
\[
a = n + \lambda_1, \quad s = \lambda_1 + 1,
\]  
(9)

it follows from (7) and (8) that \( s(s - 1)/a \) and \( a^2/s \) are both integral-valued. This holds trivially if \( s = 1 \) (i.e. \( \lambda_1 = 0 \)), in which case by (4)-(8), the parameters of the design are of the form
\[
v = mn, \quad b = n^2, \quad r = n, \quad k = m, \quad \lambda_1 = 0, \quad \lambda_2 = 1.
\]  
(10)

Consider now the further case \( s > 1 \) (i.e. \( \lambda_1 > 1 \)). Let \( d \) represent the integer \( s(s - 1)/a \). Then
\[
a = s(s - 1)/d.
\]  
(11)

Evidently, there exists a unique factorization of \( d \) such that
\[
d = pq,
\]  
(12)

and \( s/p \) and \( (s - 1)/q \) are integral-valued. Here \( p \) and \( q \) are relatively prime since so are \( s \) and \( s - 1 \). Let
\[ \frac{s}{p} = x, \frac{(s - 1)}{q} = y. \]  

Note that \( x \) and \( y \) have to be positive integers, since \( s > 1 \). Under (13), \( px - qy = 1 \), and, therefore, by our earlier discussion \( x \) and \( y \) must be of the form

\[ x = x_0 + tq, \; y = y_0 + tp \quad (t = 0, 1, 2, \ldots), \]  

where \((x_0, y_0)\) is the minimal solution of (1b). By (11)-(14),

\[ s = px = p(x_0 + tq), \]  

\[ s - 1 = qy = q(y_0 + tp), \]  

\[ a = s(s - 1)/d = (x_0 + tq)(y_0 + tp). \]

In the above \( t > 1 \), for \( t = 0 \) implies that \( a/s = y_0/p \leq 1 \) (by (2)), i.e. \( a \leq s \), which is impossible from (9) and the fact \( n \geq 2 \).

Now by (15a), (16),

\[ a^2/s = (x_0 + tq)(y_0 + tp)^2/p, \]

which must be integral-valued. As noted earlier, \( y_0 \) and \( p \) and hence \( y_0 + tp \) and \( p \) are relatively prime. Therefore, \( x_0 + tq \) must be an integral multiple of \( p \). Let \( z = (x_0 + tq)/p \). Then \( pz - qt = x_0 \), and comparing this with (1c), \( z \) and \( t \) are of the form

\[ z = g_0 + fq, \; t = h_0 + fp \quad (f = 0, 1, 2, \ldots), \]

\( g_0 \) and \( h_0 \) being as defined earlier. Hence

\[ (x_0 + tq)/p = [x_0 + (h_0 + fp)q]/p = (x_0 + h_0 q)/p + fq = g_0 +fq, \]

since \((g_0, h_0)\) is a solution of (1c).

By (15)-(18),

\[ s = p^2(g_0 + fq), \]  

\[ s - 1 = q(y_0 + (h_0 + fp)p), \]  

\[ a = p(g_0 + fq)(y_0 + (h_0 + fp)p). \]

Hence by (4)-(9),

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\[ n = a - (s - 1) = [y_o + (h_o + fp)p] [p(g_o + fq) - q], \]  
\[ v = mn = m[y_o + (h_o + fp)p] [p(g_o + fq) - q], \]  
\[ b = a^2/s = (g_o + fq) [y_o + (h_o + fp)p]^2, \]  
\[ r = a = p(g_o + fq) [y_o + (h_o + fp)p], \]  
\[ c = s - s(s - 1)/a = p[p(g_o + fq) - q], \]  
\[ k = cm = mp[p(g_o + fq) - q], \]  
\[ \lambda_1 = s - l = q[y_o + (h_o + fp)p], \]  
\[ \lambda_2 = s = p^2(g_o + fq), \]

where \( m \geq 2, f \geq 0, p \geq 1, q \geq 1 \) are integral-valued, \( p \) and \( q \) are relatively prime and \( y_o, g_o, h_o \) are functions of \( p \) and \( q \) as defined earlier.

Thus for an SRGD design with \( \lambda_2 = \lambda_1 + 1 \), the parameters must be of the form (10) or (19a-g). It is seen that the parameters of the design can be expressed in a closed form in terms of at most four integer parameters. It may, further, be remarked that the four parameters involved in (19a-g) are again not all independent since \( p \) and \( q \) have to be relatively prime. The series (10) occurs frequently in the available literature as one of the main series of GD designs.

The relations (10) and (19a-g) provide a natural classification of SRGD designs with \( \lambda_2 = \lambda_1 + 1 \). The designs with parameters as in (19a-g) may be further subclassified according to \( m, f, p \) and \( q \). Incidentally, from (10) and (19a-g), an SRGD design with \( \lambda_1 = 1 \) and \( \lambda_2 = 2 \) is non-existent.

In a large number of SRGD designs with \( \lambda_2 = \lambda_1 + 1 \), \( v \) is an integral multiple of \( k \) and it may be interesting to investigate this situation as a special case of (10) and (19a-g). For the series in (10), \( v \) is trivially an integral multiple of \( k \). Consider, therefore, the series described in (19a-g). Note that by (6), (7), (9), (14) and (16),

\[ v/k = (n + \lambda_1)/(\lambda_1 + 1) = a/s = (y_o + tp)/p, \]
and hence the integrality of \(v/k\) implies that \(y_o/p\) is an integer. Now by (2), and the fact that \(y_o\) and \(p\) are relatively prime, one must have \(p = 1\).

If \(p = 1\), then for arbitrary positive integer \(q\), it is easy to check that

\[
\begin{align*}
x_o &= q + 1, \quad y_o = 1, \quad g_o = 2q + 1, \quad h_o = 1, \quad \text{and hence (19a-g) reduce to}
\end{align*}
\]

\[
\begin{align*}
n &= (f + 2)[(f + 1)q + 1], \quad v = m(f + 2)[(f + 1)q + 1], \\
b &= (f + 2)^2[(f + 2)q + 1], \quad r = (f + 2)[(f + 2)q + 1], \\
k &= m[(f + 1)q + 1], \quad \lambda_1 = (f + 2)q, \quad \lambda_2 = (f + 2)q + 1,
\end{align*}
\]

(20)

where \(m(\geq 2), \quad f(\geq 0), \quad q(\geq 1)\) are integers. Combining (10) and (20), the parameters of an SRGD design with \(\lambda_2 = \lambda_1 + 1\), and, further, with \(v\) as an integral multiple of \(k\), may be expressed in a compact form as

\[
\begin{align*}
n &= (\ell + 1)(\ell + 1) + 1, \quad v = m(\ell + 1)(\ell + 1) + 1, \\
b &= (\ell + 1)^2(\ell + 2)(\ell + a + 1), \\
r &= (\ell + 1)(\ell + a + 1), \quad k = m(\ell + a + 1), \quad \lambda_1 = (\ell + 1)\alpha, \quad \lambda_2 = k\alpha + \alpha + 1,
\end{align*}
\]

where \(m(\geq 2), \quad \ell(\geq 1), \quad a(\geq 0)\) are integers.

Acknowledgment. The first author is thankful to the Indian National Science Academy, Japan Society for the Promotion of Science, Faculty of School Education, Hiroshima University and Indian Statistical Institute for grants that enabled him to carry out the work at the Hiroshima University.

References


付記：本論文は目下 "Discrete Mathematics" へ投稿中である。

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