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Some results on reflection principles in fragments of Peano arithmetic

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This is an abstract of the paper [3]. Let $I\Sigma_k$ denote the fragment of Peano arithmetic, whose axioms are Peano's axioms with induction restricted to Σ_k-formulas. We will develop a proof-theoretic study of various principles of fragments of Peano arithmetic such as reflection principles, transfinite inductions, well-ordering principles and large set principles, and compare their proof-theoretic strength.

Let $Pr_k(x)$ be a canonical representation of the provability predicate for $I\Sigma_k$. Let $I\Sigma_k + T_m$ be the arithmetic obtained from $I\Sigma_k$ by adding all true Π_m-sentences as additional axioms. In the following, the function symbol β represents the Gödel's β-function. So, $\beta(x,i) = y$ means that y is the i-th element of the sequence coded by x. We will define ordinals ω_n by $\omega_0 = 1$ and $\omega_{n+1} = \omega^{\omega_n}$. For each positive integer n, let $\prec_n$ be a canonical primitive recursive well-ordering of natural numbers of order-type ω_n. Sometimes, we will omit the subscript n of $\prec_n$, when no confusions will occur. For each natural number x, let $|x|_n$ denote the ordinal α represented by x in the well-ordering $\prec_n$. By abuse of notations, we will often write the ordinal α in place of x, when $\alpha = |x|_n$.

We will consider the following seven principles:

1) $RFN_{\Sigma_m}(I\Sigma_k)$ (*Σ_m-uniform reflection principle of $I\Sigma_k$*):

For any Σ_m-formula $\varphi(x)$, $\forall x Pr_k(\ulcorner\varphi(\dot{x})\urcorner) \supset \forall x\varphi(x)$.

2) $Con(\ I\Sigma_k + T_m\ )$ (*consistency of $I\Sigma_k + T_m$*).

3) $TI_{\Pi_m}[\omega_n]$ (*transfinite induction up to ω_n for Π_m-formulas*):

For any Π_m-formula $\psi(x)$,

$$\forall\alpha \prec \omega_n[\ \forall\gamma(\ \forall\beta(\ \beta \prec_n \gamma \supset \psi(\beta)\ ) \supset \psi(\gamma)\ ) \supset \psi(\alpha)\ ].$$

4) $WOP_{\Sigma_m}[\omega_n]$ (*well-ordering principle of ω_n for Σ_m-formulas*):

Let θ be a formula containing at least two free variables and let $F(\theta)$ denote the formula $\forall x\exists! y\theta(x,y)$. Then, $WOP_{\Sigma_m}[\omega_n]$ is the schema: *for any Σ_m-formula θ containing at least two free variables,*

$$F(\theta) \supset \exists x\exists y\exists z(\ \theta(x,y) \wedge \theta(x+1,z) \wedge \neg(\ z \prec_n y\ )\ ).$$

Roughly speaking, $WOP_{\Sigma_m}[\omega_n]$ means that if a function $f : \mathbb{N} \longrightarrow \omega_n$ is represented by some Σ_m-formula then the sequence f(0), f(1), f(2),... is not strictly descending with respect to $\prec_n$.

5) $LSP_{\Sigma_m}[\omega_n]$ (*ω_n-large set principle for Σ_m-formulas*).

Let [x,y] denote the set { z ; $x \le z \le y$ } of natural numbers. Suppose that θ is a formula containing at least two free variables. Then, $SIF(\theta)$ is the formula

$$\forall x\exists! y\theta(x,y) \wedge \forall x\forall y\forall z(\ (\ \theta(x,y) \wedge \theta(x+1,z)\ ) \supset y < z\ ),$$

which means that θ is the graph of a strictly increasing function. Let $\{\gamma\}(x)$ be the fundamental sequence defined in [1]. Let fs_n be the function symbol which represents the primitive recursive function fs_n such that $fs_n(u,x) = w$ if and only if $\{|u|_n\}(x) = |w|_n$. Now, for each $\alpha \prec \omega_n$ and each formula θ, we will abbreviate the following formula

$$\exists z[\ \beta(z,0) = \alpha \wedge \forall w<(y\dot{-}x)\exists u\exists t(\ \beta(z,w) = u \wedge \theta(x+w,t) \wedge \beta(z,w+1) = fs_n(u,t)\ ) \wedge \beta(z,y\dot{-}x) = 0\ ],$$

to '$[x,y]$ *is* (α,θ)-*large*'. Then, $LSP_{\Sigma_m}[\omega_n]$ is the schema;

for any Σ_m-*formula* θ *containing at least two free variables,*

$$SIF(\theta) \supset \forall\alpha \prec \omega_n \forall x\exists y(\ [x,y] \text{ is } (\alpha,\theta)\text{-large}\ ).$$

Clearly, $LSP_{\Sigma_m}[\omega_n]$ means that if a function f is represented by some Σ_m-formula then for any $\alpha \prec \omega_n$ $\forall x\exists y(\ [x,y]$ is (α,f)-large) holds. Here, we say that $[x,y]$ is (α,f)-large if the set $f([x,y])$ is α-large (see [2]).

6) $WOP^*_{\Sigma_m}[\omega_n]$ (*well-ordering principle of* ω_n *for* Σ_m-*definable functions*): *For any* Σ_m-*formula* θ *containing at least two free variables,*

$$Pr_m(\ulcorner F(\theta)\urcorner) \supset \exists x\exists y\exists z(\ \theta(x,y) \wedge \theta(x+1,z) \wedge \neg(\ z \prec_n y\ )\ ).$$

7) $LSP^*_{\Sigma_m}[\omega_n]$ (ω_n-*large set principle for* Σ_m-*definable functions*): *For any* Σ_m-*formula* θ *containing at least two free variables.*

$$Pr_m(\ulcorner SIF(\theta)\urcorner) \supset \forall\alpha \prec \omega_n \forall x\exists y(\ [x,y] \text{ is } (\alpha,\theta)\text{-large}\ ).$$

Then, we have the following theorems.

THEOREM 1. *Let* m *be positive integer.*

1) $I\Sigma_1 + RFN_{\Sigma_{m+1}}(I\Sigma_{m+n-1}) \vdash TI_{\Pi_m}[\omega_{n+1}]$ *for* $n > 0$.

2) *The following three theories are equivalent* ($n \geq 0$);

a. $I\Sigma_1 + TI_{\Pi_m}[\omega_{n+1}]$.

b. $I\Sigma_m + WOP_{\Sigma_m}[\omega_{n+1}]$.

c. $I\Sigma_m + LSP_{\Sigma_m}[\omega_{n+1}]$.

3) $I\Sigma_1 + TI_{\Pi_m}[\omega_{n+1}] \vdash I\Sigma_m + RFN_{\Sigma_m}(I\Sigma_{m+n-1})$ *for* $n > 0$.

4) *The following four principles are equivalent in* $I\Sigma_m$ ($n > 0$);

a. $RFN_{\Sigma_m}(I\Sigma_{m+n-1})$. *b.* $Con(I\Sigma_{m+n-1} + T_m)$.

c. $LSP^*_{\Sigma_m}[\omega_{n+1}]$. *d.* $WOP^*_{\Sigma_m}[\omega_{n+1}]$.

THEOREM 2. 1) *For each* $m > 0$, $I\Sigma_m \vdash RFN_{\Sigma_{m+1}}(I\Sigma_{m-1})$.
2) *For each* $k > 0$ *and* $m \geq 0$, $I\Sigma_k + T_m \nvdash RFN_{\Sigma_m}(I\Sigma_k)$ if $I\Sigma_k + T_m$ *is consistent.*

To prove Theorem 1. 3), we need to introduce Skolem functions, and reduce the original fragments of arithmetic to fragments in the extended language, having weaker mathematical induction. (As for details, see §3 of [3].)

In the following, we will give a proof of Theorem 1. 2). We assume the familiarity with Ketonen and Solovay [1] and Kurata [2]. We remark that both implications c. ==> b. and b. ==> a. can be proved in the same way as Theorems 2.5.5 and 2.5.6 in [2]. In either case, we need the Σ_m-*mathematical induction* $Ind\Sigma_m$ or the Σ_m-*least number principle* $L\Sigma_m$, which is equivalent to $\mathrm{Ind}\Sigma_m$. To show this, we will give here a detailed proof of the implication b. ==> a.

Let T be the theory obtained from $I\Sigma_m$ by adding

(1) $\forall x(\forall y(y \prec_n x \supset \varphi(y)) \supset \varphi(x))$

and

(2) $\exists z \neg\varphi(z)$

as additional axioms, where $\varphi(z)$ is a Π_m-formula. We can suppose that $\varphi(z)$ is $\forall u\psi(z,u)$ for a Σ_{m-1}-formula $\psi(z,u)$. Then, it follows from (1) that

(3) $T \vdash \forall x\forall u\exists y\exists v[\neg\psi(x,u) \supset (y \prec_n x \wedge \neg\psi(y,v))]$.

Let J be the primitive recursive pairing function defined by $J(x,y) = \frac{1}{2}[(x+y)^2+3x+y]$ and both K(z) and L(z) are primitive recursive projection functions satisfying that

i. $J(K(z),L(z)) = z$,

ii. $K(J(x,y)) = x$ and $L(J(x,y)) = y$.

Now, define $\theta(z)$ by $\neg\psi(K(z),L(z))$. Clearly, $\theta(z)$ is a Π_{m-1}-formula. From (3) it follows that

$$T \vdash \forall z\exists w(\theta(z) \supset (K(w) \prec_n K(z) \wedge \theta(w))).$$

Let $\xi(z,w)$ denote the formula

$$\theta(z) \supset (K(w) \prec_n K(z) \wedge \theta(w)).$$

Then, $\xi(z,w)$ belongs to Δ_m. Since

$$\exists w\xi(z,w) \supset \exists w(\xi(z,w) \wedge \forall u{<}w\neg\xi(z,u))$$

follows from $L\Sigma_m$,

$$T \vdash \forall z\exists!w(\xi(z,w) \wedge \forall u{<}w\neg\xi(z,u)).$$

Similarly, since $T \vdash \exists w\theta(w)$ and moreover $\exists w\theta(w) \supset \exists w(\theta(w) \wedge \forall u{<}w\neg\theta(u))$ follows from $L\Sigma_m$, we have

$$T \vdash \exists!w(\theta(w) \wedge \forall u{<}w\neg\theta(u)).$$

Now define Σ_m-formulas $\tau(x,t)$ and $\sigma(x,s)$ by

$$\tau(x,t) \equiv \exists z[\ \exists y(\ \beta(z,0) = y \wedge \theta(y) \wedge \forall u<y\neg\theta(u)\ )$$
$$\wedge \forall u<x\exists v\exists w(\ \beta(z,u) = v \wedge \xi(v,w) \wedge \forall r<w\neg\xi(v,r)$$
$$\wedge \beta(z,u+1) = w\ ) \wedge \beta(z,x) = t\ ],$$

and

$$\sigma(x,s) \equiv \exists t(\ \tau(x,t) \wedge s = K(t)\ ).$$

(Notice that τ and σ represent the graphs of functions g and f in the proof of Theorem 2.5.6 in [2], respectively.) By using $\mathrm{Ind}\Sigma_m$, both $F(\tau)$ and $F(\sigma)$ are provable in T. On the other hand,

$$T \vdash (\ \sigma(x,s) \wedge \sigma(x+1,s')\ ) \supset \exists t\exists t'[\ s = K(t)$$
$$\wedge\ s' = K(t') \wedge \tau(x,t) \wedge \tau(x+1,t') \wedge \xi(t,t')\ ].$$

Clearly, $\xi(t,t')$ implies $\theta(t) \supset K(t') \prec_n K(t)$, i.e., $\theta(t) \supset s' \prec_n s$. But by using $\mathrm{Ind}\Sigma_m$, $T \vdash \tau(x,t) \supset \theta(t)$. Therefore,

$$T \vdash \forall x\forall s\forall s'(\ (\ \sigma(x,s) \wedge \sigma(x+1,s')\ ) \supset s' \prec_n s\ ).$$

Hence, $\mathrm{WOP}_{\Sigma_m}[\omega_n]$ for σ fails in T. By taking the contra-position, we have

$$I\Sigma_m + \mathrm{WOP}_{\Sigma_m}[\omega_n] \vdash \mathrm{TI}_{\Pi_m}[\omega_n].$$

Next, we will show that $I\Sigma_m + \mathrm{TI}_{\Pi_m}[\omega_n] \vdash \mathrm{LSP}_{\Sigma_m}[\omega_n]$. We remark here that $\mathrm{TI}_{\Pi_m}[\omega_n]$ is equivalent in $I\Sigma_1$ to the schema

$$(4)\quad \exists x\psi(x) \supset \exists y[\ \psi(y) \wedge \forall z(\ z \prec_n y \supset \neg\psi(z)\ )\ ],$$

where $\psi(x)$ is a Σ_m-formula. Let T be the theory obtained from $I\Sigma_m$ by adding the above schema (4) and the formula $\mathrm{SIF}(\theta)$ for a Σ_m-formula θ, as additional axioms. Let α be an ordinal such that $\alpha \prec \omega_n$. For a given number x, let $\theta^*(s,v)$ denote the following Σ_m-formula;

$$\exists z[\ \beta(z,0) = \alpha \wedge \forall w{<}s \exists u \exists t(\ \beta(z,w) = u \wedge \theta(x+w,t)$$
$$\wedge\ \beta(z,w+1) = fs_n(u,t)\ ) \wedge \beta(z,s) = v\ ].$$

Clearly, $\theta^*(s,0)$ means that $[x,x+s]$ is (α,θ)-large. When $\alpha = 0$, it is obvious that $T \vdash \exists s\theta^*(s,0)$. So, suppose otherwise. Let $\psi(r)$ be $\exists s \exists v(\ \theta^*(s,v) \wedge v \prec_n r\ )$. Then, $T \vdash \exists r\psi(r)$. Thus, by the schema (4)

$$T \vdash \exists r[\ \psi(r) \wedge \forall z(\ z \prec_n r \supset \neg\psi(z)\ )\ ].$$

Take such an r. Then, $\exists s \exists v(\ \theta^*(s,v) \wedge v \prec_n r\ )$. Take also such s and v. Then, $\neg\psi(v)$ holds. Hence

(5) $\quad T \vdash \forall s' \forall v'(\ \theta^*(s',v') \supset \neg(\ v' \prec_n v\ )\ ).$

On the other hand,

$$T \vdash \forall s' \exists w \theta^*(s',w)$$

by using $\mathrm{Ind}\Sigma_m$. In particular, $T \vdash \exists w\theta^*(s+1,w)$. Thus we have

$$T \vdash \exists s \exists v \exists w(\ \theta^*(s,v) \wedge \theta^*(s+1,w) \wedge \neg(\ w \prec_n v\ )\ )$$

by (5). But, fundamental sequences have the property;

$$T \vdash \forall s \forall v \forall w(\ (\ \theta^*(s,v) \wedge \theta^*(s+1,w) \wedge 0 \prec_n v\ ) \supset w \prec_n v\ ).$$

Hence, $T \vdash \exists s\theta^*(s,0)$. Therefore,

$$\forall \alpha \prec \omega_n\ \forall x \exists y(\ [x,y] \text{ is } (\alpha,\theta)\text{-large}\ )$$

is provable in T.

Remark here that $I\Sigma_1 + TI_{\Pi_m}[\alpha] \vdash L\Sigma_m$ if $\omega \preccurlyeq \alpha$. Thus, $I\Sigma_m + TI_{\Pi_m}[\omega_n]$ is equivalent to $I\Sigma_1 + TI_{\Pi_m}[\omega_n]$ when $n > 0$. Therefore, we have Theorem 1. 2).

REFERENCES

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